Journal of Integer Sequences, Vol. 16 (2013),
Article 13.6.3

An Alternating Sum Involving the Reciprocal of Certain Multiplicative Functions

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To the memory of Patrick Sargos

Abstract

We establish an asymptotic formula for an alternating sum of the reciprocal of a class of multiplicative functions. The proof is straightforward and uses classical convolution techniques. Numerous examples are given.

1 Introduction and main results

In 1900, E. Landau [2] proved that

$$\sum_{n\le x}\frac{1}{\varphi(n)}=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\left(\log x+\gamma-\sum_{p}\frac{\log p}{p^2-p+1}\right)+O\left(\frac{\log x}{x}\right)$$

where φ is the classic Euler totient function and γ is Euler's constant, but it seems that in the usual literature there is not any similar result for the alternating sum

$$\sum_{n\le x}\frac{(-1)^n}{\varphi(n)}.$$

It is fairly easy to show that there exists an absolute constant $c>0$ such that this sum is $> c\log x + O(1)$, and therefore the question of an asymptotic formula arises naturally. It should be mentioned that Landau's result could eventually provide such an asymptotic formula via the simple identity

$$\sum_{n\le x}\frac{(-1)^n}{\varphi(n)} = \sum_{n\le x}\frac{1}{\varphi(n)} - 2\sum_{\substack{n\le x\\(n,2)=1}}\frac{1}{\varphi(n)}$$

and the use of known estimates as in [3, Theorem 1], but this method does not seem to be easily generalized to a larger class of multiplicative functions than that of [3]. In this article, we will follow a slightly different approach working with the set of non-zero complex-valued multiplicative functions f satisfying the following assumptions: there exist constants $\lambda_1>0$ and $0\le\lambda_2<2$ such that, for each prime power p^α, we have

$$p^2\left|\frac{1}{f(p)}-\frac{1}{p}\right|\le\lambda_1 \quad\text{and}\quad p^{2\alpha-1}\left|\frac{1}{f(p^\alpha)}-\frac{1}{pf(p^{\alpha-1})}\right|\le\lambda_1\lambda_2^{\alpha-1}\quad(\alpha\ge2). \tag{1}$$

It will be convenient to set

$$\lambda := \frac{\lambda_1(2+\lambda_2)}{2-\lambda_2}. \tag{2}$$

For any prime number p, define

$$S_f(p) := \sum_{\alpha=1}^{\infty}\frac{1}{f(p^\alpha)}\quad\text{and}\quad s_f(p) := \sum_{\alpha=1}^{\infty}\frac{\alpha}{f(p^\alpha)} \tag{3}$$

and set

$$\mathcal{P}_f := \prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{1}{f(p^\alpha)}\right) \tag{4}$$

where the product is absolutely convergent.

We are now in a position to state our main result.

Theorem 1. *Let f be a non-zero multiplicative function satisfying (1). Then, for $x\ge e$, we have*

$$\sum_{n\le x}\frac{(-1)^n}{f(n)} = \frac{S_f(2)-1}{S_f(2)+1}\mathcal{P}_f\log x + C_f + O\left(\frac{(\log x)^{\lambda+1}}{x}\right)$$

where

$$C_f := \frac{S_f(2)-1}{S_f(2)+1}\,\mathcal{P}_f\left(\gamma+\sum_p \log p\left(\frac{1}{p-1}-\frac{s_f(p)}{1+S_f(p)}\right)-\frac{2s_f(2)\log 2}{S_f(2)^2-1}\right)$$

and $S_f(p)$*,* $s_f(p)$ *and* $\mathcal{P}_f$ *are given in* (3) *and* (4)*.*

Corollary 2. *Let* f *be a non-zero multiplicative function satisfying* (1) *and assume that, for any prime* p *and any integer* $\alpha \geq 1$*,* $f(p^\alpha) = p^{\alpha-1} f(p)$*. Then, for* $x \geq e$*, we have*

$$\sum_{n\leq x}\frac{(-1)^n}{f(n)} = \frac{2-f(2)}{2+f(2)}\prod_p\left(1-\frac{1}{p}+\frac{1}{f(p)}\right)\left(\log x+\gamma+\sum_p\frac{(f(p)-p)\log p}{(p-1)f(p)+p}-\frac{8f(2)\log 2}{4-f(2)^2}\right) + O\left(\frac{(\log x)^{\lambda_1+1}}{x}\right).$$

Remark 3. If $S_f(2)=1$ in Theorem 1, i.e. $f(2)=2$ in Corollary 2, then the constant C_f reduces to

$$C_f = -\mathcal{P}_f\, s_f(2)\,\log\sqrt{2}.$$

Corollary 4. *For* $x \geq e$*, the following estimates hold.*

(i)

$$\sum_{n\leq x}\frac{(-1)^n}{\varphi(n)} = \frac{\zeta(2)\zeta(3)}{3\zeta(6)}\left(\log x+\gamma-\sum_p\frac{\log p}{p^2-p+1}-\frac{8\log 2}{3}\right)+O\left(\frac{(\log x)^3}{x}\right).$$

(ii) *Let* χ *be a non-principal Dirichlet character modulo* $q>2$ *and* $\varphi(n,\chi)$ *be the twisted Euler function attached to* χ *given in* (6)*. Then*

$$\sum_{n\leq x}\frac{(-1)^n}{\varphi(n,\chi)} = \frac{\chi(2)L(1,\chi)}{4-\chi(2)}\prod_p\left(1-\frac{\chi(p)}{p}+\frac{\chi(p)}{p^2}\right)\left(\log x+\gamma-\sum_p\frac{\chi(p)\log p}{p^2-p\chi(p)+\chi(p)}-\frac{8(2-\chi(2))\log 2}{\chi(2)(4-\chi(2))}\right)+O\left(\frac{(\log x)^3}{x}\right).$$

(iii) *Let* Ψ *be the Dedekind arithmetic function defined in* (5)*. Then*

$$\sum_{n\leq x}\frac{(-1)^n}{\Psi(n)} = -\frac{1}{5}\prod_p\left(1-\frac{1}{p(p+1)}\right)\left(\log x+\gamma+\sum_p\frac{\log p}{p^2+p-1}+\frac{24\log 2}{5}\right) + O\left(\frac{(\log x)^2}{x}\right).$$

(iv) *Let* $\gamma(n) := \prod_{p|n} p$ *be squarefree kernel of* n. *Then*

$$\sum_{n\le x}\frac{(-1)^n\gamma(n)}{\varphi(n)^2} = \frac{5}{11}\prod_p\left(1+\frac{p^2+p-1}{p(p+1)(p-1)^2}\right)\Bigg(\log x+\gamma - \sum_p\frac{(p^3-2p^2-p+1)\log p}{(p^2-1)(p^4-p^3+2p-1)}-\frac{64\log 2}{55}\Bigg)+O\left(\frac{(\log x)^7}{x}\right).$$

(v) *Let* σ *be the sum-of-divisors function. Then*

$$\sum_{n\le x}\frac{(-1)^n}{\sigma(n)} = \frac{S_\sigma(2)-1}{S_\sigma(2)+1}\prod_p\left(1-\frac{1}{p}+\frac{(p-1)^2}{p}\sum_{\alpha=2}^\infty\frac{1}{p^\alpha-1}\right)\Bigg(\log x+\gamma + \sum_p\log p\left(\frac{1}{p-1}-\frac{s_\sigma(p)}{1+S_\sigma(p)}\right)+\kappa\log 2\Bigg)+O\left(\frac{(\log x)^4}{x}\right)$$

where $\kappa \doteq 3.6$. *Note that the leading constant is* $\doteq -0.16\,468$.

(vi) *Let* σ^* *be the sum-of-unitary-divisors function. Then*

$$\sum_{n\le x}\frac{(-1)^n}{\sigma^*(n)} = \frac{S_{\sigma^*}(2)-1}{S_{\sigma^*}(2)+1}\prod_p\left(1-\frac{1}{p}+\frac{p-1}{p}\sum_{\alpha=1}^\infty\frac{1}{p^\alpha+1}\right)\Bigg(\log x+\gamma + \sum_p\log p\left(\frac{1}{p-1}-\frac{s_{\sigma^*}(p)}{1+S_{\sigma^*}(p)}\right)+\nu\log 2\Bigg)+O\left(\frac{(\log x)^4}{x}\right)$$

where $\nu \doteq 8.04$. *Note that the leading constant is* $\doteq -0.10\,259$.

2 Notation

The celebrated Möbius function is as always denoted by μ, $\mathrm{Id}(n) = n$, φ is the Euler totient function, Ψ is the Dedekind function defined by

$$\Psi(n) = n\prod_{p|n}\left(1+\frac{1}{p}\right) \tag{5}$$

and, for each fixed non-principal Dirichlet character χ modulo $q > 2$, the χ-twisted Euler function $\varphi(n,\chi)$ recently introduced in [1] is defined by

$$\varphi(n,\chi) = n\prod_{p|n}\left(1-\frac{\chi(p)}{p}\right). \tag{6}$$

For any two arithmetic functions u and v, $u \star v$ is the usual Dirichlet convolution product of u and v defined by

$$(u \star v)(n) = \sum_{d|n} u(d)v(n/d).$$

The *Eratosthenes transform* of u is the arithmetic function $u \star \mu$. Finally, f is a non-zero multiplicative function satisfying (1) and g is the Eratosthenes transform of $\operatorname{Id} f^{-1}$. Note that the assumption (1) can be written as

$$p|g(p)| \leq \lambda_1 \quad \text{and} \quad p^{\alpha-1} \left| g\left(p^\alpha\right) \right| \leq \lambda_1 \lambda_2^{\alpha-1} \quad (\alpha \geq 2)\,. \tag{7}$$

3 Tools

Lemma 5. *Let $\delta > 0$. The Dirichlet series of the arithmetic function $g(n)$ is absolutely convergent in the half-plane $\sigma > \delta$.*

Proof. Let $s = \delta + it \in \mathbb{C}$ with $\delta > 0$. The function g is multiplicative and using (7) we get for all $z \geq e$

$$\begin{aligned}
\sum_{p \leq z} \sum_{\alpha=1}^{\infty} \left| \frac{g\left(p^\alpha\right)}{p^{s\alpha}} \right| &= \sum_{p \leq z} \left(\frac{|g(p)|}{p^\delta} + \sum_{\alpha=2}^{\infty} \left| \frac{g\left(p^\alpha\right)}{p^{s\alpha}} \right| \right) \\
&\leq \lambda_1 \left(\sum_{p \leq z} \frac{1}{p^{\delta+1}} + \sum_{p \leq z} \frac{1}{p^\delta} \sum_{\alpha=2}^{\infty} \left(\frac{\lambda_2}{p^{\delta+1}} \right)^{\alpha-1} \right) \\
&= \lambda_1 \sum_{p \leq z} \left(\frac{1}{p^{\delta+1}} + \frac{\lambda_2}{p^\delta (p^{\delta+1} - \lambda_2)} \right) \\
&\leq \lambda_1 \sum_{p \leq z} \left(\frac{1}{p^{\delta+1}} + \frac{2\lambda_2}{p^\delta (2 - \lambda_2)} \frac{1}{p^{\delta+1}} \right) \\
&\leq \lambda \sum_{p \leq z} \frac{1}{p^{\delta+1}}
\end{aligned}$$

where we used the inequality

$$\frac{1}{p^\theta - \lambda_2} \leq \frac{2}{2 - \lambda_2} \frac{1}{p^\theta} \quad (\theta \geq 1)\,.$$

This implies the asserted result. □

Lemma 6. *For all real numbers $z \geq e$ and $a \in \{0, 1\}$*

$$\text{(i)} \quad : \quad \sum_{n \leq z} |g(n)| \ll (\log z)^\lambda\,.$$

$$\text{(ii)} \quad : \quad \sum_{n > z} \frac{|g(n)|(\log n)^a}{n} \ll z^{-1} (\log z)^{\lambda + a}\,.$$

Proof.

(i) As in the proof of Lemma 5, we get for all $z \geq e$

$$\begin{aligned}\sum_{n\leq z}|g(n)| &\leq \exp\left\{\sum_{p\leq z}|g(p)|+\sum_{p\leq z}\sum_{\alpha=2}^{\infty}|g(p^{\alpha})|\right\}\\ &\leq \exp\left\{\lambda_1\left(\sum_{p\leq z}\frac{1}{p}+\sum_{p\leq z}\sum_{\alpha=2}^{\infty}\left(\frac{\lambda_2}{p}\right)^{\alpha-1}\right)\right\}\\ &\leq \exp\left\{\lambda_1\left(\sum_{p\leq z}\frac{1}{p}+\frac{2\lambda_2}{2-\lambda_2}\sum_{p\leq z}\frac{1}{p}\right)\right\}\\ &= \exp\left(\lambda\sum_{p\leq z}\frac{1}{p}\right)\ll(\log z)^{\lambda}\end{aligned}$$

as asserted.

(ii) Follows from Lemma 5, (i) and partial summation. We leave the details to the reader.

The proof is complete. □

Lemma 7. *For any real number $x \geq 1$ and any integer $1 \leq d \leq 2x$, we have*

$$\sum_{\substack{n\leq x\\ d|2n}}\frac{1}{n}=\frac{\rho(d)}{d}\left(\log\frac{\rho(d)x}{d}+\gamma\right)+O\left(\frac{1}{x}\right)$$

where

$$\rho(d):=\begin{cases}2, & \textit{if } d \textit{ is even;}\\ 1, & \textit{if } d \textit{ is odd.}\end{cases}\tag{8}$$

Proof. Let $S(x,d)$ be the sum of the left-hand side. If $x < d \leq 2x$, then

$$S(x,d)=\frac{2}{d}<\frac{2}{x}.$$

If $d \leq x$ is odd, then by Gauss's theorem we have

$$S(x,d)=\sum_{\substack{n\leq x\\ d|n}}\frac{1}{n}=\frac{1}{d}\sum_{1\leq k\leq x/d}\frac{1}{k}=\frac{1}{d}\left(\log\frac{x}{d}+\gamma+O\left(\frac{d}{x}\right)\right)$$

as required.

Now assume that d is even. If $\frac{2x}{3} < d \leq x$, then $\frac{x}{d} - \frac{1}{2} < 1$ and therefore

$$\begin{aligned} S(x,d) &= \sum_{\substack{n\leq x \\ d\mid n}} \frac{1}{n} + \sum_{\substack{n\leq x \\ n\equiv d/2 \pmod d}} \frac{1}{n} \\ &= \frac{1}{d}\sum_{1\leq k\leq x/d}\frac{1}{k} + \frac{1}{d}\sum_{0\leq k\leq x/d-1/2}\frac{1}{k+1/2} \\ &= \frac{1}{d}\left(\log\frac{x}{d} + \gamma + O\left(\frac{d}{x}\right)\right) + \frac{2}{d} \\ &= \frac{1}{d}\left(\log\frac{x}{d} + \gamma\right) + O\left(\frac{1}{x}\right) \end{aligned}$$

since the assumption $d > \frac{2x}{3}$ implies $\frac{2}{d} < \frac{3}{x}$. On the other hand, since $1 \leq \frac{x}{d} < \frac{3}{2}$, we have

$$\left|\frac{2}{d}\log\frac{2x}{d} + \frac{\gamma}{d} - \frac{1}{d}\log\frac{x}{d}\right| \leq \frac{1}{d}\left(\log\frac{4x}{d} + \gamma\right) < \frac{\log 6 + \gamma}{d} < \frac{3\left(\log 6 + \gamma\right)}{2x}$$

and thus in this case we also get

$$S(x,d) = \frac{2}{d}\left(\log\frac{2x}{d} + \gamma\right) + O\left(\frac{1}{x}\right).$$

If $1 \leq d \leq \frac{2x}{3}$, since

$$\sum_{1\leq k\leq x/d-1/2}\frac{1}{k+1/2} = \log\frac{x}{d} + \gamma - 2 + \log 4 + O\left(\frac{d}{x}\right)$$

where we used the fact that $d \leq \frac{2x}{3}$ implies $\frac{x}{d} - \frac{1}{2} \geq \frac{2x}{3d}$, then

$$\begin{aligned} S(x,d) &= \frac{1}{d}\left(\log\frac{x}{d} + \gamma + O\left(\frac{d}{x}\right)\right) + \frac{2}{d} \\ &\quad + \frac{1}{d}\left(\log\frac{x}{d} + \gamma - 2 + \log 4 + O\left(\frac{d}{x}\right)\right) \\ &= \frac{2}{d}\left(\log\frac{2x}{d} + \gamma\right) + O\left(\frac{1}{x}\right) \end{aligned}$$

completing the proof. □

4 Sums of reciprocals

Lemma 8. *Let f satisfying the conditions* (1). *Then*

$$\sum_{n\leq x}\frac{1}{f(n)} = \log x \sum_{d=1}^{\infty}\frac{g(d)}{d} + K_f + O\left(\frac{(\log x)^{\lambda+1}}{x}\right)$$

where

$$K_f := \sum_{d=1}^{\infty} \frac{g(d)}{d} (\gamma - \log d) \tag{9}$$

and

$$\sum_{\substack{n \le 2x \\ n \,\text{even}}} \frac{1}{f(n)} = \frac{\log x}{2} \sum_{d=1}^{\infty} \frac{g(d)\rho(d)}{d} + L_f + O\left(\frac{(\log x)^{\lambda+1}}{x}\right)$$

where

$$L_f := \frac{1}{2} \sum_{d=1}^{\infty} \frac{g(d)\rho(d)}{d} \left(\gamma + \log \frac{\rho(d)}{d}\right). \tag{10}$$

and where $\rho(d)$ *is given in* (8) *and* λ *is defined in* (2)*.*

Proof. The proof of the first estimate follows the classical lines. We have

$$\begin{aligned}
\sum_{n \le x} \frac{1}{f(n)} &= \sum_{n \le x} \frac{(g \star \mathbf{1})(n)}{n} = \sum_{n \le x} \frac{1}{n} \sum_{d \mid n} g(d) \\
&= \sum_{d \le x} \frac{g(d)}{d} \sum_{k \le x/d} \frac{1}{k} \\
&= \sum_{d \le x} \frac{g(d)}{d} \left(\log \frac{x}{d} + \gamma + O\left(\frac{d}{x}\right)\right) \\
&= (\log x + \gamma) \sum_{d \le x} \frac{g(d)}{d} - \sum_{d \le x} \frac{g(d) \log d}{d} + O\left(\frac{1}{x} \sum_{d \le x} |g(d)|\right)
\end{aligned}$$

and by Lemma 6 we get

$$\sum_{n \le x} \frac{1}{f(n)} = (\log x + \gamma) \sum_{d=1}^{\infty} \frac{g(d)\rho(d)}{d} - \sum_{d=1}^{\infty} \frac{g(d) \log(d)}{d} + O\left(\frac{(\log x)^{\lambda+1}}{x}\right)$$

as asserted.

The second estimate is similar, with the additional use of Lemma 7 which gives

$$\begin{aligned}
\sum_{\substack{n \le 2x \\ n \,\text{even}}} \frac{1}{f(n)} &= \sum_{n \le x} \frac{1}{f(2n)} = \sum_{n \le x} \frac{(g \star \mathbf{1})(2n)}{2n} \\
&= \frac{1}{2} \sum_{n \le x} \frac{1}{n} \sum_{d \mid 2n} g(d) = \frac{1}{2} \sum_{d \le 2x} g(d) \sum_{\substack{n \le x \\ d \mid 2n}} \frac{1}{n}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{d\le 2x} g(d)\left(\frac{\rho(d)}{d}\left(\log\frac{\rho(d)x}{d}+\gamma\right)+O\left(\frac{1}{x}\right)\right)\\
&= \frac{1}{2}(\log x+\gamma)\sum_{d\le 2x}\frac{g(d)\rho(d)}{d}+\frac{1}{2}\sum_{d\le 2x}\frac{g(d)\rho(d)}{d}\log\frac{\rho(d)}{d}\\
&\quad + O\left(\frac{1}{x}\sum_{d\le 2x}|g(d)|\right)\\
&= \frac{1}{2}(\log x+\gamma)\sum_{d=1}^{\infty}\frac{g(d)\rho(d)}{d}-\frac{1}{2}(\log x+\gamma)\sum_{d>2x}\frac{g(d)\rho(d)}{d}\\
&\quad + \frac{1}{2}\sum_{d=1}^{\infty}\frac{g(d)\rho(d)}{d}\log\frac{\rho(d)}{d}-\frac{1}{2}\sum_{d>2x}\frac{g(d)\rho(d)}{d}\log\frac{\rho(d)}{d}\\
&\quad + O\left(\frac{1}{x}\sum_{d\le 2x}|g(d)|\right)
\end{aligned}
$$

and by Lemma 6 we get

$$
\sum_{\substack{n\le 2x\\ n\,\text{even}}}\frac{1}{f(n)}=\frac{1}{2}(\log x+\gamma)\sum_{d=1}^{\infty}\frac{g(d)\rho(d)}{d}+\frac{1}{2}\sum_{d=1}^{\infty}\frac{g(d)\rho(d)}{d}\log\frac{\rho(d)}{d}+O\left(\frac{(\log x)^{\lambda+1}}{x}\right)
$$

completing the proof. □

5 Proof of Theorem 1

5.1 First step: Asymptotic formula

From Lemma 8 we get

$$
\begin{aligned}
\sum_{n\le x}\frac{(-1)^n}{f(n)} &= 2\sum_{\substack{n\le x\\ n\,\text{even}}}\frac{1}{f(n)}-\sum_{n\le x}\frac{1}{f(n)}\\
&= \log\frac{x}{2}\sum_{d=1}^{\infty}\frac{g(d)\rho(d)}{d}-\log x\sum_{d=1}^{\infty}\frac{g(d)}{d}+2L_f-K_f\\
&\quad + O\left(\frac{(\log x)^{\lambda+1}}{x}\right)
\end{aligned}
$$

where K_f et L_f are given in (9) et (10), so that setting

$$\begin{aligned}
C_f &:= 2L_f - K_f - \log 2 \sum_{d=1}^{\infty} \frac{g(d)}{d} \\
&= \sum_{d=1}^{\infty} \frac{g(d)}{d} \left((\rho(d) - 1)(\gamma - \log d) + \rho(d) \log \frac{\rho(d)}{2} \right) \\
&= \sum_{\substack{d=1 \\ d \text{ even}}}^{\infty} \frac{g(d)}{d} (\gamma - \log d) + \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \frac{g(d)\rho(d)}{d} \log \frac{\rho(d)}{2} \\
&= \sum_{d=1}^{\infty} \frac{g(2d)}{2d} (\gamma - \log 2d) - \log 2 \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \frac{g(d)}{d} \\
&= \frac{1}{2} \sum_{d=1}^{\infty} \frac{g(2d)}{d} (\gamma - \log d) - \log 2 \sum_{d=1}^{\infty} \frac{g(d)}{d}
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{n \le x} \frac{(-1)^n}{f(n)} &= \log x \sum_{d=1}^{\infty} \frac{g(d)(\rho(d) - 1)}{d} + C_f + O\left(\frac{(\log x)^{\lambda+1}}{x} \right) \\
&= \frac{\log x}{2} \sum_{d=1}^{\infty} \frac{g(2d)}{d} + C_f + O\left(\frac{(\log x)^{\lambda+1}}{x} \right)
\end{aligned}$$

completing the proof. □

5.2 Second step: Series expansions

The unique decomposition $d = 2^\alpha m$ with $\alpha \in \mathbb{Z}^+$ and $m \ge 1$ odd provides

$$\begin{aligned}
\sum_{d=1}^{\infty} \frac{g(d)}{d} &= \sum_{\alpha=0}^{\infty} \frac{g(2^\alpha)}{2^\alpha} \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{g(m)}{m} \\
&= \left(1 + \sum_{\alpha=1}^{\infty} \frac{g(2^\alpha)}{2^\alpha} \right) \prod_{p \ge 3} \left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)}{p^\alpha} \right) \\
&= \frac{1}{2} \sum_{\alpha=0}^{\infty} \frac{1}{f(2^\alpha)} \prod_{p \ge 3} \left(1 - \frac{1}{p} \right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{f(p^\alpha)} \right) \\
&= \mathcal{P}_f
\end{aligned}$$

and

$$\begin{aligned}
\sum_{d=1}^{\infty} \frac{g(2d)}{d} &= \sum_{\alpha=0}^{\infty} \frac{g\left(2^{\alpha+1}\right)}{2^\alpha} \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m)}{m} \\
&= \left(g(2) + 2 \sum_{\alpha=2}^{\infty} \frac{g\left(2^\alpha\right)}{2^\alpha} \right) \prod_{p \geq 3} \left(1 + \sum_{\alpha=1}^{\infty} \frac{g\left(p^\alpha\right)}{p^\alpha} \right) \\
&= \left(\sum_{\alpha=1}^{\infty} \frac{1}{f\left(2^\alpha\right)} - 1 \right) \prod_{p \geq 3} \left(1 - \frac{1}{p} \right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{f\left(p^\alpha\right)} \right) \\
&= \frac{2\left(S_f(2) - 1\right)}{S_f(2) + 1} \mathcal{P}_f.
\end{aligned}$$

Similarly

$$\begin{aligned}
\sum_{d=1}^{\infty} \frac{g(2d) \log d}{d} &= \sum_{\alpha=0}^{\infty} \frac{g\left(2^{\alpha+1}\right)}{2^\alpha} \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m) \log(2^\alpha m)}{m} \\
&= \sum_{\alpha=0}^{\infty} \frac{g\left(2^{\alpha+1}\right)}{2^\alpha} \left(\alpha \log 2 \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m)}{m} + \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m) \log m}{m} \right) \\
&= \log 2 \sum_{\alpha=0}^{\infty} \frac{\alpha g\left(2^{\alpha+1}\right)}{2^\alpha} \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m)}{m} + \sum_{\alpha=0}^{\infty} \frac{g\left(2^{\alpha+1}\right)}{2^\alpha} \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m) \log m}{m} \\
&= \log 2 \left(\sum_{\alpha=1}^{\infty} \frac{\alpha - 2}{f\left(2^\alpha\right)} \right) \prod_{p \geq 3} \left(1 - \frac{1}{p} \right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{f\left(p^\alpha\right)} \right) \\
&\quad + \left(\sum_{\alpha=1}^{\infty} \frac{1}{f\left(2^\alpha\right)} - 1 \right) \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m) \log m}{m} \\
&= \frac{2 \log 2 \left(s_f(2) - 2S_f(2)\right)}{S_f(2) + 1} \mathcal{P}_f + \left(S_f(2) - 1\right) \sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m) \log m}{m}.
\end{aligned}$$

Now since for $s \in \mathbb{C}$ such that $\operatorname{Re} s > \delta > 0$, we have

$$\sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m)}{m^s} = \prod_{p \geq 3} \left(1 - \frac{1}{p^s} \right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^{\alpha(s-1)} f(p^\alpha)} \right) := G(s)$$

we infer

$$\sum_{\substack{m=1 \\ m \,\mathrm{odd}}}^{\infty} \frac{g(m) \log m}{m} = -G'(1).$$

Using the logarithmic derivative, we get

$$\frac{G'_f(s)}{G(s)} = \sum_{p\geq 3} \log p \left(\frac{1}{p^s - 1} - \frac{\sum_{\alpha=1}^{\infty} \frac{\alpha}{p^{\alpha(s-1)} f(p^\alpha)}}{1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^{\alpha(s-1)} f(p^\alpha)}} \right)$$

and therefore

$$\sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{g(m) \log m}{m} = -\frac{2\mathcal{P}_f}{S_f(2) + 1} \sum_{p\geq 3} \log p \left(\frac{1}{p-1} - \frac{s_f(p)}{1 + S_f(p)} \right)$$

thus completing the proof of Theorem 1. □

6 Acknowledgments

We express our gratitude to the referee for his careful reading of the manuscript and the many valuable suggestions and corrections he made.

References

[1] J. Kaczorowski and K. Wiertelak, On the sum of the twisted Euler function, *Internat. J. Number Theory* **8** (2012), 1741–1761.

[2] E. Landau, Über die Zahlentheoretische Function $\varphi(n)$ und ihre Beziehung zum Goldbachschen Satz, *Nachrichten der Koniglichten Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1900, 177–186.

[3] K. Wooldridge, Mean value theorems for arithmetic functions similar to Euler's phi-function, *Proc. Amer. Math. Soc.* **58** (1976), 73–78.

2010 *Mathematics Subject Classification*: Primary 11A25; Secondary 11N37.
Keywords: alternating sum, average order over a polynomial sequence.

(Concerned with sequences A001615, A058026, A065463, A082695, A211117, and A211178.)

Received February 9 2013; revised versions received March 3 2013; June 12 2013. Published in *Journal of Integer Sequences*, June 16 2013.

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