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Alternating Sums in the Hosoya Polynomial Triangle

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Abstract

The *Hosoya polynomial triangle* is a triangular arrangement of polynomials where each entry is a product of two polynomials. The geometry of this triangle is a good tool to study the algebraic properties of polynomial products. In particular, we find closed formulas for the alternating sum of products of polynomials such as Fibonacci polynomials, Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell polynomials, Fermat polynomials, Jacobsthal polynomials, and other familiar sequences of polynomials.

1 Introduction

The *generalized Fibonacci polynomial* is a recursive sequence that generalizes the Fibonacci numbers. The *Hosoya triangle* [6, 11, 12] consists of a triangular array of numbers where each of its entries is a product of two Fibonacci numbers (see Table 5). If in this triangle instead of Fibonacci numbers we put generalized Fibonacci polynomials, then we obtain the Hosoya polynomial triangle (see Table 3). The new triangle provides a good geometry to study algebraic and combinatorial properties of products of recursive sequences of polynomials. Here we use the Hosoya polynomial triangle to analyze the behavior of alternating sums of products of generalized Fibonacci polynomials. In particular, we analyze alternating sums of products of Fibonacci, Lucas, and other well-known sequences of polynomials.

Some known examples of generalized Fibonacci polynomials are the Fibonacci polynomials, Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell polynomials, Jacobsthal polynomials, and Fermat polynomials. Since the entries of the Hosoya polynomial triangle are products of two generalized Fibonacci polynomials, the triangle varies depending on the type of polynomials that its entries have (see Tables 2, 3, and 4). All these polynomials also have representations using Binet formulas.

In this paper we find two closed formulas for the alternating sum of the entries in the nth row of the Hosoya polynomial triangle. There are two Binet formulas for generalized Fibonacci polynomials, depending on the Binet representation of each polynomial we obtain a formula for its alternating sum. For instance, if we consider that the entries of the nth row of the Hosoya polynomial triangle to be products of Fibonacci numbers (see Table 5), then

$$F_nF_1 - F_{n-1}F_2 + F_{n-1}F_2 - \cdots + F_1F_n = F_{n+1}.$$

Evaluating the alternating sums found here, at integers, we obtain several well-known sequences (see Table 7). Those sequences can be found, for example, in Sloane [16].

2 Preliminaries and examples

In this section we give some examples, introduce notation, and some definitions that are going to be used throughout the paper. Some of them are well-known; however we prefer to restate them here as it will avoid ambiguities.

2.1 Generalized Fibonacci polynomials and generalized Hosoya polynomials

In the literature there are several definitions of generalized Fibonacci polynomials. However, the definition that we introduce here is simpler and covers other definitions. The *generalized Fibonacci polynomial* sequence is defined by the recurrence relation

$$G_0(x) = p_0(x),\ G_1(x) = p_1(x),\ \text{and } G_n(x) = d(x)G_{n-1}(x) + g(x)G_{n-2}(x) \text{ for } n \geq 2 \qquad (1)$$

where $p_0(x)$, $p_1(x)$, $d(x)$, and $g(x)$ are polynomials in $\mathbb{Z}[x]$.

For example, if we take $p_0(x) = 1$, $p_1(x) = x$, $d(x) = x$, and $g(x) = 1$ we obtain the regular Fibonacci polynomial sequence. That is, the Fibonacci polynomial sequence $F_n(x)$, is defined by the recurrence relation

$$F_0(x) = 1,\ F_1(x) = x,\ \text{and } F_n = xF_{n-1}(x) + F_{n-2}(x) \text{ for } n \geq 2.$$

Letting $x = 1$ and choosing the correct values for $p_0(x)$, $p_1(x)$, $d(x)$, and $g(x)$, the generalized Fibonacci polynomial sequence gives rise to three classical numerical sequences, the Fibonacci sequence, the Lucas sequence, and the generalized Fibonacci sequence.

In Table 1 there are more familiar examples of generalized Fibonacci polynomials (see [9, 10, 12]). Hoggatt and Bicknell-Johnson [7] show that Schechter polynomials are another example of generalized Fibonacci polynomials.

We now give a formal definition of some of those polynomials. The *Chebyshev polynomials* $T_n(x)$ of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equations. This polynomial can be defined as the contour integral where the contour encloses the origin and is traversed in a counterclockwise direction,

$$T_n(x) = \frac{1}{4\pi i} \oint \frac{(1-t^2)t^{-n-1}}{1-2tz+t^2} dt.$$

The Chebyshev polynomials of the first kind satisfy the recurrence relation given in the generalized Fibonacci polynomial sequence. Indeed,

$$T_0(x) = 1, \qquad T_1(x) = x, \quad \text{and} \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \text{ for } n \geq 2.$$

For instance,

$$T_2(x) = 2x^2 - 1; \quad T_3(x) = 4x^3 - 3x; \quad T_4(x) = 8x^4 - 8x^2 + 1; \quad T_5(x) = 16x^5 - 20x^3 + 5x.$$

Morgan-Voyce [14] introduced two type of polynomial sequences. Here we use one of them and we also use a generalized version of the Morgan-Voyce polynomials given by André-Jeannin [2, 3, 4]. We note that Morgan-Voyce polynomials are well known in the study of electrical networks ([12, pp. 480–495] and [13, 14, 15]). The *Morgan-Voyce polynomials* are defined recursively as

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$$
$$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x)$$

for $n \geq 2$, where $B_0(x) = 1$, $C_0(x) = 2$, $B_1(x) = C_1(x) = x + 2$. For example,

$$B_2(x) = x^2 + 4x + 3; \quad B_3(x) = x^3 + 6x^2 + 10x + 4; \quad B_4(x) = x^4 + 8x^3 + 21x^2 + 20x + 5,$$

$$C_2(x) = x^2 + 4x + 2; \quad C_3(x) = x^3 + 6x^2 + 9x + 2; \quad C_4(x) = x^4 + 8x^3 + 20x^2 + 16x + 2.$$

Polynomial	Initial value $G_0(x) = p_0(x)$	Initial value $G_1(x) = p_1(x)$	Recursive Formula $G_n(x) = d(x)G_{n-1}(x) + g(x)G_{n-2}(x)$
Fibonacci	1	x	$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$
Lucas	2	x	$D_n(x) = xD_{n-1}(x) + D_{n-2}(x)$
Pell	1	$2x$	$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$
Pell-Lucas	2	$2x$	$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$
Fermat	1	$3x$	$\Phi_n(x) = 3x\Phi_{n-1}(x) - 2\Phi_{n-2}(x)$
Fermat-Lucas	2	$3x$	$\vartheta_n(x) = 3x\vartheta_{n-1}(x) - 2\vartheta_{n-2}(x)$
Chebyshev first kind	1	x	$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
Chebyshev second kind	1	$2x$	$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$
Jacobsthal	1	1	$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x)$
Jacobsthal-Lucas	2	1	$j_n(x) = j_{n-1}(x) + 2xj_{n-2}(x)$
Morgan-Voyce	1	$x + 2$	$B_n(x) = (x + 2)B_{n-1}(x) - B_{n-2}(x)$
Morgan-Voyce	2	$x + 2$	$C_n(x) = (x + 2)C_{n-1}(x) - C_{n-2}(x)$

Table 1: Recurrence relation of some generalized Fibonacci polynomials.

2.2 Hosoya polynomials and the Hosoya polynomial triangle

We now give a precise definition of both the Hosoya polynomial sequence and the Hosoya polynomial triangle. The *Hosoya polynomial sequence* $\{H(r,k)\}_{r,k\geq 0}$ is defined using the double recursion

$$H(r,k) = \delta(x)H(r-1,k) + \gamma(x)H(r-2,k)$$

and

$$H(r,k) = \delta(x)H(r-1,k-1) + \gamma(x)H(r-2,k-2)$$

with initial conditions

$$H(0,0) = p_0(x)^2; \quad H(1,0) = p_0(x)p_1(x); \quad H(1,1) = p_0(x)p_1(x); \quad H(2,1) = p_1(x)^2,$$

where $r > 1$ and $0 \leq k \leq r-1$, and $\delta(x), \gamma(x)$, $p_0(x)$, and $p_1(x)$ are polynomials in $\mathbb{Z}[x]$. This sequence gives rise to the *Hosoya polynomial triangle*, where the entry in position k (taken from left to right) of the rth row is equal to $H(r,k)$ (see Table 2).

Proposition 1 is a generalization of [12, (15.4), p. 188] and [5, Proposition 1]. The proof of Proposition 1 has some similarities to the proof of [5, Proposition 1] which was done for integers. In Proposition 1 we prove that Table 2 is equal to Table 3. Here it is worth mentioning that in Table 3, for brevity, we use the notation G_k instead of $G_k(x)$. We are

					$H(0,0)$					
				$H(1,0)$		$H(1,1)$				
			$H(2,0)$		$H(2,1)$		$H(2,2)$			
		$H(3,0)$		$H(3,1)$		$H(3,2)$		$H(3,3)$		
	$H(4,0)$		$H(4,1)$		$H(4,2)$		$H(4,3)$		$H(4,4)$	
$H(5,0)$		$H(5,1)$		$H(5,2)$		$H(5,3)$		$H(5,4)$		$H(5,5)$

Table 2: Hosoya polynomial triangle.

				$G_0\,G_0$				
			$G_0\,G_1$		$G_1\,G_0$			
		$G_0\,G_2$		$G_1\,G_1$		$G_2\,G_0$		
	$G_0\,G_3$		$G_1\,G_2$		$G_2\,G_1$		$G_3\,G_0$	
$G_0\,G_4$		$G_1\,G_3$		$G_2\,G_2$		$G_3\,G_1$		$G_4\,G_0$

Table 3: $H(r,k) = G_k(x)G_{r-k}(x)$.

however interested in particular cases of Proposition 1 which we state as Corollaries 2, 3, and 4.

In this paper we are interested in the relationship between the points of the Hosoya polynomial triangle and the products of generalized Fibonacci polynomials. To establish this relation we need that $\delta(x) = d(x)$ and $\gamma(x) = g(x)$ where $d(x)$ and $g(x)$ are polynomials as defined in (1) and $\delta(x)$ and $\gamma(x)$ are polynomials as defined in the Hosoya polynomial sequence. So, for the rest of the paper we assume that $\delta(x) = d(x)$ and $\gamma(x) = g(x)$.

Proposition 1. *$H(r,k) = G_k(x)G_{r-k}(x)$.*

Proof. We prove this proposition using the method of mathematical induction. Let $E(k)$ be the statement that $H(r,k) = G_k(x)G_{r-k}(x)$ for $k \in \mathbb{Z}_{\geq 0}$ and some fixed $r \geq k$.

In order to prove the basis step we replace the entries of the generalized Hosoya triangle with generalized Fibonacci polynomials in the proof of [5, Proposition 1] to obtain $H(r,0) = p_0(x)G_r(x)$ and $H(r,1) = p_1(x)G_{r-1}(x)$ for some fixed value of r. Note that here $p_0(x)$ is the polynomial $G_0(x)$ and $p_1(x)$ is the polynomial $G_1(x)$ defined in the generalized Fibonacci polynomials and therefore the basis step is proved.

Now for the inductive step, we assume that the statements $E(k-1)$ and $E(k)$ are true for $k \geq 2$ and prove that $E(k+1)$ is true. Let r be a non-negative integer such that $r \geq k$, since $E(k-1)$ and $E(k)$ are true, we can write,

$$H(r-2,k-1) = G_{k-1}(x)G_{r-k-1}(x) \text{ and } H(r-1,k) = G_k(x)G_{r-k-1}(x).$$

This and the recurrence relation $H(r,k+1) = \delta(x)H(r-1,k) + \gamma(x)H(r-2,k-1)$ imply that

$$\begin{aligned}H(r,k+1) &= \delta(x)G_k(x)G_{r-k-1}(x)+\gamma(x)G_{k-1}(x)G_{r-k-1}(x)\\ &= G_{r-k-1}(x)(\delta(x)G_k(x)+\gamma(x)G_{k-1}(x))\\ &= G_{r-k-1}(x)G_{k+1}(x).\end{aligned}$$

Thus, $E(k+1)$ is true. This proves the proposition. □

We prove Corollary 2, the proofs of Corollaries 3 and 4 are similar and we omit them. The notation used in Corollaries 2, 3, and 4 are defined in Table 1. We recall that $\delta(x)$ and $\gamma(x)$ represent the polynomials used to define $H(r,k)$, and $p_0(x)$ and $p_1(x)$ represent the polynomials used to define $G_n(x)$.

Corollary 2. *If $p_0(x)=1$ and $p_1(x)=\delta(x)$, then for $r,k\geq 0$,*

$$H(r,k)=\begin{cases}F_k(x)F_{r-k}(x), & \text{if } \gamma(x)=1 \quad \text{and } p_1(x)=x;\\ U_k(x)U_{r-k}(x), & \text{if } \gamma(x)=-1 \quad \text{and } p_1(x)=2x;\\ B_k(x)B_{r-k}(x), & \text{if } \gamma(x)=-1 \quad \text{and } p_1(x)=x+2;\\ P_k(x)P_{r-k}(x), & \text{if } \gamma(x)=1 \quad \text{and } p_1(x)=2x;\\ J_k(x)J_{r-k}(x), & \text{if } \gamma(x)=2x \quad \text{and } p_1(x)=1.\end{cases}$$

Proof. We prove the corollary only for the first case; the proof of other cases are similar and we omit their proofs.

Suppose that $p_0(x)=1$, $p_1(x)=\delta(x)=x$, and $\gamma(x)=1$. We show that $H(r,k)=F_k(x)F_{r-k}(x)$. If $G_n(x)=F_n(x)$, then this along with the initial conditions given above in the proof of Proposition 1 imply that $H(r,k)=F_k(x)F_{r-k}(x)$. □

Corollary 3. *If $p_0(x)=2$ and $p_1(x)=\delta(x)$, then for $r,k\geq 0$,*

$$H(r,k)=\begin{cases}Q_k(x)Q_{r-k}(x), & \text{if } \gamma(x)=1 \quad \text{and } p_1(x)=2x;\\ C_k(x)C_{r-k}(x), & \text{if } \gamma(x)=-1 \quad \text{and } p_1(x)=x+2;\\ D_k(x)D_{r-k}(x), & \text{if } \gamma(x)=1 \quad \text{and } p_1(x)=x;\\ j_k(x)j_{r-k}(x), & \text{if } \gamma(x)=2x \quad \text{and } p_1(x)=1.\end{cases}$$

Corollary 4. *If $r,k\geq 0$, then*

$$H(r,k)=\begin{cases}\Phi_k(x)\Phi_{r-k}(x), & \text{if } p_0(x)=\gamma(x)+3=1 \quad \text{and } 3\delta(x)=p_1(x)=3x;\\ T_k(x)T_{r-k}(x), & \text{if } p_0(x)=-\gamma(x)=1 \quad \text{and } \delta(x)=2p_1(x)=2x;\\ \vartheta_k(x)\vartheta_{r-k}(x), & \text{if } p_0(x)=-\gamma(x)=2 \quad \text{and } 3\delta(x)=p_1(x)=3x.\end{cases}$$

As a consequence of the corollaries above, we obtain a Hosoya polynomial triangle for each of the specific polynomials mentioned in Table 1. For example, using the first result of Corollary 2 and Proposition 1 we obtain the Hosoya polynomial triangle where the entry $H(r,k)$ is equal to $F_k(x)F_{r-k}(x)$ (see Table 4). Similarly, we can obtain the Hosoya polynomial triangles with entries $H(r,k)$ as described in Corollaries 2, 3, and 4.

				1				
			x		x			
		x^2+1		x^2		x^2+1		
	x^3+2x		x^3+x		x^3+x		x^3+2x	
x^4+3x^2+1		$x(x^3+2x)$		$(x^2+1)^2$		$x(x^3+2x)$		x^4+3x^2+1

Table 4: The Hosoya polynomial triangle where $H(r,k) = F_k(x)F_{r-k}(x)$.

3 Alternating sum of points in a row of the Hosoya polynomial triangle

We begin this section by presenting some numerical examples of alternating sums. We follow this up with a discussion on Binet formulas which play an important role in the proof of the main results of this paper. We finally state and prove the main results at the end of this section.

3.1 Examples

If we set $x = 1$ in Table 4, then we obtain the regular Hosoya triangle, see Table 5. Note that the alternating sum of all points in an odd row of this triangle is zero, due to the symmetry of the triangle. However, the alternating sum of all points in an even row does not satisfy this property. For instance, if we take the alternating sum of all points in the second, fourth, and sixth row of Table 5, the results are F_3, F_5, and F_7, respectively, (see the following example).

$2-1+2=3=F_3;\quad 5-3+4-3+5=8=F_5;\quad 13-8+10-9+10-8+13=21=F_7.$

From this example and Table 5 we can see that

$$\sum_{i=0}^{2t}(-1)^i H(2t,i) = \sum_{i=0}^{2t}(-1)^i F_i \cdot F_{2t-i} = F_{2t+1}.$$

We now give two examples that generalize this fact to polynomials. If we take $H(r,k) = F_k(x)F_{r-k}(x)$ then the alternating sum of the terms in an even row of the Hosoya polynomial triangle is the quotient of two Fibonacci polynomials. The following is the alternating sum of all points on the fourth row of Table 4.

$$\begin{aligned}
\sum_{i=0}^{4}(-1)^i H(4,i) &= \sum_{i=0}^{4}(-1)^i F_i(x)\cdot F_{4-i}(x)\\
&= (x^4+3x^2+1)-(x^4+2x^2)+(x^2+1)^2-(x^4+2x^2)+(x^4+3x^2+1)\\
&= x^4+4x^2+3\\
&= \frac{F_5(x)}{F_1(x)}.
\end{aligned}$$

						1						
					1		1					
				2		1		2				
			3		2		2		3			
		5		3		4		3		5		
	8		5		6		6		5		8	
13		8		10		9		10		8		13

Table 5: The Hosoya triangle.

If we take $H(r,k) = B_k(x)B_{r-k}(x)$, then we obtain

$$\begin{aligned}\sum_{i=0}^{4}(-1)^i H(4,i) &= \textstyle\sum_{i=0}^{4}(-1)^i B_i(x)\cdot B_{4-i}(x)\\ &= x^4+8x^3+20x^2+16x+5\\ &= \frac{B_5(x)}{B_1(x)}.\end{aligned}$$

Those examples will be stated in Corollary 8. Note that this property does not hold for every choice of $H(r,k)$. For example, if we take $H(r,k) = T_k(x)T_{r-k}(x)$, then it is easy to see that $\sum_{i=0}^{2}(-1)^i T_i(x)\cdot T_{2t-i}(x) \neq T_3(x)/T_1(x)$. We analyze this case in Corollary 10.

3.2 Binet formulas

A generalized Fibonacci polynomial which satisfies the Binet formula (2) is said to be of *first type* and it is of *second type* if it satisfies the Binet formula (3). Horadam [8] and André-Jeannin [1] have studied these polynomials in detail. The following are the Binet formulas mentioned above,

$$L_n(x) = \frac{a^n(x)+b^n(x)}{\alpha}, \tag{2}$$

$$R_n(x) = \frac{a^{n+1}(x)-b^{n+1}(x)}{a(x)-b(x)}, \tag{3}$$

where $a(x)+b(x) = d(x)$ and $a(x)b(x) = -g(x)$. If $d^2(x)+4g(x) \geq 0$, then we have $a(x)-b(x) = \sqrt{d^2(x)+4g(x)}$ where $d(x)$ and $g(x)$ are the polynomials defined on the generalized Fibonacci polynomials.

The sequence of polynomials that have Binet representations $R_n(x)$ or $L_n(x)$ depend only on $d(x)$ and $g(x)$ defined on the generalized Fibonacci polynomials. We say that a generalized Fibonacci sequence of first (second) type is *equivalent* to a sequence of the second (first) type, if their recursive sequences are determined by the same polynomials $d(x)$ and $g(x)$. Notice that two equivalent polynomials have the same $a(x)$ and $b(x)$ in their Binet representations.

For example, the Lucas polynomial is of first type, whereas the Fibonacci polynomial is of second type. Lucas and Fibonacci polynomials are equivalent because $d(x) = x$ and $g(x) = 1$ (see Table 1). Note that in their Binet representations they both have $a(x) = (x+\sqrt{x^2+4})/2$ and $b(x) = (x - \sqrt{x^2+4})/2$. We use the results in [1] and [8] to classify all polynomials in Table 1 as pairs of equivalent polynomials (see Table 6). The leftmost polynomials in Table 6 are of the first type their equivalent polynomials are in the third column on the same line. In the last two columns of Table 6 we can see the $a(x)$ and $b(x)$ that the pairs of equivalent polynomials share. It is easy to obtain the characteristic equations of the sequences given in Table 1, and the roots of the equations are $a(x)$ and $b(x)$.

Polynomial First type	$L_n(x)$	Polynomial of Second type	$R_n(x)$	$a(x)$	$b(x)$
Lucas	$D_n(x)$	Fibonacci	$F_n(x)$	$(x+\sqrt{x^2+4})/2$	$(x-\sqrt{x^2+4})/2$
Pell-Lucas	$Q_n(x)$	Pell	$P_n(x)$	$x+\sqrt{x^2+1}$	$x-\sqrt{x^2+1}$
Fermat-Lucas	$\vartheta_n(x)$	Fermat	$\Phi_n(x)$	$(3x+\sqrt{9x^2-8})/2$	$(3x-\sqrt{9x^2-8})/2$
Chebyshev first kind	$T_n(x)$	Chebyshev second kind	$U_n(x)$	$x+\sqrt{x^2-1}$	$x-\sqrt{x^2-1}$
Jacobsthal-Lucas	$j_n(x)$	Jacobsthal	$J_n(x)$	$(1+\sqrt{1+8x})/2$	$(1-\sqrt{1+8x})/2$
Morgan-Voyce	$C_n(x)$	Morgan-Voyce	$B_n(x)$	$(x+2+\sqrt{x^2+4x})/2$	$(x+2-\sqrt{x^2+4x})/2$

Table 6: $R_n(x)$ equivalent to $L_n(x)$.

3.3 Main results

Lemma 5. *The equivalent sequence of a generalized Fibonacci polynomial sequence always exists.*

Proof. Without loss of generality we suppose that $G_n(x)$ is a generalized Fibonacci polynomial sequence of the first type. From the Binet formula (2) we know that there are $a(x)$ and $b(x)$ such that $a(x) + b(x) = d(x)$, $a(x)b(x) = -g(x)$, and when $d^2(x) + 4g(x) \geq 0$, we have $a(x) - b(x) = \sqrt{d^2(x) + 4g(x)}$. It is easy to see that $a(x)$ and $b(x)$ give rise to a polynomial of the second type. □

Lemma 6. *If $G_k(x)$ is a generalized Fibonacci polynomial that satisfies the Binet formula given in (3), then*

$$\sum_{k=0}^{\infty} G_k(x)t^k = \frac{1}{(1-a(x)t)(1-b(x)t)}.$$

Proof. Let $f(t,x)$ be $\sum_{k=0}^{\infty} G_k(x)t^k$. Since the sequence $G_n(x)$ satisfies the Binet formula given in (3), $G_n(x) = [a^{n+1}(x) - b^{n+1}(x)] \,/\, [a(x) - b(x)]$. Therefore,

$$f(t,x) = \sum_{k=0}^{\infty} \frac{a^{k+1}(x) - b^{k+1}(x)}{a(x) - b(x)} t^k. \tag{4}$$

It is known that $\sum_{k=0}^{\infty} \alpha t^k = \alpha/(1-t)$. This and the functions $a(x)$ and $b(x)$, imply that

$$\sum_{k=0}^{\infty} a(x)\left(a(x)t\right)^k = \frac{a(x)}{1-a(x)t}, \quad \text{and} \quad \sum_{k=0}^{\infty} b(x)\left(b(x)t\right)^k = \frac{b(x)}{1-b(x)t}.$$

Therefore, these two generating functions, and (4) imply that

$$f(t,x) = \frac{1}{a(x)-b(x)} \sum_{k=0}^{\infty} \left(a^{k+1}(x) - b^{k+1}(x)\right) t^k = \frac{1}{a(x)-b(x)} \left(\frac{a(x)}{1-a(x)t} - \frac{b(x)}{1-b(x)t}\right).$$

So, simplifying we obtain that $f(t,x) = 1/\left[(1-a(x)t)(1-b(x)t)\right]$. □

Theorem 7. *If $G_n(x)$ is a generalized Fibonacci polynomial that satisfies the Binet formula given in (3), then*

$$\sum_{k=0}^{r} (-1)^k H(r,k) = \begin{cases} G_{r+1}(x)/G_1(x), & \text{if } r \text{ is even;} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$$

Proof. If r is odd, by symmetry of the Hosoya polynomial triangle, it is easy to see that $\sum_{k=0}^{r}(-1)^k G_k(x) G_{r-k}(x) = 0$. To prove the case where r is even, we use generating functions. Let $f(t,x)$ be $\sum_{k=0}^{\infty} G_k(x) t^k$. Since $G_k(x)$ satisfies the Binet formula given in (3), by Lemma 6 we have that $f(t,x)$ is equal to $1/\left[(1-a(x)t)(1-b(x)t)\right]$. Therefore,

$$\begin{aligned} f(t,x) - f(-t,x) &= \frac{1}{(1-a(x)t)(1-b(x)t)} - \frac{1}{(1+a(x)t)(1+b(x)t)} \\ &= \frac{2t(a(x)+b(x))}{(1-a(x)t)(1-b(x)t)(1+a(x)t)(1+b(x)t)}. \end{aligned}$$

Simplifying we have that

$$\begin{aligned} f(t,x) - f(-t,x) &= 2(a(x)+b(x))tf(t,x)f(-t,x) \\ &= \frac{2(a^2(x)-b^2(x))}{a(x)-b(x)} tf(t,x)f(-t,x) \\ &= 2G_1(x)tf(t,x)f(-t,x) \\ &= 2G_1(x)t \left(\sum_{k=0}^{\infty} G_k(x)t^k\right)\left(\sum_{k=0}^{\infty} G_k(x)(-t)^k\right). \end{aligned}$$

Thus,

$$f(t,x) - f(-t,x) = 2G_1(x) \sum_{r=0}^{\infty} \left(\sum_{k=0}^{r} (-1)^{r-k} G_k(x) G_{r-k}(x)\right) t^{r+1}. \tag{5}$$

We note that $f(t,x) - f(-t,x)$ is also equal to

$$\sum_{j=0}^{\infty} G_j(x)t^j - \sum_{j=0}^{\infty} G_j(x)(-t)^j = \sum_{j=0}^{\infty}(1-(-1)^j)G_j(x)t^j = 2\sum_{j=0}^{\infty} G_{2j+1}(x)t^{2j+1}.$$

From the right side of this last equality and from the right side of (5) we have that

$$\sum_{r=0}^{\infty} 2G_1(x)\left(\sum_{k=0}^{r}(-1)^{r-k}G_k(x)G_{r-k}(x)\right)t^{r+1} = \sum_{j=0}^{\infty} 2G_{2j+1}(x)t^{2j+1}.$$

It is known that two generating functions are equal if their corresponding coefficients of t^i are equal for all i. We are only interested in the cases when r is even. So, we analyze the case in which $r+1 = 2j+1$ in the last equality above. Thus, if $r+1 = 2j+1$, then

$$2G_1(x)\sum_{k=0}^{r}(-1)^{r-k}G_k(x)G_{r-k}(x) = 2G_{2j+1}(x),$$

simplifying,

$$\sum_{k=0}^{r}(-1)^{r-k}G_k(x)G_{r-k}(x) = \frac{G_{r+1}(x)}{G_1(x)}.$$

Since $H(r,k) = G_k(x)G_{r-k}(x)$, this completes the proof of the theorem. □

The notation used in Corollaries 8 and 10 are defined in Tables 1 and 6.

Corollary 8. *If t is a positive integer, then $\sum_{i=0}^{2t+1}(-1)^i H(2t+1,i) = 0$, and*

$$\sum_{i=0}^{2t}(-1)^i H(2t,i) = \begin{cases} F_{2t+1}(x)/F_1(x), & \text{if } H(2t,i) = F_i(x)F_{2t-i}(x); \\ B_{2t+1}(x)/B_1(x), & \text{if } H(2t,i) = B_i(x)B_{2t-i}(x); \\ P_{2t+1}(x)/P_1(x), & \text{if } H(2t,i) = P_i(x)P_{2t-i}(x); \\ J_{2t+1}(x)/J_1(x), & \text{if } H(2t,i) = J_i(x)J_{2t-i}(x); \\ \Phi_{2t+1}(x)/\Phi_1(x), & \text{if } H(2t,i) = \Phi_i(x)\Phi_{2t-i}(x); \\ U_{2t+1}(x)/U_1(x), & \text{if } H(2t,i) = U_i(x)U_{2t-i}(x). \end{cases}$$

Proof. We prove the corollary only for the case in which $H(2t,i) = F_i(x)F_{2t-i}(x)$, the proof of the other five cases are similar. Therefore, omit their proofs.

From Table 6 we know that $F_n(x)$ is of second type, therefore it satisfies (3) where $a(x) = (x+\sqrt{x^2+4})/2$ and $b(x) = (x-\sqrt{x^2+4})/2$. This and Theorem 7 with $G_k(x) = F_k(x)$ imply that

$$\sum_{i=0}^{2t}(-1)^i H(2t,i) = \frac{F_{2t+1}(x)}{F_1(x)}.$$

This proves the corollary. □

Theorem 9. *If $G_n(x)$ is a generalized Fibonacci polynomial that satisfies the Binet formula (2), then for $r \geq 2$*

$$\sum_{k=0}^{r}(-1)^k H(r,k) = \begin{cases} \dfrac{(G_0(x))^2 R_{r+1}(x) - (G_1(x))^2 R_{r-1}(x)}{R_1(x)}, & \text{if } r \text{ is even;} \\ 0; & \text{if } r \text{ is odd.} \end{cases}$$

where $R_n(x)$ is equivalent to $G_n(x)$.

Proof. If r is odd, by symmetry of the Hosoya polynomial triangle, it is easy to see that $\sum_{k=0}^{r}(-1)^k G_k(x)G_{r-k}(x) = 0$. To prove the case in which r is even, we use generating functions. We suppose that $G_n(x)$ is a sequence of generalized Fibonacci polynomials which satisfy Binet formula (2) and let $R_n(x)$ be its equivalent sequence of polynomials which satisfy Binet formula (3). Let $g(t,x)$ be $\sum_{k=0}^{\infty} G_k(x)t^k$ and let $f(t,x)$ be $\sum_{k=0}^{\infty} R_k(x)t^k$. Since $R_n(x)$ satisfies the Binet formula given in (3), by Lemma 6 we have that $f(t,x)$ is equal to $1/\left[(1-a(x)t)(1-b(x)t)\right]$.

Since $G_n(x)$ satisfies (2), taking $\alpha = 1$ we obtain

$$\begin{aligned} g(t,x) &= \sum_{k=0}^{\infty}(a(x)^k + b(x)^k)t^k \\ &= \sum_{k=0}^{\infty}(a(x)t)^k + \sum_{k=0}^{\infty}(b(x)t)^k \\ &= \frac{1}{1-a(x)t} + \frac{1}{1-b(x)t} \\ &= \frac{2-(a(x)+b(x))t}{(1-a(x)t)(1-b(x)t)}. \end{aligned}$$

Since $f(t,x) = 1/\left[(1-a(x)t)(1-b(x)t)\right]$, we have that $g(t,x) = (2-(a(x)+b(x))t)\, f(t,x)$. This implies that

$$g(t,x)g(-t,x) = \left(4-(a(x)+b(x))^2t^2\right) f(t,x)f(-t,x).$$

This, the definition of $g(t,x)$, and the definition of $f(t,x)$, imply that

$$\sum_{r=0}^{\infty} G_r(x)t^r \sum_{r=0}^{\infty} G_r(x)(-t)^r = \left(4-(a(x)+b(x))^2t^2\right) \sum_{r=0}^{\infty} R_r(x)t^r \sum_{r=0}^{\infty} R_r(x)(-t)^r.$$

Thus,

$$\sum_{r=0}^{\infty}\left(\sum_{k=0}^{r}(-1)^k G_k(x)G_{r-k}(x)\right)t^r = (4-(a(x)+b(x))^2t^2)\sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}(-1)^k R_k(x)R_{l-k}(x)\right)t^l$$

for l even. Since $R_k(x)$ satisfies that Binet formula (3), Theorem 7 implies

$$\begin{aligned}
\sum_{r=0}^{\infty}\left(\sum_{k=0}^{r}(-1)^k G_k(x)G_{r-k}(x)\right)t^r &= (4-(a(x)+b(x))^2t^2)\sum_{j=0}^{\infty}\frac{R_{2j+1}(x)}{R_1(x)}t^{2j}\\
&= \sum_{j=0}^{\infty}\frac{4R_{2j+1}(x)}{R_1(x)}t^{2j}-\sum_{j=0}^{\infty}(a(x)+b(x))^2\frac{R_{2j+1}(x)}{R_1(x)}t^{2j+2}\\
&= 4+\sum_{j=1}^{\infty}\left(\frac{4R_{2j+1}(x)-(a(x)+b(x))^2R_{2j-1}(x)}{R_1(x)}\right)t^{2j}.
\end{aligned}$$

Since $G_1(x)=a(x)+b(x)$ we have that

$$\sum_{r=0}^{\infty}\left(\sum_{k=0}^{r}(-1)^k G_k(x)G_{r-k}(x)\right)t^r = 4t^0+\sum_{j=1}^{\infty}\left(\frac{4R_{2j+1}(x)-(G_1(x))^2R_{2j-1}(x)}{R_1(x)}\right)t^{2j}.$$

It is known that two generating functions are equal if the corresponding coefficients of t^i are equal for all i. We are only interested in even values of r. Therefore, we analyze the case when $r=2j$ in the last equality above. If $r=2j$ with $j>0$ and $G_0(x)=2$, then

$$\sum_{k=0}^{r}(-1)^k G_k(x)G_{r-k}(x)=\frac{(G_0(x))^2R_{r+1}(x)-(G_1(x))^2R_{r-1}(x)}{R_1(x)}.$$

If $\alpha\neq 1$, we let $G_k^{'}(x)$ be $(a^k(x)+b^k(x))/\alpha$, we recall from the previous analysis that $G_k(x)=a^k(x)+b^k(x)$. Note that the equivalent polynomial $R_k(x)$ of $G_k(x)$ is the same for $G_k^{'}(x)$, because they contain the same $a^k(x)$ and $b^k(x)$. Therefore,

$$\begin{aligned}
\sum_{k=0}^{r}(-1)^k G_k^{'}(x)G_{r-k}^{'}(x) &= \frac{1}{\alpha^2}\sum_{k=0}^{r}(-1)^k G_k(x)G_{r-k}(x)\\
&= \frac{1}{\alpha^2}\frac{(G_0(x))^2R_{r+1}(x)-(G_1(x))^2R_{r-1}(x)}{R_1(x)}\\
&= \frac{(G_0(x)/\alpha)^2R_{r+1}(x)-(G_1(x)/\alpha)^2R_{r-1}(x)}{R_1(x)}\\
&= \frac{(G_0^{'}(x))^2R_{r+1}(x)-(G_1^{'}(x))^2R_{r-1}(x)}{R_1(x)}.
\end{aligned}$$

This proves the theorem. □

Corollary 10. *If t is a positive integer and $r=2t$, then $\sum_{i=0}^{r+1}(-1)^i H(r+1,i)=0$, and*

$$\sum_{i=0}^{r}(-1)^i H(r,i)=\begin{cases}(U_{r+1}(x)-x^2U_{r-1}(x))/U_1(x), & \text{if } H(r,i)=T_i(x)T_{r-i}(x);\\ (4F_{r+1}(x)-x^2F_{r-1}(x))/F_1(x), & \text{if } H(r,i)=D_i(x)D_{r-i}(x);\\ (4\Phi_{r+1}(x)-9x^2\Phi_{r-1}(x))/\Phi_1(x), & \text{if } H(r,i)=\vartheta_i(x)\vartheta_{r-i}(x);\\ (4P_{r+1}(x)-4x^2P_{r-1}(x))/P_1(x), & \text{if } H(r,i)=Q_i(x)Q_{r-i}(x);\\ (4J_{r+1}(x)-J_{r-1}(x))/J_1(x), & \text{if } H(r,i)=j_i(x)j_{r-i}(x);\\ (4B_{r+1}(x)-(x+2)^2B_{r-1}(x))/B_1(x), & \text{if } H(r,i)=C_i(x)C_{r-i}(x).\end{cases}$$

Proof. We prove the corollary only for the case in which $H(r,i)=T_i(x)T_{r-i}(x)$; the proof of the other five cases are similar. Therefore, we omit their proofs.

From Table 6 we know that $T_n(x)$ is of first type and satisfies the Binet formula (2). So, $G_n(x)=T_n(x)$ and $R_n(x)=U_n(x)$. Thus, $T_n(x)=((x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n)/2$. This and Theorem 9 where $H(r,k)=T_k(x)T_{r-k}(x)$ imply that

$$\sum_{k=0}^{r}(-1)^k H(r,k)=\begin{cases}\dfrac{U_{r+1}(x)-x^2U_{r-1}(x)}{U_1(x)}, & \text{if } r \text{ nonzero even;}\\ 0, & \text{if } r \text{ is odd.}\end{cases}$$

This proves the corollary. □

4 Particular cases and examples

In this section we study some numerical cases of the Hosoya polynomial triangles. We show that when we evaluate the entries of a Hosoya polynomial triangle at some integer numbers, the alternating sums found in Section 3 give rise to several well-known sequences published in The On-Line Encyclopedia of Integer Sequences [16] by different authors.

We define an n-initial Hosoya triangle as the finite triangular arrangement formed by the first n-rows of the Hosoya triangle. Note that the initial triangle is the equilateral subtriangle of any Hosoya triangle that contains the vertex $H(0,0)$. In Proposition 11 we prove that the sum of all alternating sums of any initial triangle of the classical Hosoya triangle (see Table 5) is a closed formula depending on a Fibonacci number. We recall that in this paper the Fibonacci numbers are

$$F_0=1,\quad F_1=1,\quad F_2=2,\quad F_3=3,\quad F_4=5,\quad F_5=8,\ldots.$$

Proposition 11. *Let n be a positive integer and $t=\lfloor n/2\rfloor+1$. If $H(r,k)(1)=F_kF_{r-k}$ then*

$$\sum_{0\le k\le r\le n}(-1)^k H(r,k)(1)=\sum_{i=0}^{t-1}(t-i)F_{2i}=F_{2t}-1.$$

Proof. We prove $\sum_{0\le k\le r\le n}(-1)^k H(r,k)(1) = F_{2n+1}-1$, the other equality is known. From Corollary 8 we know that $\sum_{i=0}^{2t}(-1)^k H(r,k)(1) = F_{2t+1}$ and $\sum_{i=0}^{2t+1}(-1)^k H(r,k)(1) = 0$. This implies that $\sum_{0\le k\le r\le n}(-1)^k H(r,k) = \sum_{t=0}^{n} F_{2(n-i)} = F_{2n}-1$. □

We now introduce some notation that are used in Table 7. We define

$$C(\cdot)(x) = \sum_{r=0}^{n}(-1)^k H(r,k)$$

where $(\cdot)$ depends on the polynomials used in the sequences. For example, $CF_n(x)$ is the notation that we use to indicate the first result in Corollary 8. If we evaluate $CF_n(x)$ at $x=2$ for $n\in\mathbb{N}$ we obtain the sequence $1, 6, 35, 204, 1189, 6930$ which is in `http://oeis.org/A001109` (see Sloane [16]). We use $CM_n(x)$ to indicate the second result in Corollary 8. Similarly we use $CP_n(x)$ and $CJ_n(x)$, to indicate the third and fourth results in Corollary 8, respectively. In Table 7 we present some known sequences. Thus, the Corollaries 8 and 10 provide another way to generate sequences. Note that for the case Fermat and Chebyshev in Corollary 8 we only found the sequences A002450 and A007954, respectively.

$CF_n(x)$	Sloane	$CM_n(x)$	Sloane	$CP_n(x)$	Sloane	$CJ_n(x)$	Sloane
$CF_n(1)$	A001906	$CM_n(1)$	A04187	$CP_n(1)$	A001109	$CJ_n(1)$	A002450
$CF_n(2)$	A001109	$CM_n(2)$	A007655	$CP_n(2)$	A049660	$CJ_n(2)$	A102902
$CF_n(3)$	A004190	$CM_n(3)$	A097778	$CP_n(3)$	A078987	$CJ_n(3)$	A016153
$CF_n(4)$	A049660	$CM_n(4)$	A029547	$CP_n(4)$	A097316		
$CF_n(5)$	A097781	$CM_n(5)$	A049668	$CP_n(5)$	A097725		
$CF_n(6)$	A078987			$CP_n(6)$	A097728		
$CF_n(7)$	A097836			$CP_n(7)$	A097731		
$CF_n(8)$	A097316	$CM_n(8)$	A173205	$CP_n(8)$	A097734		
$CF_n(9)$	A097839			$CP_n(9)$	A097737		
$CF_n(10)$	A097725			$CP_n(10)$	A097740		

Table 7: Some sequences from Corollary 8 that are in [16].

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