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Some Identities for Fibonacci and Incomplete Fibonacci p-Numbers via the Symmetric Matrix Method

M. Cetin Firengiz
Department of Mathematics
Faculty of Education
University of Baskent
Baglica, Ankara
Turkey
mcetin@baskent.edu.tr

Dursun Tasci and Naim Tuglu
Department of Mathematics
University of Gazi
Teknikokullar, 06500, Ankara
Turkey
dtasci@gazi.edu.tr
naimtuglu@gazi.edu.tr

Abstract

We obtain some new formulas for the Fibonacci and Lucas p-numbers, by using the symmetric infinite matrix method. We also give some results for the Fibonacci and Lucas p-numbers by means of the binomial inverse pairing.

1 Introduction

Dil and Mező [3] defined the symmetric infinite matrix method. For sequences (a_n) and (a^n), the recursive formula

$$\begin{aligned} a_n^0 &= a_n, \quad a_0^n = a^n \qquad (n \geq 0) \\ a_n^k &= a_{n-1}^k + a_n^{k-1} \qquad (n \geq 1,\ k \geq 1) \end{aligned} \tag{1}$$

gives the associated symmetric infinite matrix [3]:

$$\begin{pmatrix} \cdot & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \begin{matrix} a_n^{k-1} \\ \downarrow \end{matrix} & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{n-1}^k & & \rightarrow & a_n^k & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Proposition 1. *[3] If the relation (1) holds, the entry a_n^k of the corresponding symmetric infinite matrix is*

$$a_n^k = \sum_{i=1}^{k} \binom{n+k-i-1}{n-1} a_0^i + \sum_{j=1}^{n} \binom{k+n-j-1}{k-1} a_j^0. \tag{2}$$

For two sequences (a_n) and (b_n), the well-known binomial inverse pair [9] is given by the relations

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \tag{3}$$

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}\, b_k. \tag{4}$$

Stakhov and Rozin [6] defined the Fibonacci p-numbers $F_p(n)$ by the following recurrence relation for $n > p$

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{5}$$

with initial conditions

$$F_p(0) = 0,\ F_p(n) = 1 \quad (n = 1, 2, \ldots, p)$$

and the Lucas p-numbers $L_p(n)$ by the following recurrence relation for $n > p$

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \tag{6}$$

with initial conditions

$$L_p(0) = p + 1,\ L_p(n) = 1 \quad (n = 1, 2, \ldots, p).$$

Note that for the case $p = 1$, the sequences of Fibonacci and Lucas p-numbers reduce to the well-known Fibonacci and Lucas sequences $\{F_n\}$, $\{L_n\}$, respectively. See [1, 5, 8] for more details about the Fibonacci and Lucas p-numbers.

Tasci and Cetin-Firengiz [7] introduced the incomplete Fibonacci and Lucas p-numbers. The incomplete Fibonacci p-numbers $F_p^k(n)$ and the incomplete Lucas p-numbers $L_p^k(n)$ are defined by

$$F_p^k(n) = \sum_{j=0}^{k} \binom{n - jp - 1}{j} \qquad \left(n = 1, 2, \ldots;\ 0 \le k \le \left\lfloor \tfrac{n-1}{p+1} \right\rfloor\right)$$

and

$$L_p^k(n) = \sum_{j=0}^{k} \frac{n}{n - jp} \binom{n - jp}{j} \qquad \left(n = 1, 2, \ldots;\ 0 \le k \le \left\lfloor \tfrac{n}{p+1} \right\rfloor\right).$$

We note that $F_1^{\lfloor \frac{n-1}{2} \rfloor}(n) = F_n$, $L_1^{\lfloor \frac{n}{2} \rfloor}(n) = L_n$ and $F_1^k(n) = F_n(k)$, $L_1^k(n) = L_n(k)$, where $\{F_n(k)\}$ and $\{L_n(k)\}$ are the sequences of incomplete Fibonacci and Lucas numbers, respectively. The same authors [7] gave the following properties of the incomplete Fibonacci and Lucas p-numbers:

$$\sum_{j=0}^{h} \binom{h}{j} F_p^{k+j}(n + p(j-1)) = F_p^{k+h}(n + (p+1)h - p) \tag{7}$$

for $0 \le k \le \frac{n-p-h-1}{p+1}$,

$$\sum_{j=0}^{h} \binom{h}{j} L_p^{k+j}(n + p(j-1)) = L_p^{k+h}(n + (p+1)h - p) \tag{8}$$

for $0 \le k \le \frac{n-p-h}{p+1}$.

In this paper, we give the generalization of some results of [3]. Some properties for the Fibonacci and Lucas p-numbers are obtained via the symmetric method. The results of incomplete Fibonacci and Lucas p-numbers are given by using binomial inverse pair as used for the Euler-Seidel matrix [2, 3].

2 Applications of symmetric infinite matrix

2.1 Applications for the Fibonacci and Lucas p-numbers

Let us consider the initial sequences $a_n^0 = F_p(n-1)$ and $a_0^n = F_p(n(p+1) - 1)$, $n \ge 1$. Thus the following infinite matrix is given for the special case

$$\begin{pmatrix} 0 & F_p(0) & F_p(1) & F_p(2) & \cdots \\ F_p(p) & F_p(p+1) & F_p(p+2) & F_p(p+3) & \cdots \\ F_p(2p+1) & F_p(2p+2) & F_p(2p+3) & F_p(2p+4) & \cdots \\ F_p(3p+2) & F_p(3p+3) & F_p(3p+4) & F_p(3p+5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 2. *The Fibonacci p-numbers satisfy the relation*

$$\sum_{i=1}^{n} F_p(i(p+1)-1) = F_p(n(p+1)). \tag{9}$$

Proof. For $a_n^0 = F_p(n-1)$ and $a_0^n = F_p(n(p+1)-1)$, $n \geq 1$. We have $a_1^1 = F_p(p+1)$, $a_1^2 = F_p(2p+2)$. Suppose that the equation holds for $n > 1$. Now we show that the equation holds for $(n+1)$. Thus we get using (1) and (5)

$$\begin{aligned} a_1^{n+1} &= a_0^{n+1} + a_1^n \\ &= F_p((n+1)(p+1)-1) + F_p(n(p+1)) \\ &= F_p(n(p+1)+p) + F_p(n(p+1)) \\ &= F_p(n(p+1)+p+1) \\ &= F_p((n+1)(p+1)). \end{aligned}$$

By considering (2), we have

$$\begin{aligned} a_1^n &= \sum_{i=1}^{n} \binom{n-i}{0} a_0^i + \sum_{j=1}^{1} \binom{n-j}{n-1} a_j^0 \\ &= \sum_{i=1}^{n} F_p(i(p+1)-1) + a_1^0 \\ &= \sum_{i=1}^{n} F_p(i(p+1)-1) + F_p(0). \end{aligned}$$

Then, we can obtain

$$F_p(n(p+1)) = \sum_{i=1}^{n} F_p(i(p+1)-1).$$

□

Taking $p = 1$ in (9), we get $F_{2n} = \sum\limits_{i=1}^{n} F_{2i-1}$ in [4].

Stakhov and Rozin [6] gave the equation $F_p(1) + F_p(2) + \cdots + F_p(n) = F_p(n+p+1) - 1$ for the Fibonacci p-numbers. The following proposition shows that the formula can be obtained via the symmetric method.

Proposition 3. *The Fibonacci p-numbers are*

$$\sum_{j=1}^{n} F_p\left(j-1\right) = F_p\left(p+n\right) - 1. \tag{10}$$

Proof. Let $a_n^0 = F_p\left(n-1\right)$ and $a_0^n = F_p\left(n\left(p+1\right)-1\right),\ n \geq 1$. If we take $n=1$ and $n=2$, then $a_1^1 = F_p\left(p+1\right),\ a_2^1 = F_p\left(p+2\right)$. Suppose that the equation holds for $n>1$. We show that the equation holds for $(n+1)$. We have by (1) and (5)

$$\begin{aligned} a_{n+1}^1 &= a_n^1 + a_{n+1}^0 \\ &= F_p\left(p+n\right) + F_p\left(n\right) \\ &= F_p\left(p+n+1\right). \end{aligned}$$

Using (2), we can write

$$\begin{aligned} a_n^1 &= \sum_{i=1}^{1} \binom{n-i}{n-1} a_0^i + \sum_{j=1}^{n} \binom{n-j}{0} a_j^0 \\ &= a_0^1 + \sum_{j=1}^{n} a_j^0 \\ &= F_p\left(p\right) + \sum_{j=1}^{n} F_p\left(j-1\right), \end{aligned}$$

which completes the proof. □

When $p=1$ in (10), we obtain $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ in [4].

In particular, let $a_n^0 = L_p\left(n-1\right)$ and $a_0^n = L_p\left(n\left(p+1\right)-1\right),\ n \geq 1$. This case gives the following infinite matrix

$$\begin{pmatrix} 0 & L_p\left(0\right) & L_p\left(1\right) & L_p\left(2\right) & \cdots \\ L_p\left(p\right) & L_p\left(p+1\right) & L_p\left(p+2\right) & L_p\left(p+3\right) & \cdots \\ L_p\left(2p+1\right) & L_p\left(2p+2\right) & L_p\left(2p+3\right) & L_p\left(2p+4\right) & \cdots \\ L_p\left(3p+2\right) & L_p\left(3p+3\right) & L_p\left(3p+4\right) & L_p\left(3p+5\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Similar results for the Lucas p-numbers can be obtained likewise. Therefore we omit the proofs of Proposition 4 and 5

Proposition 4. *The Lucas p-numbers $L_p\left(n\right)$ satisfy the following relation*

$$\sum_{i=1}^{n} L_p\left(i\left(p+1\right)-1\right) = L_p\left(n\left(p+1\right)\right) - \left(p+1\right). \tag{11}$$

Proposition 5. *We have*

$$\sum_{j=1}^{n} L_p(j-1) = L_p(p+n) - 1. \tag{12}$$

If $p = 1$ in (11) and (12), we get the well known identities $\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2$ and $\sum_{i=1}^{n} L_i = L_{n+2} - 3$.

2.2 Applications for the incomplete Fibonacci and Lucas p-numbers

In this subsection, we get similar formulas for (7) and (8) by using the binomial inverse pair.

Let $a_n^0 = F_p^{k+n}(t + p(n-1))$. From (3) we have

$$a_0^n = \sum_{j=0}^{n} \binom{n}{j} F_p^{k+j}(t + p(j-1)).$$

By using (7)

$$a_0^n = F_p^{k+n}(t + (p+1)n - p).$$

Therefore, the dual formula of (7) is obtained from (4)

$$F_p^{k+n}(t + p(n-1)) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} F_p^{k+j}(t + (p+1)j - p) \tag{13}$$

for $0 \leq k \leq \frac{t-p-n-1}{p+1}$. Similarly, let us take $a_n^0 = L_p^{k+n}(t + p(n-1))$. Then (3) can be rewritten as

$$a_0^n = \sum_{j=0}^{n} \binom{n}{j} L_p^{k+j}(t + p(j-1)).$$

By (8)

$$a_0^n = L_p^{k+n}(t + (p+1)n - p).$$

Finally, using (4), we obtain the dual formula (8)

$$L_p^{k+n}(t + p(n-1)) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} L_p^{k+j}(t + (p+1)j - p) \tag{14}$$

for $0 \leq k \leq \frac{t-p-n}{p+1}$.

For $p = 1$ in (13) and (14), we get the properties of incomplete Fibonacci and Lucas numbers in [3].

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