# On the Diophantine Equation
# $x^4 + y^4 + z^4 + t^4 = w^2$

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**Abstract**

To our knowledge, only three parametric solutions to the equation $x^4 + y^4 + z^4 + t^4 = w^2$ were previously known. In this paper, we study the equation $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$. We prove that it is possible to obtain infinitely many parametric solutions by finding points on an elliptic curve over a field $\mathbb{Q}(m)$ and we give several new parametric solutions.

## 1 Introduction

Jacobi and Madden [3] considered the equation

$$x^4 + y^4 + z^4 + t^4 = (x + y + z + t)^4. \tag{1}$$

They showed the existence of infinitely many integral solutions to (1). This is a special case of the equation

$$x^4 + y^4 + z^4 + t^4 = w^4, \tag{2}$$

for which Elkies [1] found an infinite family of integral solutions when $t = 0$. In this paper, we consider a special case of a similar equation

$$x^4 + y^4 + z^4 + t^4 = w^2. \tag{3}$$

## 2 Background

Consider the equation (3). We say that a solution is *trivial* if at least three of the numbers $x, y, z, t, w$ are zero, for instance $(x, y, z, t, w) = (x, 0, 0, 0, x^2)$. If two and only two of the numbers $x, y, z, t$ are zero, the equation has no nontrivial solution since Fermat proved that the equation $x^4 + y^4 = w^2$ has no solution in nonzero integers.

The first known parametric solution is nontrivial but very elementary:

$$(x, y, z, t, w) = (a^2, ab, b^2, ab, a^4 + b^4).$$

In the next solution, found by Fauquembergue [2], one of the numbers $x, y, z, t, w$ is zero, for instance $z = 0$:

$$(x, y, z, t, w) = (ac, bc, 0, ab, a^4 + a^2b^2 + b^4)$$

where $a^2 + b^2 = c^2$. The following solution was also found by Fauquembergue [2], again assuming $a^2 + b^2 = c^2$:

$$(x, y, z, t, w) = \left(2a^2bc^3, 2ab^2c^3, (a^2 - b^2)c^4, 2ab(a^4 + b^4), \right.$$
$$\left. (a^6 + 2a^5b + 3a^4b^2 + 3a^2b^4 + 2ab^5 + b^6)(a^6 - 2a^5b + 3a^4b^2 + 3a^2b^4 - 2ab^5 + b^6)\right).$$

These three parametric solutions yield nontrivial numerical solutions, unless $ab = 0$.

## 3 The Equation $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$

While investigating solutions to equation (3), the second author noticed some interesting properties. After some numerical results, he considered the following three cases.

- If $w = x^2 + y^2 + z^2 + t^2$ then $x^2y^2 + x^2z^2 + z^2t^2 + y^2z^2 + y^2t^2 + z^2t^2 = 0$; hence, there are only trivial solutions.

- If $w = x^2 + y^2 - z^2 - t^2$ then $x^2y^2 - x^2z^2 - y^2z^2 = t^2(x^2 + y^2 - z^2)$. This is an interesting but complicated case. We leave this for future work.

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- If $w = x^2 + y^2 + z^2 - t^2$ then $x^2 y^2 + x^2 z^2 + y^2 z^2 = t^2 (x^2 + y^2 + z^2)$. This case also looked interesting and will be discussed below, beginning with the following proposition. We believe our analysis of this case is new.

**Proposition 1.** *If $x^2 + y^2 + z^2 \neq 0$, then $(x, y, z, t)$ satisfies*

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \tag{4}$$

*if and only if*

$$(x^2 + y^2 + z^2)(y^2 z^2 + z^2 x^2 + x^2 y^2) = \square \tag{5}$$

*and*

$$t^2 = \frac{y^2 z^2 + z^2 x^2 + x^2 y^2}{x^2 + y^2 + z^2}. \tag{6}$$

*Proof.* We have

$$x^4 + y^4 + z^4 + t^4 - (x^2 + y^2 + z^2 - t^2)^2 = 2\big((x^2 + y^2 + z^2)t^2 - (y^2 z^2 + z^2 x^2 + x^2 y^2)\big).$$

$\square$

Since (4) is homogeneous of degree four, from now on we will write solutions to this equation as $(x : y : z : t)$. The equation represents a surface in $\mathbb{P}^3$,

$$\mathcal{S}' = \big\{ (x : y : z : t) \in \mathbb{P}^3 \mid x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \big\}.$$

Assuming $t \neq 0$, we can view the surface in affine form by corresponding $(x, y, z) \leftrightarrow (x : y : z : 1)$. This gives a rather interesting looking surface in three dimensions; see Figure 1.

The main focus of this paper is to answer the following question.

*Question* 2. How many rational curves $m \mapsto (x(m), y(m), z(m))$ are on the surface $\mathcal{S}'$?

If $xyz \neq 0$, then (5) can be expressed as

$$(x^2 + y^2 + z^2)\left( \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) = \square.$$

This leads to the following lemma.

**Lemma 3.** *If $(x : y : z : t)$ is a solution to (4) such that $xyzt \neq 0$ then $\left( \dfrac{1}{x} : \dfrac{1}{y} : \dfrac{1}{z} : \dfrac{1}{t} \right)$ is also a solution to (4).*
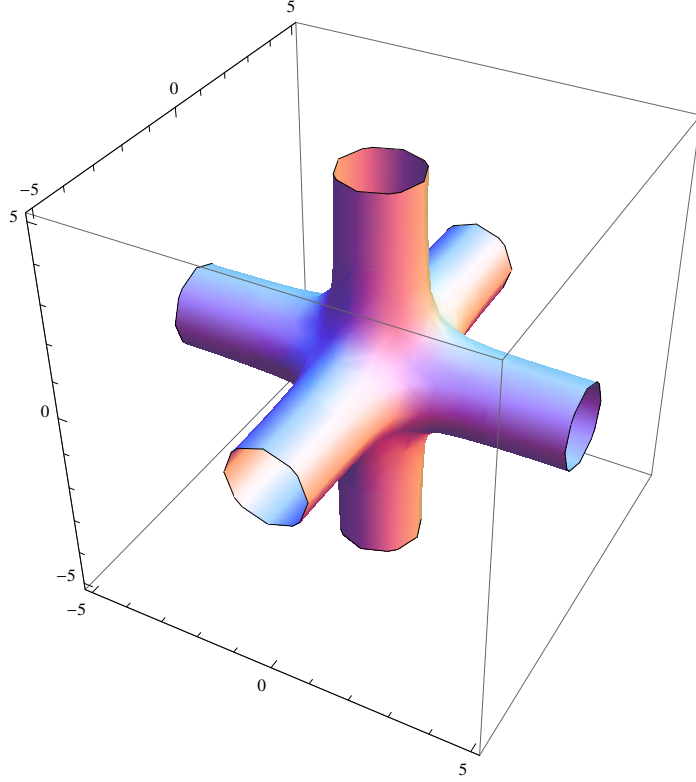
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Figure 1: Plot of $\mathcal{S}'$ in affine space

*Proof.* Let $(X : Y : Z : T) = \left(\dfrac{1}{x} : \dfrac{1}{y} : \dfrac{1}{z} : \dfrac{1}{t}\right)$. Then

$$X^2 + Y^2 + Z^2 = \frac{1}{x^2 y^2 z^2} \left(y^2 z^2 + z^2 x^2 + x^2 y^2\right) \text{ and}$$

$$Y^2 Z^2 + Z^2 X^2 + X^2 Y^2 = \frac{1}{x^2 y^2 z^2}(x^2 + y^2 + z^2)$$

Hence

$$\frac{Y^2 Z^2 + Z^2 X^2 + X^2 Y^2}{X^2 + Y^2 + Z^2} = \frac{x^2 + y^2 + z^2}{y^2 z^2 + z^2 x^2 + x^2 y^2} = \frac{1}{t^2} = T^2$$

$\square$

The following examples can be shown to be solutions to (4).

- The elementary solution $\mathcal{F}_0 = (a^2 : ab : b^2 : ab)$.

- The first solution of Fauquembergue $\mathcal{F}_1 = (ac : bc : 0 : ab)$ where $a^2 + b^2 = c^2$.

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- The second solution of Fauquembergue $\mathcal{F}_2 = (2a^2bc^3 : 2ab^2c^3 : (a^2-b^2)c^4 : 2ab(a^4+b^4))$ where $a^2 + b^2 = c^2$ .

An application of the previous lemma to the second solution of Fauquembergue, we deduce a new solution to (4).

**Proposition 4.** *If* $a^2 + b^2 = c^2$ *and*

$$x = ac(a^2 - b^2)(a^4 + b^4)$$
$$y = bc(a^2 - b^2)(a^4 + b^4)$$
$$z = 2a^2b^2(a^4 + b^4)$$
$$t = ab(a^2 + b^2)^2(a^2 - b^2)$$

*then* $(x : y : z : t)$ *is a solution to (4).*

We will label this solution $\mathcal{D}_1$. For example, if $(a : b : c) = (4 : 3 : 5)$ then $(x : y : z : t) = (47180 : 35385 : 97056 : 52500)$ is a solution to (4).

# 4   An Elliptic Curve over $\mathbb{Q}(m)$

We begin this section by providing some background on elliptic surfaces, which can be defined as a one-parameter algebraic family of elliptic curves. See Silverman [5, Chapter 3].

Let $C$ be a curve defined over a field $k$. Consider a rational map $C \to \mathbb{P}^1$. The collection of all such maps is denoted by $K = k(C)$. For example, if $C : a^2 + b^2 = c^2$ is defined over $\mathbb{Q}$, we have an isomorphism $C \to \mathbb{P}^1$ given by

$$(a : b : c) \mapsto m = \frac{a}{c - b} \iff (a : b : c) = (2m : m^2 - 1 : m^2 + 1).$$

Hence, $K = \mathbb{Q}(C) = \mathbb{Q}(m)$. This is the field we will consider.

Consider a family of curves

$$E_m : x_2^2 = x_1^3 + A(m)x_1 + B(m)$$

with rational functions $A(m), B(m) \in K$. If we rewrite our equation in homogeneous form, we form the elliptic surface

$$\mathcal{E} = \left\{ ((x_1 : x_2 : x_3), m) \in \mathbb{P}^2 \times C | x_2^2 x_3 = x_1^3 + A(m)x_1 x_3^2 + B(m)x_3^3 \right\}.$$

We have a map $\pi : \mathcal{E} \to C$ defined by $((x_1 : x_2 : x_3), m) \mapsto m$. For $m \in \mathbb{Q}$ such that $4A(m)^3 + 27B(m)^2 \neq 0$, the *fiber*

$$\mathcal{E}_m = \pi^{-1}(m) = \left\{ (x_1 : x_2 : 1) \in \mathbb{P}^2 | x_2^2 x_3 = x_1^3 + A(m)x_1 x_3^2 + B(m)x_3^3 \right\}$$

is the elliptic curve $E_m$ over $\mathbb{Q}$.

We say that the elliptic surface is *non-split* if the $j$-invariant

$$j : C \to \mathbb{P}^1 \text{ defined by } j(m) = 1728 \frac{4A(m)^3}{4A(m)^3 + 27B(m)^2}$$

is a non-constant function. A *parametrization of $\mathcal{E}$ by $C$*, or a *section to $\pi$*, is a map $\sigma : C \to \mathcal{E}$ such that the composition $\pi \circ \sigma : m \mapsto m$ is the identity map on $C$. There is always a trivial section on $\mathcal{E}$, namely the map $\sigma_0 : m \mapsto \mathcal{O}_m = ((0 : 1 : 0), m)$. In general, the collection $\mathcal{E}(C)$ of all sections is an abelian group, where we define

$$\sigma_1(m) = (P(m), m)$$
$$\implies \quad (\sigma_1 \oplus \sigma_2)(m) = (P(m) \oplus Q(m), m).$$
$$\sigma_2(m) = (Q(m), m)$$

We often abuse notation and write $E : x_2^2 = x_1^3 + Ax_1 + B$ as an elliptic curve over the function field $K = k(C)$. In fact, we have an isomorphism between the group of points of $E$ over $K$ and the group of sections of $\mathcal{E}$ over $\mathbb{Q}$.

$$E(K) \qquad\qquad \overset{\sim}{\longrightarrow} \qquad \mathcal{E}(C)$$
$$P(m) = (x_1(m) : x_2(m) : x_3(m)) \qquad \mapsto \qquad [\sigma : m \mapsto ((x_1(m) : x_2(m) : x_3(m)), m)]$$

It will be helpful to consider $\mathcal{E}$ as a two-dimensional surface, where each section maps to a one-dimensional curve. The result [5, Theorem 6.1, Chapter 3] asserts that $E(K) \simeq \mathcal{E}(C)$ is a finitely generated abelian group whenever $\mathcal{E}$ is a non-split surface.

Let $m_0 \in \mathbb{Q}$ such that $E_0 = \pi^{-1}(m_0)$ is an elliptic curve over $k = \mathbb{Q}$. Silverman's "specialization theorem" [5, Theorem 11.4, Chapter 3] asserts that the map $\mathcal{E}(C) \to E_0(k)$ which sends a section $\sigma : m \mapsto (P(m), m)$ to the point $P_0 = P(m_0)$ is injective for all but finitely many $m_0 \in k$. In particular, if $P_0$ is a point of finite (infinite) order in $E_0(k)$ for some $m_0 \in k$, then $P(m)$ must be a point of finite (infinite) order in $E(K)$ as a function of $m$.

Let us return to the condition

$$(x^2 + y^2 + z^2)(y^2 z^2 + z^2 x^2 + x^2 y^2) = \square.$$

If $a^2 + b^2 = c^2$ nonzero then we can express $(a : b : c) = (2m : m^2 - 1 : m^2 + 1)$. For the sake of space, we will leave our work below in terms of $a, b, c$. If we impose the condition $(x, y) = (a, b)$, we obtain

$$(c^2 + z^2)(c^2 z^2 + a^2 b^2) = \square.$$

Dividing by $c^2$, we can consider the following equation,

$$h^2 = z^4 + \frac{a^4 + 3a^2 b^2 + b^4}{c^2} z^2 + a^2 b^2 \tag{7}$$

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From the preceding examples $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions to (8),

$$(z, h)_{\mathcal{F}_0} = \left( \frac{b^2}{a}, \frac{b(a^4 + a^2 b^2 + b^4)}{a^2 c} \right);$$

$$(z, h)_{\mathcal{F}_1} = (0, ab);$$

$$(z, h)_{\mathcal{F}_2} = \left( \frac{(a^2 - b^2)c}{2\ ab}, \frac{(a^4 + b^4)c^2}{4\ a^2 b^2} \right);$$

$$(z, h)_{\mathcal{D}_1} = \left( \frac{2\ a^2 b^2}{(a^2 - b^2)c}, \frac{(a^4 + b^4)ab}{(a^2 - b^2)^2} \right).$$

*Comment 5.* If we impose the condition $(x, y) = (a, b) = (2m, m^2 - 1)$, we obtain

$$\left((m^2 + 1)^2 + z^2\right)\left((m^2 + 1)^2 z^2 + ((2m)(m^2 - 1))^2\right) = \square.$$

Dividing by $(m^2 + 1)^2$, we can consider the following equation,

$$h^2 = z^4 + \frac{m^8 + 8m^6 - 2m^4 + 8m^2 + 1}{(m^2 + 1)^2} z^2 + \left((2m)(m^2 - 1)\right)^2 \qquad (8)$$

From the preceding examples, $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions $(z, h)$ to (8), imposing the condition $a^2 + b^2 = c^2$ and $(x, y) = (a, b)$. In terms of the parameter $m$ we obtain

$$(z, h)_{\mathcal{F}_0} = \left( \frac{(m^2 - 1)^2}{2m}, \frac{(m^2 - 1)(m^4 - 2m^3 + 2m^2 + 2m + 1)(m^4 + 2m^3 + 2m^2 - 2m + 1)}{4m^2(m^2 + 1)} \right);$$

$$(z, h)_{\mathcal{F}_1} = (0, 4m(m^2 - 1));$$

$$(z, h)_{\mathcal{F}_2} = \left( \frac{-(m^2 + 1)(m^2 + 2m - 1)(m^2 - 2m - 1)}{4m(m^2 - 1)}, \frac{(m^2 + 1)^2(m^8 - 4m^6 + 22m^4 - 4m^2 + 1)}{16m^2(m^2 - 1)^2} \right);$$

$$(z, h)_{\mathcal{D}_1} = \left( \frac{8m^2(m^2 - 1)^2}{-(m^2 + 1)(m^2 + 2m - 1)(m^2 - 2m - 1)}, \frac{2m(m^2 - 1)(m^8 - 4m^6 + 22m^4 - 4m^2 + 1)}{((m^2 + 2m - 1)(m^2 - 2m - 1))^2} \right).$$

We will show that, in fact, there are infinitely many parametric solutions to equation (4) by showing there are infinitely many parametric solutions to (8).

**Theorem 6.** *Parametric solutions of the equation*

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$$

*may be obtained by finding points on an elliptic curve over the field $\mathbb{Q}(m)$.*

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*Proof.* By Proposition 1 and assuming $a^2 + b^2 = c^2$ nonzero, consider equation (8). Let $A = \dfrac{a^4 + 3a^2b^2 + b^4}{4(a^2 + b^2)}$ and $B = \dfrac{a^2b^2}{4}$, so that (8) can be expressed as

$$h^2 = z^4 + 4Az^2 + 4B \tag{9}$$

If we have a rational solution to (9) then we get a rational solution to

$$v^2 = u^3 + \alpha u^2 + \beta u \tag{10}$$

where $\alpha = -2A$ and $\beta = A^2 - B$, by

$$u = \frac{1}{2}(z^2 + 2A - h) \quad \text{and} \quad v = \frac{1}{2}z(z^2 + 2A - h).$$

Conversely, assuming $(u, v) \neq (0, 0)$, a rational solution to (10) leads to a rational solution to (9) by

$$z = \frac{v}{u} \quad \text{and} \quad h = \frac{v^2}{u^2} + 2A - 2u.$$

The discriminant of (10) is

$$(\alpha^2 - 4\beta)\beta^2 = 4B(A^2 - B)^2 = \frac{a^2b^2(a^4 + a^2b^2 + b^4)^4}{256(a^2 + b^2)^4}$$

which is nonzero since at least one of $a, b$ are nonzero. Hence (10) defines an elliptic curve over the field $\mathbb{Q}(m)$ and every point $(u, v)$ on this elliptic curve yields a point $(z, h)$ satisfying (8), which implies a solution $(x, y, z, t)$ to (4). $\qquad \square$

We will show how to obtain new solutions, by adding points on the elliptic curve. We write $E$ for the elliptic curve (10) over $\mathbb{Q}(m)$, $+$ for the addition of points on the curve (10), and $P_S$ will denote a point on $E$ yielding a solution $S$ to (4). The addition of points on an elliptic curve is described in Silverman [6]. Note that instead of writing $P + P$ we will write $2P$.

**Example 7.** The solutions $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{D}_1$ to (4) are provided by the following points on (10):

$$P_{\mathcal{F}_0} = \left( \frac{(c - b)\,(a^2 - ab + b^2)\,(a^2 + ab + b^2)}{4\,(c + b)\,(a^2 + b^2)}, \; \frac{b^2\,(c - b)\,(a^2 - ab + b^2)\,(a^2 + ab + b^2)}{4\,a\,(c + b)\,(a^2 + b^2)} \right)$$

$$P_{\mathcal{F}_1} = \left( \frac{(a^2 - ab + b^2)^2}{4\,(a^2 + b^2)}, \; 0 \right)$$

$$P_{\mathcal{F}_2} = \left( \frac{a^2b^2}{4\,(a^2 + b^2)}, \; \frac{ab\,(a^2 - b^2)}{8c} \right)$$

$$P_{\mathcal{D}_1} = \left( \frac{(a - b)^2\,(a^2 - ab + b^2)^2}{4\,(a + b)^2\,(a^2 + b^2)}, \; \frac{a^2b^2\,(a - b)\,(a^2 + ab + b^2)^2}{2c\,(a + b)^3\,(a^2 + b^2)} \right)$$

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Let us remind the reader of the Lutz-Nagell theorem, which will be used in the proof of Theorem 8 along with the "specialization theorem".

**Theorem 8.** *Let $E$ be given by $y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$. Let $P = (x, y) \in E(\mathbb{Q})$. Suppose $P$ has finite order. Then $x, y \in \mathbb{Z}$. If $y \neq 0$ then $y^2$ divides $4A^3 + 27B^2$.*

**Theorem 9.** *There exists infinitely many points on (10).*

*Proof.* Let $N = \dfrac{a^2 - ab + b^2}{2c}$ and $L = \dfrac{a^2 + ab + b^2}{2c}$. Then (10) can be expressed as

$$E: \quad v^2 = u\left(u - N^2\right)\left(u - L^2\right) \tag{11}$$

It can be shown that the rank of this elliptic curve over $\mathbb{Q}(m)$ is at least one. To show the rank is at least one, specialize at say, $(a, b) = (3, 4)$. Then the point $P = (36/25, 21/10)$ is on the curve

$$E_1: \quad v^2 = u^3 - \frac{769}{50}u^2 + \frac{231361}{10000}u.$$

In order to use the Lutz-Nagell theorem, we need to express $E_1$ in Weierstrass form with integral coefficients:

$$E_1': \quad y^2 = x^3 - 1538x^2 + 231361x.$$

The point $P$ on $E_1$ corresponds to $P' = (144, 2100)$ on $E_1'$, and thus

$$2P' = (70980625/28224, 389867877575/4741632).$$

Since $2P'$ is not an integral point, $P'$ is not a point of finite order on $E_1'$, so by the "specialization theorem" the rank of (11) is of positive rank, these are the same ideas as in Ulas [7]. Also note, calculations found at least 17 distinct points on $E$. The maximum number of torsion points is 16, so the rank must be at least one. $\qquad\square$

*Remark* 10. On $E$, the torsion points are $(0, 0)$, $(N^2, 0)$, $(L^2, 0)$, and $\mathcal{O}$, the point at infinity.

**Theorem 11.** *Every solution to the equation*

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \qquad (4)$$

*such that $(x, y) = (a, b)$, $a$ and $b$ nonzero, proceeds from exactly two points on $E$ different from $(0, 0)$. If one of them is $P = (u, v)$, with $u \neq 0$, then the other one is $P' = (u', v') = \lambda(u, v)$, with $\lambda = \dfrac{\beta}{u^2}$.*

*Proof.* Although $E$ is defined over a function field, Figure 2 provides some intuition for our proof. Let $P = (u, v)$ be a point on $E$, with $u \neq 0$, which yields a solution to (4). If there exists a point $P' = (u', v')$ on $E$ different from $(0, 0)$ yielding the same solution to (4) as
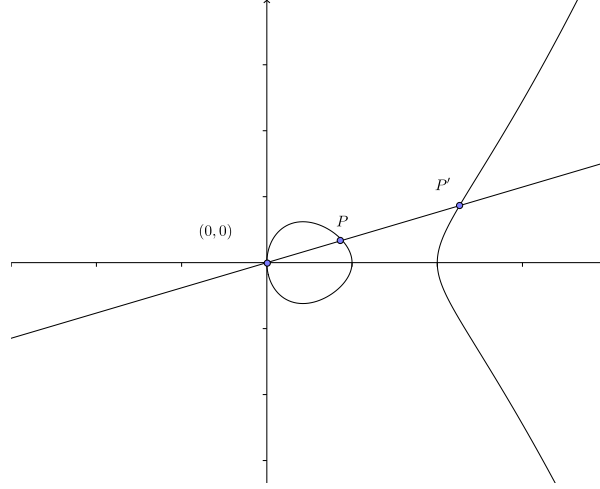
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Figure 2: $E : v^2 = u^3 + \alpha u^2 + \beta u$

$(u, v)$, then $z' = z$ so $u' \neq 0$ and $\dfrac{v'}{u'} = \dfrac{v}{u}$. Thus there exists a rational $\lambda$ such that $u' = \lambda u$ and $v' = \lambda v$ and

$$0 = -v'^2 + u'^3 + \alpha u'^2 + \beta u' = -\lambda^2 v^2 + \lambda^3 u^3 + \alpha \lambda^2 u^2 + \beta \lambda u.$$

By substituting for $v^2$, we have

$$0 = -\lambda^2(u^3 + \alpha u^2 + \beta u) + \lambda^3 u^3 + \alpha^2 \lambda^2 u^2 + \beta \lambda u = \lambda \left(\lambda - 1\right) u \left(\lambda u^2 - \beta\right).$$

Since $u' \neq 0$, then $\lambda \neq 0$. Assuming $(u', v') \neq (u, v)$ implies $\lambda \neq 1$. Thus $\lambda = \dfrac{\beta}{u^2}$.

To show $(u', v')$ is on $E$, notice

$$u'^3 + \alpha u'^2 + \beta u' = \frac{\beta^3}{u^3} + \alpha \frac{\beta^2}{u^2} + \frac{\beta^2}{u} = \frac{\beta^2}{u^4}\left(\beta u + \alpha u^2 + u^3\right) = \frac{\beta^2}{u^4} v^2 = \left(\frac{\beta}{u^2} v\right)^2 = v'^2.$$

Both of these points yield the same solution to (4) since $x = a, y = b$ and $z = z'$. This defines a solution $(x : y : z : t)$, except possibly for the sign of $t$. $\qquad\square$

*Remark* 12. If $P_{\mathcal{F}_2} = (u, v)$, we find that $2P_{\mathcal{F}_0} = (u', v')$, both yielding the same solution $\mathcal{F}_2$.

From the previous two theorems we deduce the following corollary.

**Corollary 13.** *There exists infinitely many parametric solutions to* $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$.

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# 5  Obtaining new parametric solutions

Before obtaining new parametric solutions, we can interpret Lemma 3 in terms of points in $E(K)$ where $K = \mathbb{Q}(m)$ and $E$ as described earlier.

**Proposition 14.** *Let $(u, v) \in E(K)$ such that $(u, v) \notin \{\mathcal{O}, (0,0), (N^2, 0), (L^2, 0)\}$, and let $(u', v'), (u'', v'') \in E(K)$ such that:*

$$(u, v) + (u', v') = (N^2, 0) \text{ and } (u, v) + (u'', v'') = (L^2, 0).$$

*If $(u, v)$ yields a solution $(x : y : z : t)$ to (4) such that $xyzt \neq 0$, then $(u', v')$ and $(u'', v'')$ both yield $\left( \dfrac{1}{y} : \dfrac{1}{x} : \dfrac{1}{z} : \dfrac{1}{t} \right)$, except perhaps the signs of $\dfrac{1}{z}$ and $\dfrac{1}{t}$.*
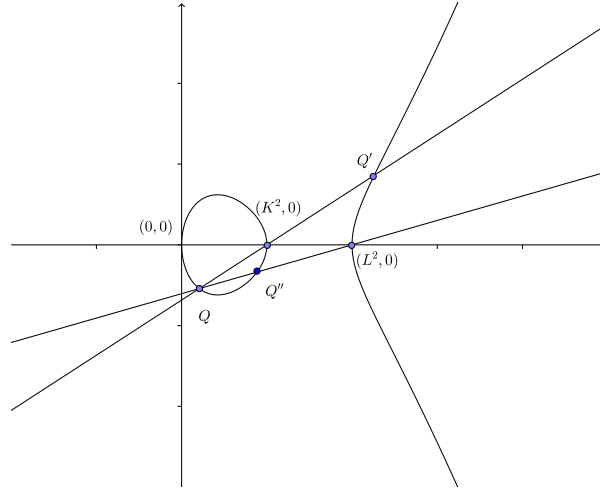


Figure 3: $E : v^2 = u^3 + \alpha u^2 + \beta u$

*Proof.* From the group law on equation (10), we deduce

$$(u', v') = \left( \frac{N^2(u - L^2)}{u - N^2}, \frac{N^2(N^2 - L^2)v}{(u - N^2)^2} \right), \quad (u'', v'') = \left( \frac{L^2(u - N^2)}{u - L^2}, \frac{L^2(L^2 - N^2)v}{(u - L^2)^2} \right).$$

If $(u', v')$ yields $(x' : y' : z' : t')$ then $z' = \dfrac{v'}{u'} = \dfrac{(N^2 - L^2)v}{(u - N^2)(u - L^2)}$. From the relations $N^2 - L^2 = -ab$ and $v^2 = u(u - N^2)(u - L^2)$, we conclude

$$z' = -ab\frac{u}{v} = -\frac{ab}{z}.$$

Thus if $(u, v)$ yields $(x : y : z : t) = (a : b : z : t)$, then $(u', v')$ yields

$$(x' : y' : z' : t') = \left(a : b : \frac{ab}{z} : \frac{ab}{t}\right) = \left(\frac{1}{b} : \frac{1}{a} : \frac{1}{z} : \frac{1}{t}\right) = \left(\frac{1}{y} : \frac{1}{x} : \frac{1}{z} : \frac{1}{t}\right),$$

where the signs of $z'$ and $t'$ may be positive or negative. The proof for $z''$ is similar. □

Next let $P_{\mathcal{D}_2} = P_{\mathcal{F}_0} + P_{\mathcal{D}_1}$. If $a^2 + b^2 = c^2$, we find $P_{\mathcal{D}_2} = (u, v)$ with

$$u = \frac{(c + a)(a^4 + a^2 b^2 + b^4)(2a^3 - a^2 c + b^2 c)^2}{4(c - a)(a^2 + b^2)(2a^3 + a^2 c - b^2 c)^2}$$

$$v = \frac{a^2 b(a^4 + a^2 b^2 + b^4)(2a^3 - a^2 c + b^2 c)(2b^3 - a^2 c + b^2 c)(2b^3 + a^2 c - b^2 c)}{4(c - a)^2 (a^2 + b^2)(2a^3 + a^2 c - b^2 c)^3}$$

By the same methods used in the proof of Theorem 6, we deduce the following solution $\mathcal{D}_2$:

**Proposition 15.** *If $a^2 + b^2 = c^2$ and if*

$$
\begin{aligned}
x &= ab(2\,a^3 - a^2 c + b^2 c)(2\,a^3 + a^2 c - b^2 c)(2\,ab^2 + a^2 c + b^2 c)(-2\,ab^2 + a^2 c + b^2 c) \\
y &= b^2(2\,a^3 - a^2 c + b^2 c)(2\,a^3 + a^2 c - b^2 c)(2\,ab^2 + a^2 c + b^2 c)(-2\,ab^2 + a^2 c + b^2 c) \\
z &= a^2(2\,b^3 - a^2 c + b^2 c)(2\,b^3 + a^2 c - b^2 c)(2\,ab^2 + a^2 c + b^2 c)(-2\,ab^2 + a^2 c + b^2 c) \\
t &= ab(2\,a^3 - a^2 c + b^2 c)(2\,a^3 + a^2 c - b^2 c)(2\,a^2 b + a^2 c + b^2 c)(-2\,a^2 b + a^2 c + b^2 c)
\end{aligned}
$$

*then $\mathcal{D}_2 = (x : y : z : t)$ is a solution to (4).*

**Example 16.** Since $(a : b : c) = (2m : m^2 - 1 : m^2 + 1)$, if $m = 2$, then

$$
\begin{aligned}
x &= 1\ 899\ 301\ 428 \\
y &= 1\ 424\ 476\ 071 \\
z &= 282\ 491\ 696 \\
t &= 1\ 165\ 848\ 372
\end{aligned}
$$

Next let $P_{\mathcal{D}_3} = 2P_{\mathcal{D}_1}$. We find that $P_{\mathcal{D}_3} = (u, v)$ with:

$$u = \frac{(a^2 + b^2)(a^4 + b^4)^2}{16a^2 b^2 (a - b)^2 (a + b)^2}, \qquad v = \frac{PQRS(a^4 + b^4)}{64a^3 b^3 c(a - b)^3 (a + b)^3}$$

where

$$
\begin{aligned}
P &= -a^3 + a^2 b + ab^2 + b^3, & Q &= a^3 - a^2 b + ab^2 + b^3 \\
R &= a^3 + a^2 b - ab^2 + b^3, & S &= a^3 + a^2 b + ab^2 - b^3.
\end{aligned}
$$

From this we deduce $\mathcal{D}_3$:

**Proposition 17.** *If $a^2 + b^2 = c^2$ and if*

$$x = a^2bc(a^2 - b^2)(a^2 + b^2)(a^4 + b^4)G$$
$$y = ab^2c(a^2 - b^2)(a^2 + b^2)(a^4 + b^4)G$$
$$z = PQRSG$$
$$t = 4ab(a^2 - b^2)(a^2 + b^2)^2(a^4 + b^4) \; H$$

*with $P, Q, R, S$ as above, and*

$$G = a^{12} + 6a^{10}b^2 - a^8b^4 + 4a^6b^6 - a^4b^8 + 6a^2b^{10} + b^{12}$$
$$H = a^{12} - 2a^{10}b^2 + 7a^8b^4 + 4a^6b^6 + 7a^4b^8 - 2a^2b^{10} + b^{12},$$

*then $\mathcal{D}_2 = (x : y : z : t)$ is a solution to (4).*

Now put $P_{\mathcal{D}_4} = P_{\mathcal{D}_3} + P_{\mathcal{F}_1}$. We find $P_{\mathcal{D}_4} = (u, v)$, with

$$u = \frac{(a^2 - ab + b^2)^2 P^2 S^2}{4(a^2 + b^2)^2 Q^2 R^2}, \qquad v = \frac{a^2 b^2 c(a^2 - b^2)(a^2 - ab + b^2)^2(a^4 + b^4)PS}{Q^3 R^3}$$

From this we deduce the following solution $\mathcal{D}_4$:

**Proposition 18.** *If $a^2 + b^2 = c^2$ and if*

$$x = acPQRSH$$
$$y = bcPQRSH$$
$$z = 4a^2b^2(a^2 - b^2)(a^2 + b^2)^2(a^4 + b^4)H$$
$$t = abPQRSG$$

*with the same $P, Q, R, S, G, H$ as in Proposition 17, then $\mathcal{D}_4 = (x : y : z : t)$ is a solution to (4).*

The degree of this solution is 26 in $(a : b : c)$, and hence 52 in the homogeneous coordinates $(m : n)$ if we express $(a : b : c) = (2mn : m^2 - n^2 : m^2 + n^2)$ for nonzero integers $m, n$.

*Remark* 19. The values $z$ for $P_{\mathcal{D}_3}$ and $z'$ for $P_{\mathcal{D}_4}$ satisfy $zz' = ab$, so except perhaps exchanging $(x', y')$ and $(y', x')$, $\mathcal{D}_4$ is deduced from $\mathcal{D}_3$ by replacing $(x, y, z)$ by $\left(\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}\right)$.

In summary, parametric solutions $(x : y : z : t)$ to (4) with their degree are shown in Table 1.

| solution | $\mathcal{F}_0$ | $\mathcal{F}_1$ | $\mathcal{F}_2$ | $\mathcal{D}_1$ | $\mathcal{D}_2$ | $\mathcal{D}_3$ | $\mathcal{D}_4$ |
|---|---|---|---|---|---|---|---|
| degree | 4 | 4 | 12 | 16 | 28 | 48 | 52 |

Table 1: Degree of parametric solutions

13

# 6 Acknowledgment

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