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# On the Entropy of Curves

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*To Jean-Paul Allouche for his 60th birthday*

**Abstract**

Using geometric probability, we apply the formal definitions of Shannon entropy and Rényi's generalization to study the complexity of planar curves of finite length within a convex set. The bounds for the Shannon and Rényi entropies depend on the arc length of the curve and on that of the boundary of the convex set; they involve a Gibbs distribution and a power law distribution, respectively. We also obtain explicit formulae for the two entropies and determine convex sets that maximize the entropy of curves.

## 1 Introduction

Planar curves range from a simple straight line segment or a circle to a complicated tangle. Their complexity has been analysed using both the thermodynamic concept of Shannon entropy [9, 10] and the geometrical concept of dimension [7, 8]. Examples can be found in

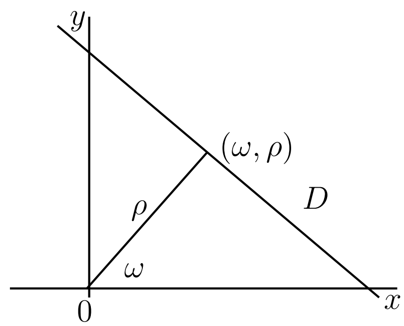


Figure 1: A straight line $D$ determined by polar coordinates $(\omega, \rho)$

various other domains [1, 2, 3, 4, 5, 6, 11, 12]. A brief overview of the fundamental ideas is given and the Shannon and the more general Shannon-Rényi entropies are defined. The question of bounds for these entropies is considered; the analysis of the general Rényi entropy turns out to be quite different from that of the Shannon entropy and involves a modified Hurwitz zeta function. A more precise viewpoint which gives exact formulae for the entropies is then developed.

## 2 Geometric probability

Given a plane curve $\Gamma$ of finite length $|\Gamma|$, we recall the classical definition of the probability that a straight line $D$ intersects $\Gamma$ in exactly $n$ points [9, 15]. In the plane, a straight line $D$ that does not pass through the origin $O$ is determined by the polar coordinates $(\omega, \rho)$, where $\omega$, $0 \leqslant \omega < 2\pi$, is the angle the normal to $D$ makes with the $x$-axis and $\rho > 0$ is the distance from $O$. Straight lines that pass through the origin are not counted since, as we will see, the family of such lines has measure 0.

In the $(\omega, \rho)$ plane, a point in the strip $\mathcal{S} := [0, 2\pi) \times (0, \infty)$ represents a straight line not passing through the origin in the usual $(x, y)$-plane. The measure $\mu$ in the strip is defined by

$$d\mu := d\rho\, d\omega,$$

i.e., the usual Lebesgue measure. Now let $K$ be a bounded convex set in the $(x, y)$-plane and let $F(K)$ be the set of straight lines that intersect $K$. The $\mu$-measure of $F(K)$ is known to be equal to the length $|\partial K|$ of the boundary $\partial K$ of $K$, i.e.,

$$\mu(F(K)) = |\partial K|.$$

It follows that the measure given by

$$dp = \frac{d\mu}{|\partial K|} = \frac{d\rho\, d\omega}{|\partial K|}$$

is a probability measure defined on $F(K)$.

Suppose $K_0 \subseteq K$ is a convex subset of $K$. Then $|\partial K_0| \leqslant |\partial K|$ and the probability that a straight line meeting $K$ also meets $K_0$ is

$$\frac{|\partial K_0|}{|\partial K|}.$$

Consequently the probability that a straight line meeting $K$ misses $K_0$ is

$$1 - \frac{|\partial K_0|}{|\partial K|}. \tag{1}$$

Consider a curve $\Gamma_0$ of finite length lying in the convex set $K$. Denote by $p_n(\Gamma_0, K)$ the probability that a straight line $D$ meeting $K$ intersects $\Gamma_0$ in exactly $n$ points. Observe that if $K = K_0 = \mathrm{con}(\Gamma_0)$, the convex hull of $\Gamma_0$, then $D$ must hit $\Gamma_0$, so that $p_0(\Gamma_0, K_0) = 0$.

By the definition of probability,

$$\sum_{n=0}^{\infty} p_n(\Gamma_0, K) = 1. \tag{2}$$

A classical result of H. Steinhaus [17] states that the expected number $\lambda$ say of intersection points of $\Gamma_0$ with random straight lines is

$$\lambda := \sum_{n=1}^{\infty} n\, p_n(\Gamma_0, K) = \frac{2|\Gamma_0|}{|\partial K|}. \tag{3}$$

The next moment $\sum_{n=1}^{\infty} n^2\, p_n(\Gamma_0, K)$ corresponds to the "energy" of the curve but is much less well-behaved and will not be considered.

# 3 Entropy

The celebrated Shannon entropy of thermodynamics and information theory has a generalization due to Rényi [13, 14]. Their definitions are used in conjunction with geometric probability to define formally notions of entropy for curves of finite length within a convex set $K$. Note that unlike the thermodynamic setting, there is no underlying mechanism or dynamic here that leads to the entropies increasing.

## 3.1 The Shannon entropy

The sequence $(p_n(\Gamma_0, K)\colon n = 0, 1, 2, \ldots)$ of probabilities of the number of intersection points can be substituted into the formula for Shannon entropy. To be precise, the *Shannon entropy* $h(\Gamma_0, K)$ of the curve $\Gamma_0$ relative to a convex set $K$ containing $\Gamma_0$ is defined by the formula

$$h(\Gamma_0, K) = \sum_{n=0}^{\infty} p_n(\Gamma_0, K) \log \frac{1}{p_n(\Gamma_0, K)} = \sum_{n=0}^{\infty} p_n(K) \log \frac{1}{p_n(K)}, \tag{4}$$

where as usual we put $p \log(1/p) = 0$ if $p = 0$ and where for simplicity here and elsewhere whenever the argument is clear, we suppress $\Gamma_0$ or $K$ and write the probability

$$p_n(\Gamma_0, K) = p_n(K) \text{ or } p_n.$$

Similarly we will write the Shannon entropy

$$h(\Gamma_0, K) = h(K) = h.$$

### 3.2 The Rényi entropy

Rényi [13, 14] has defined a form of entropy $h^{(s)}$, where $s \geqslant 0$, by

$$h^{(s)} := \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s, \tag{5}$$

which tends to the Shannon entropy $h$ as $s \to 1$:

$$h^{(1)} := \lim_{s \to 1} h^{(s)} = \lim_{s \to 1} \left( \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s \right) = \sum_{n=0}^{\infty} p_n \log 1/p_n = h\,. \tag{6}$$

In addition it is readily verified that $d\,h^{(s)}/ds = 0$ at $s = 1$ and that $h^{(s)}$ is minimal for $s = 1$. The Rényi entropy can also be applied formally to curves. We will write

$$h^{(s)}(\Gamma_0, K) = \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s(\Gamma_0, K) \tag{7}$$

for the Rényi entropy for a curve $\Gamma_0$ relative to a convex set $K \supseteq \Gamma_0$ and, as with the Shannon entropy, we will write

$$h^{(s)}(\Gamma_0, K) = h^{(s)}(K) = h^{(s)}$$

when the meaning is clear.

The two series have finitely many terms for an algebraic curve, since $p_n = 0$ for all $n$ larger than its degree, but they converge (absolutely) in general by the following results. A separate argument is needed for the two entropies. The Shannon entropy ($s = 1$) has been treated in this context [9, 10] and is now discussed for completeness.

### 3.3 A bound for the Shannon entropy of curves

**Theorem 1.** *The Shannon entropy $h(\Gamma_0, K)$ of a curve $\Gamma_0$ of finite length lying in the convex set $K$ satisfies*

$$h(\Gamma_0, K) \leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K|} + 1\right) + \frac{2|\Gamma_0|}{|\partial K|} \log\left(1 + \frac{|\partial K|}{2|\Gamma_0|}\right) \leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K|} + 1\right) + 1 \tag{8}$$

*and more precisely in the case $K = K_0 = \operatorname{con}(\Gamma_0)$ (so that $p_0 = 0$),*

$$h(\Gamma_0, K_0) \leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K_0|}\right) + \left(\frac{2|\Gamma_0|}{|\partial K_0|} - 1\right)\log\left(\frac{2|\Gamma_0|}{2|\Gamma_0| - |\partial K_0|}\right)$$
$$\leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K_0|}\right) + 1. \tag{9}$$

*Proof.* By (4), it suffices to bound the sum $\sum_{n=k}^{\infty} p_n \log 1/p_n$, where $p_n \in [0,1]$ and $k = 0$ in (8) and $k = 1$ in (9), subject to the two constraints (2) and (3), i.e.,

$$\sum_{n=k}^{\infty} p_n = 1$$

and

$$\sum_{n=k}^{\infty} n\, p_n = \frac{2|\Gamma_0|}{|\partial K|} = \lambda,$$

where $k = 0$ or $1$. This is done using Lagrange multipliers. Let $\alpha, \beta \in \mathbb{R}$ and consider

$$U = \sum_{n=k}^{\infty} \widehat{p}_n \log 1/\widehat{p}_n - \alpha \sum_{n=k}^{\infty} \widehat{p}_n - \beta \sum_{n=k}^{\infty} n\widehat{p}_n.$$

Then for each $n = k, k+1, \ldots$, $\partial U/\partial \widehat{p}_n = 0$ implies

$$-\log \widehat{p}_n - 1 - \alpha - n\beta = 0,$$

whence the 'maximal' probability distribution $\widehat{p}_n$ is a negative exponential with constant factor. More precisely,

$$\widehat{p}_n = e^{-1-\alpha-n\beta} = Ce^{-\beta n}, \tag{10}$$

where $C = e^{-1-\alpha}$ and $\widehat{p}_n$ is a Gibbs distribution [16]. Recall that Gibbs measure is a probability measure of importance in thermodynamics: it is the unique measure that maximizes the entropy for a given expected energy. It underlies maximum entropy methods and related algorithms and its appearance here is accordingly natural.

The constant $C = e^{-1-\alpha}$ (or $\alpha$) is determined from the power series

$$\sum_{n=k}^{\infty} \widehat{p}_n = C \sum_{n=k}^{\infty} e^{-\beta n} = 1$$

which gives $C = C(\beta) = e^{\beta k}(1 - e^{-\beta})$. In statistical mechanics, $C(\beta)$ is the reciprocal of the partition function and in physics $\beta$ corresponds to the inverse temperature [16]. Moreover $C \sum_{n=k}^{\infty} n\, e^{-\beta n} = \lambda$ whence

$$\lambda = \frac{1}{1 - e^{-\beta}}\left(k - (k-1)\, e^{-\beta}\right) = \frac{1}{e^{\beta} - 1}\left(k\, e^{\beta} - k + 1\right).$$

Thus

$$e^{\beta} = \frac{\lambda - k + 1}{\lambda - k} \text{ and } e^{\beta} - 1 = \frac{1}{\lambda - k}.$$

Hence the entropy $h$ satisfies

$$h \leqslant C \sum_{n=k}^{\infty} e^{-\beta n} \log \frac{e^{\beta n}}{C} = H$$

say, so that

$$\begin{aligned} H &= \frac{\beta}{1 - e^{-\beta}} \left(k - (k-1)e^{-\beta}\right) + \log \frac{e^{-k\beta}}{1 - e^{-\beta}} \\ &= \frac{\beta}{e^{\beta} - 1} \left(ke^{\beta} - k + 1\right) - \beta\, k + \log \frac{e^{\beta}}{e^{\beta} - 1} \\ &= \log(\lambda - k + 1) + (\lambda - k) \log \frac{\lambda - k + 1}{\lambda - k} \\ &\leqslant \log(\lambda - k + 1) + 1. \end{aligned}$$

**Case 1** $k = 0$, corresponds to the inequality (8).

**Case 2** $k = 1$ corresponds to the inequality (9).

□

*Remark* 2. The first inequality (8) in the theorem shows that the entropy vanishes as $|\partial K|$ increases to infinity, a fact we shall rediscover in §4.

*Remark* 3. In the above calculation,

$$\lambda = \sum_{n=k}^{\infty} n\, \widehat{p}_n \geqslant k \sum_{n=k}^{\infty} \widehat{p}_n = k.$$

Thus obviously if $\lambda \to k$ from above, then $H = 0$. Hence

*Remark* 4. In the case $k = 1$, if $2|\Gamma|/|\partial K| \to 1$ from above, then $h = 0$; which is otherwise obvious since $2|\Gamma| = |\partial K|$ implies $\Gamma$ is a segment of straight line.

### 3.4 Bounds for the Rényi entropy of curves

Recall from equation (7) that the Rényi entropy $h^{(s)}(\Gamma_0, K)$ of $\Gamma_0$ relative to the convex set $K \supset \Gamma_0$ is given by

$$h^{(s)}(\Gamma_0, K) := \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s(\Gamma_0, K).$$

For simplicity, we let $K = K_0 = \mathrm{con}(\Gamma_0)$, so that $p_0(\Gamma_0, K_0) = 0$, and denote the Rényi entropy $h^{(s)}(\Gamma_0, K_0)$ of $\Gamma_0$ relative to its convex hull $K_0$ by $h_0^{(s)}$. Two bounds for the Rényi entropy $h_0^{(s)}$ are obtained, one for the range $0 < s < 1$ and the other for $1/2 < s < 1$. Although the Rényi entropy coincides with Shannon entropy at $s = 1$, the bounds for the former may well be infinite at $s = 1$.

Two auxiliary functions are introduced. For each $\gamma \in [0, \infty)$ and $s \in [1/2, 1]$, we write

$$F_s(\gamma) = \frac{\zeta_1\left(\frac{s}{1-s}; \gamma\right)}{\zeta_1\left(\frac{1}{1-s}; \gamma\right)},$$

where $\zeta_1(u, c)$, $c \geqslant 0$, is the modified Hurwitz zeta function given by

$$\zeta_1(u, c) := \sum_{k=1}^{\infty} \frac{1}{(k+c)^u} = \sum_{k=0}^{\infty} \frac{1}{(k+c)^u} - \frac{1}{c^u} = \zeta(u, c) - \frac{1}{c^u}.$$

Note that $\zeta_1(u, 0) = \zeta(u)$ and that $F_s(0) > 1$.

For convenience, the function $F_s$ is simplified when $1/2 \leqslant s < 1$ by putting $u = 1/(1-s) \in [2, \infty)$, so that $s/(1-s) = u - 1$ and

$$F_s(\gamma) = \frac{\zeta_1(u-1; \gamma)}{\zeta_1(u; \gamma)} := G_u(\gamma) = G_{1/(1-s)}(\gamma).$$

Then for $u > 2$ and $\gamma \geqslant 0$,

$$\begin{aligned} (\gamma+1)\zeta_1(u; \gamma) &= \sum_{k=1}^{\infty} \frac{\gamma+1}{(k+\gamma)^u} = \sum_{k=1}^{\infty} \left( \frac{k+\gamma}{(k+\gamma)^u} - \frac{k-1}{(k+\gamma)^u} \right) \\ &< \sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{u-1}} = \zeta_1(u-1; \gamma), \end{aligned}$$

and $G_u(\gamma) := \zeta_1(u-1; \gamma)/\zeta_1(u; \gamma) > 1 + \gamma$.

A more precise estimate is possible using the familiar inequality

$$\int_1^{\infty} f(x) dx \leqslant \sum_{k=1}^{\infty} f(k) \leqslant f(1) + \int_1^{\infty} f(x) dx \tag{11}$$

for a positive increasing integrable function $f \colon \mathbb{R} \to [0, \infty)$. It follows from (11) that

$$\frac{(1+\gamma)^{-u+1}}{u-1} < \zeta_1(u, \gamma) := \sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^u} < \frac{1}{(1+\gamma)^u} + \frac{1}{(u-1)(1+\gamma)^{u-1}}. \tag{12}$$

Hence for any fixed $u > 2$,

$$\begin{aligned} G_u(\gamma) &= \frac{\zeta_1(u-1, \gamma)}{\zeta_1(u, \gamma)} > \frac{1}{(u-2)(1+\gamma)^{u-2}} \left( \frac{1}{(1+\gamma)^u} + \frac{1}{(u-1)(1+\gamma)^{u-1}} \right)^{-1} \\ &> \frac{(u-1)}{(u-2)} \frac{(1+\gamma)^2}{(u+\gamma)}. \end{aligned}$$

Thus given any $u > 2$, for all sufficiently large $\gamma$, there exists an $\varepsilon_u > 0$ such that

$$G_u(\gamma) > (1 + \varepsilon_u)(1 + \gamma).$$

It follows that given any real $\kappa$, $\varepsilon_u(1 + \gamma) > \kappa$ for all sufficiently large $\gamma$. Hence for fixed $u$ the inequality

$$G_u(\gamma) > \gamma + \kappa \tag{13}$$

holds for any $\kappa$ and all sufficiently large $\gamma$. This lower bound for the function $F_s = G_u$ is used to show that the equation $F_s(\gamma) = \gamma + \lambda$, where $\lambda = 2|\Gamma_0|/|\partial K_0| \in (0, \infty)$ (see (3)), is soluble when $\lambda \geqslant F_s(0) = G_u(0)$, i.e., when $\lambda \geqslant \zeta(u-1)/\zeta(u) > 1$.

**Lemma 5.** *Suppose $\lambda \geqslant G_u(0)$. Then there exists a unique $\gamma_0 \geqslant 0$ such that*

$$G_u(\gamma_0) = \gamma_0 + \lambda.$$

*Proof.* Suppose without loss of generality that $\lambda > G_u(0)$. By definition

$$G_u(0) - \lambda < 0.$$

But by equation (13), for fixed $u$ and sufficiently large $\gamma$,

$$G_u(\gamma) > \gamma + \lambda$$

and by continuity, there exists a $\gamma_0 > 0$ such that $G_u(\gamma_0) = \gamma_0 + \lambda$.

The uniqueness of the root $\gamma_0$ follows from considering the derivative $\partial G_u(\gamma)/\partial\gamma$ for $\gamma > 0$. The partial derivative $\partial\zeta_1(u;\gamma)/\partial\gamma = -u\,\zeta_1(u+1;\gamma)$. Thus

$$\begin{aligned} \frac{\partial G_u(\gamma)}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\frac{\zeta_1(u-1,\gamma)}{\zeta_1(u;\gamma)} = \frac{-(u-1)\,\zeta_1(u;\gamma)\zeta_1(u;\gamma) + u\,\zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma)}{\zeta_1(u;\gamma)^2} \\ &= u\left(\frac{\zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma)}{\zeta_1(u;\gamma)^2} - 1\right) + 1 > 1 \end{aligned}$$

if $\zeta_1(u;\gamma)^2 < \zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma)$. Now by the Cauchy-Schwarz inequality,

$$\begin{aligned} \zeta_1(u;\gamma) &= \sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{(u-1)/2}} \frac{1}{(k+\gamma)^{(u+1)/2}} \\ &\leqslant \left(\sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{u-1}}\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{u+1}}\right)^{1/2} = (\zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma))^{1/2} \end{aligned}$$

and $\zeta_1(u;\gamma)^2 < \zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma)$, since the inequality is readily seen to be strict. Hence $\partial G_u(\gamma)/\partial\gamma > 1$, so that the graph of $G_u$ crosses that of $\gamma \mapsto \gamma + \lambda$ no more than once. $\square$

**Theorem 6.** *Let $\Gamma_0$ be a curve of finite length with convex hull $K_0$. Then for $0 < s < 1$,*

$$h_0^{(s)} \leqslant \frac{1}{1-s} \log \left( \frac{s}{\beta(1-s)} \right) + \log \zeta_1 \left( \frac{1}{1-s}; \gamma \right), \tag{14}$$

*where $\beta = \beta(s, 2|\Gamma_0|/|\partial K_0|)$, $\gamma = \gamma(s, 2|\Gamma_0|/|\partial K_0|)$. Moreover, provided $1/2 < s < 1$ and*

$$\frac{2|\Gamma_0|}{|\partial K_0|} > \frac{\zeta(s/(1-s))}{\zeta(1/(1-s))} > 1, \tag{15}$$

*there exists a unique $\gamma_0$ such that*

$$h_0^{(s)} \leqslant \frac{1}{1-s} \log \left( \frac{2|\Gamma_0|}{|\partial K_0|} + \gamma_0 \right) + \log \zeta_1 \left( \frac{1}{1-s}; \gamma_0 \right). \tag{16}$$

*Proof.* As with Shannon entropy, we use Lagrange multipliers under the same two constraints (2) and (3) to find a bound for $h_0^{(s)}$. Consider

$$V = \frac{1}{1-s} \log \sum_{n=1}^{\infty} \widehat{p}_n^s - \alpha \sum_{n=1}^{\infty} \widehat{p}_n - \beta \sum_{n=1}^{\infty} n \widehat{p}_n.$$

Then for each $n = 1, 2, \ldots,$

$$\frac{\partial V}{\partial \widehat{p}_n} = \frac{s \widehat{p}_n^{s-1}}{(1-s) \sum_{n=1}^{\infty} \widehat{p}_n^s} - \alpha - \beta n.$$

Hence $\partial V / \partial \widehat{p}_n = 0$ implies

$$\widehat{p}_n^{s-1} = \beta \left( \frac{1-s}{s} \right) \left( \sum_{n=1}^{\infty} \widehat{p}_n^s \right) (\gamma + n),$$

where $\gamma = \alpha/\beta$. Hence the 'maximal' distribution is given by

$$\widehat{p}_n = \left( \frac{s}{\beta(1-s)} \right)^{1/(1-s)} \left( \sum_{n=1}^{\infty} \widehat{p}_n^s \right)^{-1/(1-s)} \left( \frac{1}{\gamma + n} \right)^{1/(1-s)}. \tag{17}$$

We compute $\sum_{n=1}^{\infty} \widehat{p}_n^s$ in two ways. First using (2),

$$1 \;=\; \sum_{n=1}^{\infty} \widehat{p}_n = \left( \frac{s}{\beta(1-s)} \right)^{1/(1-s)} \left( \sum_{n=1}^{\infty} \widehat{p}_n^s \right)^{-1/(1-s)} \sum_{n=1}^{\infty} \left( \frac{1}{\gamma + n} \right)^{1/(1-s)},$$

whence

$$\left( \sum_{n=1}^{\infty} \widehat{p}_n^s \right)^{1/(1-s)} = \left( \frac{s}{\beta(1-s)} \right)^{1/(1-s)} \zeta_1 \left( \frac{1}{1-s}; \gamma \right).$$

i.e.,

$$\sum_{n=1}^{\infty} \widehat{p}_n^s = \left(\frac{s}{\beta(1-s)}\right) \zeta_1 \left(\frac{1}{1-s};\gamma\right)^{1-s}, \tag{18}$$

where the right hand side converges for $s \in (0,1)$ and (14) follows.

Secondly, substitute (18) in (17) to get

$$\widehat{p}_n = \zeta_1 \left(\frac{1}{1-s};\gamma\right)^{-1} \left(\frac{1}{\gamma+n}\right)^{1/(1-s)},$$

so that, by contrast with the Shannon entropy case (10), the distribution $\widehat{p}_n$ obeys an inverse power law, with factor the reciprocal of a modified Hurwitz zeta function. Hence

$$\sum_{n=1}^{\infty} \widehat{p}_n^s = \zeta_1 \left(\frac{1}{1-s};\gamma\right)^{-s} \sum_{n=1}^{\infty} \left(\frac{1}{\gamma+n}\right)^{s/(1-s)} = \frac{\zeta_1\left(\frac{s}{1-s};\gamma\right)}{\zeta_1\left(\frac{1}{1-s};\gamma\right)^s}, \tag{19}$$

which converges when $1/2 < s < 1$ but diverges for $s \leqslant 1/2$.

To determine $\gamma$, use the constraint $2|\Gamma_0|/|\partial K_0| = \lambda$ given by (3):

$$\begin{aligned}
\frac{2|\Gamma_0|}{|\partial K_0|} = \lambda &= \sum_{n=1}^{\infty} n\widehat{p}_n = \zeta_1\left(\frac{1}{1-s};\gamma\right)^{-1} \sum_{n=1}^{\infty} n\left(\frac{1}{\gamma+n}\right)^{1/(1-s)} \\
&= \zeta_1\left(\frac{1}{1-s};\gamma\right)^{-1} \sum_{n=1}^{\infty}\left[\frac{\gamma+n}{(\gamma+n)^{1/(1-s)}} - \frac{\gamma}{(\gamma+n)^{1/(1-s)}}\right] \\
&= \zeta_1\left(\frac{1}{1-s};\gamma\right)^{-1}\left(\zeta_1\left(\frac{s}{1-s};\gamma\right) - \gamma\zeta_1\left(\frac{1}{1-s};\gamma\right)\right) \\
&= \frac{\zeta_1\left(\frac{s}{1-s};\gamma\right)}{\zeta_1\left(\frac{1}{1-s};\gamma\right)} - \gamma.
\end{aligned} \tag{20}$$

By (15) and Lemma 5 with $u = 1/(1-s)$, $1/2 < s < 1$, there is a $\gamma_0 = \gamma_0(s,\lambda)$ satisfying (20), i.e., such that

$$G_u(\gamma_0) = F_s(\gamma_0) = \frac{\zeta_1\left(\frac{s}{1-s};\gamma_0\right)}{\zeta_1\left(\frac{1}{1-s};\gamma_0\right)} = \lambda + \gamma_0.$$

Now by (19),

$$h^{(s)} = \frac{1}{1-s}\log\left(\sum_{n=1}^{\infty} p_n^s\right) \leqslant \frac{1}{1-s}\log\left(\sum_{n=1}^{\infty}\widehat{p}_n^s\right) = \widehat{h}^{(s)}$$

and for $u = 1/(1-s)$,

$$\begin{aligned}
\widehat{h}^{(s)} &= \frac{1}{1-s}\log\left[\frac{\zeta_1\left(\frac{s}{1-s};\gamma_0\right)}{\zeta_1\left(\frac{1}{1-s};\gamma_0\right)}\right] + \log\zeta_1\left(\frac{1}{1-s};\gamma_0\right) \\
&= \frac{1}{1-s}\log(\lambda+\gamma_0) + \log\zeta_1\left(\frac{1}{1-s};\gamma_0\right),
\end{aligned}$$

giving (16).

The corresponding parameters $\beta_0$ and $\alpha_0 = \beta_0\,\gamma_0$, are fixed by the equation

$$\frac{s}{\beta_0(1-s)} = \frac{\zeta_1\left(\frac{s}{1-s};\gamma_0\right)}{\zeta_1\left(\frac{1}{1-s};\gamma_0\right)} = F_s(\gamma_0) = \lambda + \gamma_0\,,$$

obtained by dividing (18) by (19) and using (20). This equation implies that as $s \to 1$ from below, $\beta_0 \to \infty$ and therefore $\alpha_0 \to \infty$, while $F_s(\gamma_0) \to \infty$ as $s \to 1/2$ from above, which implies that $\beta_0$, $\alpha_0 \to 0$. □

*Remark* 7. The bound for the Rényi entropy $h^{(s)}$ is finite for $1/2 < s < 1$. The equation $F_s(\gamma_0) = \gamma_0 + \lambda$ only holds if $\lambda = 2|\Gamma_0|/|\partial K_0| \geqslant F_s(0) = \zeta(s/(s-1))/\zeta(1/(s-1)) > 1$. This and the values $\alpha, \beta$ do not have an obvious interpretation. Nor does the bound for the Shannon entropy appear to follow from the limit of the Rényi entropy case as $s \to 1$.

# 4 Changing the viewpoint: the parameter $t$

In this section, we discuss the Shannon-Rényi entropy $h^{(s)}$ of the curve $\Gamma_0$ relative to a compact subset $K$ containing $K_0$; this involves a parameter $t$. Consider the curve $\Gamma_0$, its convex hull $K_0$ and a bounded convex set $K \supseteq K_0$. The set $K$ can be thought of as a variable set that "increases" and within which the curve can increase its entropy.

## 4.1 The parameter $t$

Let $t \geqslant 1$ be the ratio of the lengths of the boundaries of $K$ and $K_0$, i.e., let

$$t = \frac{|\partial K|}{|\partial K_0|} \in [1, \infty).$$

In this construction, the length $|\partial K|$ of the boundary of the convex set $K$ increases.

**Lemma 8.** *The probability $p_n(\Gamma_0, K)$ that a straight line $D$ meeting $K$ intersects $\Gamma_0$ in exactly $n$ points is given by*

$$p_n(\Gamma_0, K) = p_n(K) = \begin{cases} \frac{1}{t}\,p_n(K_0), & \text{if } n \geqslant 1; \\ 1 - \frac{1}{t}, & \text{if } n = 0. \end{cases} \tag{21}$$

*Proof.* If $n \geqslant 1$,

$$\begin{aligned} p_n(\Gamma_0, K) &= \frac{\mu\{D\colon \operatorname{card}(D\cap\Gamma_0) = n\}}{\mu\{D\colon D\cap K \neq \emptyset\}} \\ &= \frac{\mu\{D\colon D\cap K_0 \neq \emptyset\}}{\mu\{D\colon D\cap K \neq \emptyset\}}\,\frac{\mu\{D\colon \operatorname{card}(D\cap\Gamma_0) = n\}}{\mu\{D\colon D\cap K_0 \neq \emptyset\}} \\ &= \frac{|\partial K_0|}{|\partial K|}\,\frac{\mu\{D\colon \operatorname{card}(D\cap\Gamma_0) = n\}}{\mu\{D\colon D\cap K_0 \neq \emptyset\}} = \frac{1}{t}\,p_n(K_0, \Gamma_0). \end{aligned}$$

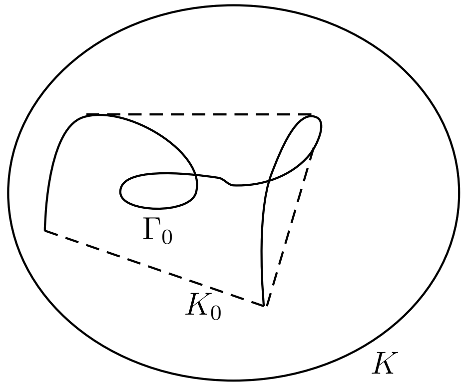


Figure 2: The curve $\Gamma_0$ inside a bounded convex set $K$.

If $n = 0$,

$$p_0(\Gamma_0, K) = 1 - \frac{|\partial K_0|}{|\partial K|} = 1 - \frac{1}{t}\,,$$

as in (1) in §2.

$\square$

Let $h_0^{(s)} := h^{(s)}(\Gamma_0, K_0)$, $0 < s \leqslant 1$, be the Shannon-Rényi entropy of $\Gamma_0$ with respect to its convex hull $K_0$. In the preceding section §3, $h_0^{(s)}$ is written by $h$ or $h^{(1)}$ when $s = 1$ (Shannon entropy) and $h^{(s)}$ when $0 < s < 1$. We begin with the case $s = 1$.

## 4.2 The Shannon entropy case

The Shannon entropy $h(\Gamma_0, K) = h$ of $\Gamma_0$ relative to $K$ is now determined in terms of the Shannon entropy $h(\Gamma_0, K_0) = h_0$ of $\Gamma_0$ relative to the convex hull $K_0$ of $\Gamma_0$ ($h_0$ is finite by Theorem 1).

**Theorem 9.** *Let $h_0$ be the Shannon entropy of $\Gamma_0$ with respect to $K_0$. The entropy $h(\Gamma_0, K)$ of $\Gamma_0$ with respect to $K$ is given by*

$$h(\Gamma_0, K) = \frac{h_0}{t} + \log t - \frac{t-1}{t}\log(t-1) = \frac{h_0}{t} + (-1 + \frac{1}{t})\log(1 - \frac{1}{t}) + \frac{\log t}{t} \to 0$$

*as $t \to \infty$. Moreover $h(\Gamma_0, K)$ is maximal when $t = t_1 = 1 + e^{-h_0}$, with value $h_1 = \log(1 + e^{h_0}) > h_0$.*

*Proof.* Let $h = h(K)$ be the entropy of $\Gamma_0$ relative to $K$ and let $h_0 = h(K_0)$ be the entropy

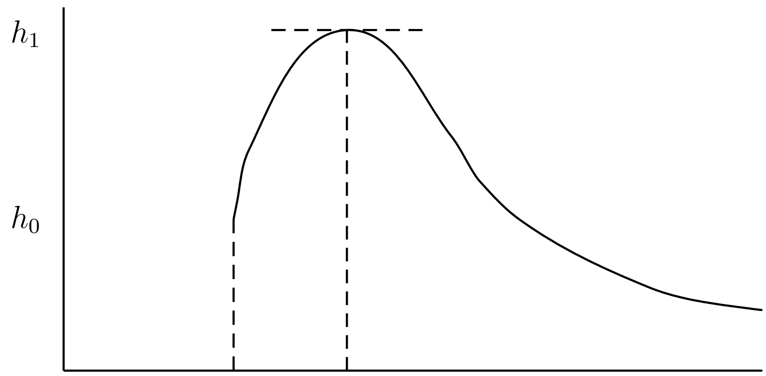


Figure 3: The graph of $h$ with maximum at $t_1$

of $\Gamma_0$ relative to $K_0$. Then by (4) and Lemma 8 (and writing $p_n(K) = p_n(\Gamma_0, K)$),

$$\begin{aligned} h &= \sum_{n=0}^{\infty} p_n(K) \log \frac{1}{p_n(K)} \\ &= \left(1 - \frac{1}{t}\right) \log \left(1 - \frac{1}{t}\right)^{-1} + \sum_{n=1}^{\infty} \frac{1}{t} p_n(K_0) \log \frac{t}{p_n(K_0)} \\ &= \left(1 - \frac{1}{t}\right) \log \frac{t}{t-1} + \frac{1}{t} \sum_{n=1}^{\infty} p_n(K_0) \log t + \frac{1}{t} \sum_{n=1}^{\infty} p_n(K_0) \log \frac{1}{p_n(K_0)} \\ &= \frac{h_0}{t} + \log t - \frac{t-1}{t} \log(t-1). \end{aligned}$$

We wish to find the maximum value of $h$ when $t \in (1, \infty)$. The derivative

$$\frac{dh}{dt} = -\frac{1}{t^2}(h_0 + \log(t-1)) = 0$$

for $t_1 = |\partial K_1|/|\partial K_0| = 1 + e^{-h_0}$, with entropy $h$ a maximum at $t = t_1$ and corresponding value

$$h_1 := \log(1 + e^{h_0}) > h_0.$$

Here $K_1$ is any convex set whose boundary has length $t_1|\partial K_0|$ and which contains $K_0$. $\square$

## 4.3 The Rényi entropy case

For convenience $h^{(s)}(\Gamma_0, K)$, the Rényi entropy with respect to the convex set $K \supset \Gamma_0$ will be written $h^{(s)}$, $p_n(\Gamma_0, K)$ will be written $p_n(K)$ and $h^{(s)}(\Gamma_0, K_0)$, the Rényi entropy with respect to the convex hull $K_0$ of $\Gamma_0$ will be written $h_0^{(s)}$. The dependence of $h^{(s)}$ on the parameter $t$ will be indicated where helpful; thus when $t = 1$, $h^{(s)}(t) = h^{(s)}(1) = h_0^{(s)}$. Note that by Theorem 6, the quantity $h_0^{(s)}$ is finite for $s \in (0, 1)$.

**Theorem 10.** *The Rényi entropy $h^{(s)}(\Gamma_0, K)$, $0 < s < 1$, of $\Gamma_0$ with respect to $K$ is given by*

$$\begin{aligned} h^{(s)}(\Gamma_0, K) &= \frac{1}{1-s} \log \left( \left(1 - \frac{1}{t}\right)^s + \sum_{n=1}^{\infty} \frac{1}{t^s} p_n^s(K_0) \right) \\ &= \frac{1}{1-s} \log \left(1 + (t-1)^{-s} e^{(1-s)h_0^{(s)}}\right) + \frac{s}{1-s} \log \left(1 - \frac{1}{t}\right), \end{aligned}$$

*so that $h^{(s)} \to 0$ as $t \to \infty$. Moreover $h^{(s)}(\Gamma_0, K)$ is maximal for*

$$t_1 = 1 + \left( \sum_{n=1}^{\infty} p_n^s(K_0) \right)^{-1/(1-s)} = 1 + e^{-h_0^{(s)}},$$

*with value $h^{(s)}(t_1) = \log(1 + e^{h_0^{(s)}})$.*

*Proof.* Write $h^{(s)}(\Gamma_0, K) = h^{(s)}$. It follows from (21) that

$$\begin{aligned} h^{(s)} &= h^{(s)}(t) = \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s(K) = \frac{1}{1-s} \log \left( \left(1 - \frac{1}{t}\right)^s + \sum_{n=1}^{\infty} \frac{1}{t^s} p_n^s(K_0) \right) \\ &= \frac{s}{1-s} \log \left(1 - \frac{1}{t}\right) + \frac{1}{1-s} \log \left(1 + \frac{e^{(1-s)h_0^{(s)}}}{(t-1)^s}\right), \end{aligned}$$

as claimed.

We now ask which $t \geqslant 1$ maximizes $h^{(s)} = h^{(s)}(t)$. The derivative

$$\frac{dh^{(s)}}{dt} = \frac{s}{(1-s)\left(\sum_{n=0}^{\infty} p_n^s(K_0)\right)} \left( (1 - \frac{1}{t})^{s-1} \frac{1}{t^2} - \frac{1}{t^{s+1}} \sum_{n=1}^{\infty} p_n^s(K_0) \right) = 0$$

iff

$$(t-1)^{s-1} = \sum_{n=1}^{\infty} p_n^s(K_0) = e^{(1-s)\, h_0^{(s)}},$$

i.e., iff

$$t = 1 + \left( \sum_{n=1}^{\infty} p_n^s(K_0) \right)^{-1/(1-s)} = 1 + e^{-h_0^{(s)}}. \tag{22}$$

The entropy $h^{(s)}(t)$ is maximal at the value $t_1$ given by (22), i.e.,

$$\begin{aligned} h^{(s)}(t_1) &= \frac{1}{1-s} \log \sum_{n=1}^{\infty} p_n^s(K_0) + \log \left( 1 + \left( \sum_{n=1}^{\infty} p_n^s(K_0) \right)^{-\frac{1}{(1-s)}} \right) \\ &= h_0^{(s)} + \log\left(1 + e^{-h_0^{(s)}}\right), \end{aligned} \tag{23}$$

whence $h^{(s)}(t_1) = \log\left(1 + e^{h_0^{(s)}}\right)$. The first term on the right hand side of (23) is the $s$-entropy $h_0^{(s)} = h^{(s)}(1)$ of $\Gamma_0$ relative to its convex hull $K_0$, so that $h^{(s)}(t_1) > h^{(s)}(1)$. □

*Remark* 11. For a certain 'dilution' corresponding to $|\partial K|/|\partial K_0| = t_1$, the Rényi $s$-entropy attains a maximal value and then decreases to 0. Suppose now $s \to 1$. Then it is easily seen that $t_1 \to 1 + e^{-h_0}$ and $h^{(s)}(t_1) \to \log(1 + e^{h_0})$, which is consistent with the results above in §4.2.

If we agree to identify entropy and complexity, we see that the complexity of a curve depends on the point of observation. Seen from a certain distance, the curve increases its complexity, while seen from infinity the curves reduces to a point with entropy 0.

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