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Dickson Polynomials, Chebyshev Polynomials, and Some Conjectures of Jeffery

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Abstract

By using Dickson polynomials in several variables and Chebyshev polynomials of the second kind, we derive the explicit expression of the entries in the array defining the sequence A185095 . As a result, we obtain a straightforward proof of some conjectures of Jeffery concerning this sequence and other related ones.

1 Proof of main conjectures

As we can find in the *Encyclopedia of Integer Sequences* [1], the sequence A185095 is a rectangular array read by antidiagonals, where the n-th row has generating function

$$F_n(z) = \frac{\sum_{r=0}^{n} (n+1-r)(-1)^r \binom{2(n+1)-r}{r} z^r}{\sum_{r=0}^{n+1} (-1)^r \binom{2(n+1)-r}{r} z^r} \tag{1}$$

for $n = 0, 1, 2, \ldots$. In this section we find the explicit expression for the entries of this rectangular array and we determine the ordinary generating functions for the columns.

Let us consider the Dickson polynomials of the first kind in several indeterminates, as defined in the book of Lidl, Mullen, and Turnwald [2].

Definition 1. The Dickson polynomials of the first kind in n indeterminates of total degree k

$$D_k^{(i)}(x_1, \dots, x_n, a), \quad i = 1, \dots, n$$

satisfy the functional equations

$$D_k^{(i)}(x_1, \dots, x_n, a) = S_i(u_1^k, \dots, u_{n+1}^k), \quad i = 1, \dots, n \tag{2}$$

where $x_i = S_i(u_1, \dots, u_{n+1})$, $S_i(y_1, \dots, y_{n+1})$ is the i-th symmetric function of $y_1, \dots, y_{n+1}$, and $u_1 \cdot \ldots \cdot u_{n+1} = a$.

In particular when $i = 1$ we have $D_k^{(1)}(x_1, \dots, x_n, a) = S_1(u_1^k, \dots, u_{n+1}^k) = \sum_{j=1}^{n+1} u_j^k$. Moreover if we pose $x_0 = 1$ and $x_{n+1} = a$, the Dickson polynomials of the first kind $D_k^{(1)}(x_1, \dots, x_n, a)$ have the following generating function (see Lidl, Mullen, and Turnwald [2, Lemma 2.23]):

$$\sum_{k=0}^{+\infty} D_k^{(1)}(x_1, \dots, x_n, a) z^k = \frac{\sum_{r=0}^{n} (n+1-r)(-1)^r x_r z^r}{\sum_{r=0}^{n+1} (-1)^r x_r z^r}, \quad k \geq 0. \tag{3}$$

Proposition 2. *The k-th entry in the n-th row $R_{n,k}$, $k, n \geq 0$, of the rectangular array which defines the sequence A185095, corresponds to $D_k^{(1)}(x_1, \dots, x_n, 1)$ where $x_r = \binom{2(n+1)-r}{r}$ for $r = 0, \dots, n+1$. Furthermore we have that*

$$R_{n,k} = 2^{2k} \sum_{j=1}^{n+1} \cos^{2k}\left(\frac{j\pi}{2n+3}\right). \tag{4}$$

Proof. After a comparison between the two generating functions (1) and (3), we immediately find that $R_{n,k} = D_k^{(1)}(x_1, \dots, x_n, 1)$. Moreover, by definition (1), the role played by symmetric functions when $x_r = \binom{2(n+1)-r}{r}$ for $r = 0 \dots, n+1$, allow us to prove that

$$R_{n,k} = D_k^{(1)}(x_1, \dots, x_n, 1) = \sum_{j=1}^{n+1} \alpha_j^k,$$

where α_j, for $j = 1, \dots, n+1$ are the zeros of the polynomial

$$P_n(x) = \sum_{j=0}^{n+1} (-1)^j \binom{2(n+1)-j}{j} x^{n+1-j}. \tag{5}$$

Now let us recall the definition of the Chebyshev polynomials of the second kind $U_h\left(\frac{x}{2}\right)$ (see for references the books of Rivlin [3] and of Mason and Hascomb [4]):

$$U_h\left(\frac{x}{2}\right) = \sum_{j=0}^{\lfloor \frac{h}{2} \rfloor} (-1)^j \binom{h-j}{j} x^{h-2j}. \tag{6}$$

It is clear that $P_n(x) = U_{2n+2}\left(\frac{\sqrt{x}}{2}\right)$. So α_j is a zero of $P_n(x)$ if and only if $\frac{\sqrt{\alpha_j}}{2}$ is a positive zero for $U_{2n+2}(x)$. Since the zeros of $U_{2n+2}(x)$ are $x_j = \cos\left(\frac{j\pi}{2n+3}\right)$, $j = 1, \ldots, 2n+3$, a simple trigonometric consideration allow us to find the positive ones when $j = 1, \ldots, n+1$, showing that $\alpha_j = 2^2 \cos^2\left(\frac{j\pi}{2n+3}\right)$. The thesis immediately follows. □

Corollary 3. *The entry $R_{n,k}$ for $n \geq 1$ has the alternative expression*

$$\begin{cases} R_{n,k} = \binom{2k}{k}\left(n+\frac{3}{2}\right) - 2^{2k-1}, & 1 \leq k < 2n+3 \\ R_{n,k} = \binom{2k}{k}\left(n+\frac{3}{2}\right) - 2^{2k-1} + (2n+3)\sum_{i=1}^{\lfloor \frac{k}{2n+3} \rfloor} \binom{2k}{k-i(2n+3)}, & k \geq 2n+3 \end{cases} \tag{7}$$

where $R_{n,0} = n+1$, $R_{0,k} = 1$.

Proof. Obviously if $k = 0$ we obtain $R_{n,0} = n+1$ and if $n = 0$ we have $R_{0,k} = 1$. When $k \geq 1$, by (4) and thanks to the summation formulas (see Gradshteyn and Ryzhik [5]):

$$\cos^{2k}(x) = \frac{1}{2^{2k}}\left\{\binom{2k}{k} + 2\sum_{h=0}^{k-1}\binom{2k}{h}\cos(2(k-h)x)\right\}$$

and

$$\sum_{j=0}^{n}\cos(jx) = \frac{1}{2}\left[1 + \frac{\sin\left(\frac{(2n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right],$$

we get

$$\begin{aligned} R_{n,k} &= \sum_{j=1}^{n+1}\left(\binom{2k}{k} + 2\sum_{h=0}^{k-1}\binom{2k}{h}\cos\left(2(k-h)\frac{j\pi}{2n+3}\right)\right) = \\ &= \binom{2k}{k}(n+1) + 2\sum_{h=0}^{k-1}\binom{2k}{h}\left(\sum_{j=0}^{n+1}\cos\left(2(k-h)\frac{j\pi}{2n+3}\right) - 1\right) = \\ &= \binom{2k}{k}(n+1) + 2\sum_{h=0}^{k-1}\binom{2k}{h}\left(\sum_{j=0}^{n+1}\cos\left(2(k-h)\frac{j\pi}{2n+3}\right)\right) - 2\sum_{h=0}^{k-1}\binom{2k}{h} \end{aligned}$$

Now we have

$$\begin{cases} \sum_{j=0}^{n+1}\cos\left(2(k-h)\frac{j\pi}{2n+3}\right) = \frac{1}{2}\left[1 + \frac{\sin\left(\frac{2n+3}{2}\frac{2(k-h)\pi}{2n+3}\right)}{\sin\left(\frac{(k-h)\pi}{2n+3}\right)}\right] = \frac{1}{2} & \text{if} \quad k \neq i(2n+3)+h \\ \sum_{j=0}^{n+1}\cos\left(2(k-h)\frac{j\pi}{2n+3}\right) = n+2 & \text{if} \quad k = i(2n+3)+h \end{cases} \tag{8}$$

for a certain positive integer i.

Recalling the following well known identity

$$\sum_{h=0}^{k-1}\binom{2k}{h}=2^{2k-1}-\frac{1}{2}\binom{2k}{k},$$

if $1\leq k<2n+3$ we clearly obtain

$$R_{n,k}=\binom{2k}{k}\left(n+\frac{3}{2}\right)-2^{2k-1},$$

on the other hand, when $k\geq 2n+3$ we find

$$R_{n,k}=\binom{2k}{k}(n+1)+2\left(\sum_{h=0}^{k-1}\binom{2k}{h}\left(-\frac{1}{2}\right)-\frac{1}{2}\sum_{i=1}^{\lfloor\frac{k}{2n+3}\rfloor}\binom{2k}{k-i(2n+3)}+(n+2)\sum_{i=1}^{\lfloor\frac{k}{2n+3}\rfloor}\binom{2k}{k-i(2n+3)}\right)=$$

$$=\binom{2k}{k}\left(n+\frac{3}{2}\right)-2^{2k-1}+(2n+3)\sum_{i=1}^{\lfloor\frac{k}{2n+3}\rfloor}\binom{2k}{k-i(2n+3)}.$$

□

Considering the rectangular array in the definition of sequence A 185095 we can also study its columns, finding their generating functions. Thanks to the previous result we have the following

Proposition 4. *The ordinary generating function $G_k(z)$ for the k-th column of the rectangular array in the definition of A 185095 is*

$$G_k(z)=\frac{\left(2^{2k-1}-\frac{3}{2}\binom{2k}{k}+1\right)z^2+\left(\frac{5}{2}\binom{2k}{k}-2^{2k-1}-2\right)z+1}{(1-z)^2}+P_{n_0}(z),\quad k\geq 0,\tag{9}$$

where $n_0=\lfloor\frac{k-3}{2}\rfloor$ and

$$P_{n_0}(z)=\sum_{n=1}^{n_0}\left((2n+3)\sum_{i=1}^{\lfloor\frac{k}{2n+3}\rfloor}\binom{2k}{k-i(2n+3)}\right)z^n\tag{10}$$

with the convention that the summation in (10) is equal to 0 when $n_0<1$.

Proof. Using the alternative expression determined in Corollary 7, and the well–known formulas $\sum_{n=0}^{+\infty}z^n=\frac{1}{1-z}$ and $\sum_{n=0}^{+\infty}(n+1)z^n=\frac{1}{(1-z)^2}$, we can compute the ordinary generating function $G_k(z)$ as follows:

$$G_k(z)=\sum_{n=0}^{+\infty}R_{n,k}z^n=1+\binom{2k}{k}\left(\sum_{n=0}^{+\infty}(n+1)z^n-1\right)-\left(2^{2k-1}-\frac{1}{2}\binom{2k}{k}\right)\left(\sum_{n=0}^{+\infty}z^n-1\right)+P_{n_0}(z)=$$

$$=1+\binom{2k}{k}\frac{(2z-z^2)}{(1-z)^2}-\frac{z}{1-z}\left(2^{2k-1}-\frac{1}{2}\binom{2k}{k}\right)+P_{n_0}(z)=$$

$$=\frac{\left(2^{2k-1}-\frac{3}{2}\binom{2k}{k}+1\right)z^2+\left(\frac{5}{2}\binom{2k}{k}-2^{2k-1}-2\right)z+1}{(1-z)^2}+P_{n_0}(z),$$

where the term $P_{n_0}(z)$ described in equation (10) appears because when $2n+3 \leq k$, or equivalently $n \leq \lfloor \frac{k-3}{2} \rfloor = n_0$, we use the second expression of $R_{n,k}$ in (7). □

2 Concluding remarks

In this section we discuss some consequences of the previous results concerning the sequence A185095. We show that the proof of the remaining conjectures and the relations with other sequences are immediate. First of all we observe that the k-th column of the array defining the sequence A185095 satisfies the recurrence relation

$$R_{n+1,k} = 2R_{n,k} - R_{n-1,k}, \text{ for } \quad n \geq \lfloor \frac{k-3}{2} \rfloor + 1,$$

which is a straightforward consequence of (7). The rectangular array which generates A186740 clearly is the transpose of the rectangular array that we have considered. In fact, if we start by numbering by 0 the columns of the rectangular array which defines A186740, we find that the entry in the k-th row and n-th column corresponds to $R_{n,k}$ by definition. Moreover by (7) and using our numeration for the columns we have

$R_{n,0} = n+1, n \geq 0$: the column 0 corresponds to the natural numbers A000027 ;

$R_{n,1} = 2n+1, n \geq 0$: the column 1 is the sequence of odd integers A005408 ;

$R_{n,2} = 6n+1, n \geq 0$: the column 2 is A016921 ;

$R_{n,3} = 20n-2, n \geq 1$: the column 3 is A114698 ;

$R_{n,4} = 70n-93, n \geq 2$: the column 4 is A114646 ,

and so on .

Finally another interesting consequence of the formula for $R_{n,k}$ concerns the sequence A198632. By considering the array $w(h,2k)$, as defined in the page related to A198632 (see OEIS [1]), we have immediately $w(2(n+1),2k) = 2R_{n,k}$. In fact

$$w(h,l) = \mathrm{Tr}(J_h^l) = \sum_{j=1}^{h} \lambda_j^l,$$

where λ_j, $j = 1, \ldots, h$, are the eigenvalues of the adjacency matrix J_h, a Jacobi matrix, whose characteristic polynomial is $U_h(\frac{x}{2})$. Thus

$$w(2(n+1),2k) = \sum_{j=1}^{2n+2} \left(2\cos\left(\frac{j\pi}{2n+3}\right)\right)^{2k} = 2^{2k+1} \sum_{j=1}^{n+1} \left(\cos\left(\frac{j\pi}{2n+3}\right)\right)^{2k} = 2R_{n,k}.$$

3 Acknowledgements

In this paper we point out some relations between many sequences and we think that a lot of interesting formulas will also rise, considering the role played by Dickson and Chebyshev polynomials, their zeros and the summation identities involved. The author sincerely thanks L. Edson Jeffery for all his interesting work and for his invitation to try a proof of his conjectures.

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