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The Frobenius Problem for Modified Arithmetic Progressions

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Abstract

For a set of positive and relatively prime integers A, let $\Gamma(A)$ denote the set of integers obtained by taking all nonnegative integer linear combinations of integers in A. Then there are finitely many positive integers that do not belong to $\Gamma(A)$. For the modified arithmetic progression $A = \{a, ha+d, ha+2d, \ldots, ha+kd\}$, $\gcd(a,d) = 1$, we determine the largest integer $\mathtt{g}(A)$ that does not belong to $\Gamma(A)$, and the number of integers $\mathtt{n}(A)$ that do not belong to $\Gamma(A)$. We also determine the set of integers $\mathcal{S}^{\star}(A)$ that do not belong to $\Gamma(A)$ which, when added to any positive integer in $\Gamma(A)$, result in an integer in $\Gamma(A)$. Our results generalize the corresponding results for arithmetic progressions.

1 Introduction

Given a finite set $A = \{a_1, \ldots, a_k\}$ of positive integers with gcd $A := \gcd(a_1, \ldots, a_k) = 1$, let $\Gamma(A) := \{a_1x_1 + \cdots + a_kx_k : x_i \geq 0\}$ and $\Gamma^{\star}(A) = \Gamma(A) \setminus \{0\}$. It is well known that $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is finite. Although it was Sylvester [9] who first asked to determine

$$\mathtt{g}(A) := \max \Gamma^c(A),$$

and who showed that $\mathtt{g}(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$, it was Frobenius who was largely instrumental in giving this problem the early recognition, and it is after him that the problem is named. Ramírez-Alfonsín's monograph on the Frobenius problem [5] gives an extensive survey. Related to the Frobenius problem is the problem of determining $\mathtt{n}(A) := |\Gamma^c(A)|$. As in the case of determining $\mathtt{g}(A)$, it was Sylvester who showed that $\mathtt{n}(a_1, a_2) = \frac{1}{2}(a_1-1)(a_2-1)$.

Tripathi [11] introduced the following variation of the Frobenius problem. Let

$$\mathcal{S}^\star(A) := \{n \in \Gamma^c(A) : n + \Gamma^\star(A) \subset \Gamma^\star(A)\}.$$

Since $\mathtt{g}(A)$ is the largest integer in $\mathcal{S}^\star(A)$, the determination of $\mathcal{S}^\star(A)$ also provides the resolution of $\mathtt{g}(A)$. For the sake of convenience, we recall the following essential result regarding $\mathcal{S}^\star(A)$ from [11]. Fix $a \in A$, and let $\mathbf{m_C}$ denote the smallest integer in $\Gamma(A) \cap \mathbf{C}$, where $\mathbf{C}$ denotes a nonzero residue class modulo a. If $\mathscr{C}$ denotes the set of all nonzero residue classes modulo a, then

$$\mathcal{S}^\star(A) \subseteq \{\mathbf{m_C} - a : \mathbf{C} \in \mathscr{C}\}. \tag{1}$$

Moreover, if (x) denotes the residue class of x modulo a and $\mathbf{m}_x$ the least integer in $\Gamma(A)\cap(x)$, then

$$\mathbf{m}_j - a \in \mathcal{S}^\star(A) \Longleftrightarrow \mathbf{m}_j - a \geq \mathbf{m}_{j+i} - \mathbf{m}_i \text{ for } 1 \leq i \leq a-1. \tag{2}$$

Observe that $\mathcal{S}^\star(A) \neq \emptyset$; in fact, $\mathtt{g}(A)$ is the largest integer in $\mathcal{S}^\star(A)$. A complete description of $\mathcal{S}^\star(A)$ would therefore lead to the determination of $\mathtt{g}(A)$.

The functions $\mathtt{g}$ and $\mathtt{n}$ are easily determined from the values of $\mathbf{m_C}$ by Lemma 1. Brauer and Shockley [2] proved (i) and Selmer [8] proved (ii); a short proof of both results may be found in [10].

Lemma 1. **([2, 8])** *Let $a \in A$. Then*

(i) $\mathtt{g}(A) = \max_{\mathbf{C}} \mathbf{m_C} - a$, *the maximum taken over all nonzero classes $\mathbf{C}$ modulo a;*

(ii) $\mathtt{n}(A) = \dfrac{1}{a}\sum_{\mathbf{C}} \mathbf{m_C} - \frac{1}{2}(a-1)$, *the sum taken over all nonzero classes $\mathbf{C}$ modulo a.*

In cases when all but one integer in A have a nontrivial divisor, the following reduction formulae given by Lemma 2 is useful. Johnson [4] gave the reduction formulae for $\mathtt{g}(A)$ and Rødseth [7] for $\mathtt{n}(A)$; a short proof of both results may be found in [12].

Lemma 2. **([4, 7])** *Let $a \in A$, let $d = \gcd\big(A \setminus \{a\}\big)$, and define $A' := \frac{1}{d}\big(A \setminus \{a\}\big)$.*

(i) $\mathtt{g}(A) = d \cdot \mathtt{g}\big(A' \cup \{a\}\big) + a(d-1)$*;*

(ii) $\mathtt{n}(A) = d \cdot \mathtt{n}\big(A' \cup \{a\}\big) + \frac{1}{2}(a-1)(d-1)$*.*

In this article, we determine $\mathtt{g}(A)$, $\mathtt{n}(A)$ and $\mathcal{S}^\star(A)$ for the modified arithmetic progression $A = \{a, ha+d, ha+2d, \ldots, ha+kd\}$ with $\gcd(a,d) = 1$.

2 The case $A = \{a, ha + d, ha + 2d, \ldots, ha + kd\}$

For arithmetic progressions, Roberts [6] determined $\mathtt{g}(A)$, later simplified by Bateman [1], while Grant [3] determined $\mathtt{n}(A)$. A simple proof for both these results using Lemma 1 can be found in [10]. In fact, it is also possible to determine $\mathcal{S}^\star(A)$ in this case; see [11]. The result about $\mathtt{g}(A)$ and $\mathtt{n}(A)$ when A consists of terms in arithmetic progression can be modified or extended in many ways. One such modification is to consider $A = \{a, ha + d, ha+2d, \ldots, ha+kd\}$ with $\gcd(a, d) = 1$ and $h, k \geq 1$. The result for $\mathtt{g}(A)$ is due to Selmer [8], but we provide a simpler proof that also leads to other results.

Henceforth let $A = \{a, ha + d, ha + 2d, \ldots, ha + kd\}$ with $\gcd(a, d) = 1$ and $h, k \geq 1$. Then $\mathtt{g}(A)$ denotes the largest N such that

$$ax_0+(ha+d)x_1+(ha+2d)x_2+\cdots+(ha+kd)x_k = a\left(x_0 + h\sum_{i=1}^k x_i\right)+d\left(\sum_{i=1}^k ix_i\right) = N \quad (3)$$

has no solution in nonnegative integers, and $\mathtt{n}(A)$ the number of such integers N.

Lemma 3. *For each x, $1 \leq x \leq a-1$, the least positive integer of the form given by equation (3) in the class dx mod a is given by $ha\left(1 + \lfloor\frac{x-1}{k}\rfloor\right) + dx$.*

Proof. Let $\mathbf{m}_{dx}$ denote the least positive integer in the class (dx) modulo a. Then $\mathbf{m}_{dx}$ is the minimum value attained by the expression on the left in equation (3) subject to $\sum_{i=1}^k ix_i = x$ and each $x_i \geq 0$. If $x = qk + r$, $0 \leq r \leq k - 1$, the sum $x_0 + h\sum_{i=1}^k x_i$ is minimized by choosing $x_k = q$, $x_r = 1$ and $x_i = 0$ for all other i, unless $r = 0$ in which case we must choose $x_r = 0$. Thus the minimum value for $x_0 + h\sum_{i=1}^k x_i$ is $h(q + 1)$ if $r \neq 0$ and hq if $r = 0$, which may be combined as $h\left(1 + \lfloor\frac{x-1}{k}\rfloor\right)$. Hence $\mathbf{m}_{dx} = ha\left(1 + \lfloor\frac{x-1}{k}\rfloor\right) + dx$. $\square$

Theorem 4. *Let a, d, h, k be positive integers, with $\gcd(a, d) = 1$. Then*

(i) $\mathtt{g}\big(a, ha + d, ha + 2d, \ldots, ha + kd\big) = ha\left\lfloor\frac{a-2}{k}\right\rfloor + (h - 1)a + d(a - 1)$;

(ii) $\mathtt{n}\big(a, ha + d, ha + 2d, \ldots, ha + kd\big) = \frac{1}{2}h(a + r)\left(1 + \left\lfloor\frac{a-2}{k}\right\rfloor\right) + \frac{1}{2}(a - 1)(d - 1)$, *where* $r \equiv a - 2$ *mod* k.

Proof.

(i)
$$\begin{aligned}\mathtt{g}\big(a, ha + d, ha + 2d, \ldots, ha + kd\big) &= \max_{\mathbf{C}\in\mathscr{C}} \mathbf{m}_{\mathbf{C}} - a \\ &= \max_{1\leq x\leq a-1}\left(ha\left(1 + \left\lfloor\frac{x-1}{k}\right\rfloor\right) + dx\right) - a \\ &= ha\left\lfloor\frac{a-2}{k}\right\rfloor + (h - 1)a + d(a - 1).\end{aligned}$$

(ii) Write $a-2=qk+r$, with $0\le r\le k-1$. Then

$$\begin{aligned}
\mathtt{n}\big(a, ha+d, ha+2d, \dots, ha+kd\big) &= \tfrac{1}{a}\sum_{\mathbf{C}\in\mathscr{C}} \mathbf{m_C} - \tfrac{1}{2}(a-1)\\
&= \tfrac{1}{a}\sum_{x=1}^{a-1}\Big(ha\left(1+\left\lfloor\tfrac{x-1}{k}\right\rfloor\right)+dx\Big)-\tfrac{1}{2}(a-1)\\
&= h\sum_{x=0}^{a-2}\left(1+\left\lfloor\tfrac{x}{k}\right\rfloor\right)+\tfrac{1}{2}d(a-1)-\tfrac{1}{2}(a-1)\\
&= h\big(k(1+2+\cdots+q)+(q+1)(r+1)\big)\\
&\quad +\tfrac{1}{2}(a-1)(d-1)\\
&= \tfrac{1}{2}h(a+r)\left(1+\left\lfloor\tfrac{a-2}{k}\right\rfloor\right)+\tfrac{(a-1)(d-1)}{2}.
\end{aligned}$$

□

Observation 5. The case when A consists of integers in arithmetic progression is the special case $h=1$ in Theorem 4.

Recall that $\mathcal{S}^\star(A) := \{n\in\Gamma^c(A) : n+\Gamma^\star(A)\subset\Gamma^\star(A)\}$. Since $\mathtt{g}(A)$ is the *largest* element in $\mathcal{S}^\star(A)$, the set $\mathcal{S}^\star(A)$ is intimately linked with the Frobenius problem.

Theorem 6. *Let a,d,h,k be positive integers, with $\gcd(a,d)=1$. Write $a-1=qk+r$, with $1\le r\le k$. Then*

$$\mathcal{S}^\star\big(a, ha+d, ha+2d, \dots, ha+kd\big) = \left\{ha\left\lfloor\tfrac{x-1}{k}\right\rfloor+(h-1)a+dx : a-r\le x\le a-1\right\}.$$

Proof. Fix $k\ge 1$, and let $A=\{a, ha+d, ha+2d, \dots, ha+kd\}$. By equation (1) and Lemma 3,

$$\mathcal{S}^\star(A)\subseteq\left\{ha\left\lfloor\tfrac{x-1}{k}\right\rfloor+(h-1)a+dx : 1\le x\le a-1\right\}.$$

By equation (2), $ha\lfloor\frac{x-1}{k}\rfloor+(h-1)a+dx\in\mathcal{S}^\star$ if and only if for each y with $1\le y\le a-1$,

$$ha\left\lfloor\tfrac{(x+y)\bmod a-1}{k}\right\rfloor+d\big((x+y)\bmod a\big)\le ha\left(\left\lfloor\tfrac{x-1}{k}\right\rfloor+\left\lfloor\tfrac{y-1}{k}\right\rfloor\right)+(h-1)a+d(x+y). \qquad (4)$$

Suppose $k\le a-1$, and write $a-1=qk+r$ with $1\le r\le k$. Then unless $x=a-1$, $x+y\le a-1$ for at least one y. For such a y, equation (4) reduces to proving the inequality

$$\left\lfloor\tfrac{x+y-1}{k}\right\rfloor\le\left\lfloor\tfrac{x-1}{k}\right\rfloor+\left\lfloor\tfrac{y-1}{k}\right\rfloor.$$

If we now write $x=q_1k+r_1$, $y=q_2k+r_2$ with $1\le r_1,r_2\le k$, the reduced inequality above fails to hold precisely when $r_1+r_2\ge k+1$. Given x, and hence r_1, the choice $y=r_2=k+1-r_1$ will thus ensure that equation (4) fails to hold provided $x+y\le a-1$. However, such a choice for y is not possible precisely when $x\ge qk+1=a-r$, so that equation (4) always holds in only these cases. Finally, it is easy to verify that equation (4) holds if $x = a - 1$. This shows $\mathcal{S}^\star = \left\{ha\lfloor \frac{x-1}{k} \rfloor + (h-1)a + dx : a - r \le x \le a - 1\right\}$ if $1 \le k \le a - 1$.

If $k \ge a$, equation (4) reduces to $d\big((x+y) \bmod a\big) \le d(x+y) + (h-1)a$. Thus $\mathcal{S}^\star(A) = \left\{(h-1)a + dx : 1 \le x \le a - 1\right\}$, as claimed, since $r = a - 1$ and $\lfloor \frac{x-1}{k} \rfloor = 0$ in this case. This completes the proof. $\square$

Observation 7. The case when A consists of integers in arithmetic progression is the special case $h = 1$ in Theorem 6. Moreover, the result in the first part of Theorem 4 follows directly from Theorem 6.

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