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New Classes of Harmonic Number Identities

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Abstract

We develop some new classes of harmonic number identities, and give an integral proof of an identity given by Sun and Zhao.

1 Introduction and preliminaries

In the paper [15], Sun and Zhao quote the following lemma.

Lemma 1. *The following identity holds*

$$\sum_{n=1}^{p}\binom{p}{n}\frac{(-1)^{n+1}\ H_n}{n}=H_p^{(2)}\ \ (p\in\mathbb{N}), \tag{1}$$

where $\mathbb{N}:=\{1,2,3,\ldots\}$.

They state that (1) is known, but it is difficult to give a reference and then proceed to give a proof by induction. The induction proof is perfectly valid however it does not shed light on how and if (1) may be generalized. In this paper we give an integral representation for the left hand side of (1) and then generalize the result in various directions. Some usual notational terms are now defined and will be utilized throughout this paper. The generalized hypergeometric function notation

$${}_pF_q\left[\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\middle|\,t\right]=\sum_{r=0}^{\infty}\frac{(a_1)_r\ldots(a_p)_r}{r!\,(b_1)_r\ldots(b_q)_r}t^r,\quad p,\ q\in\mathbb{N}_0:=\mathbb{N}\cup\{0\} \tag{2}$$

where convergence is assured for

$$p \leq q+1;\ p \leq q \text{ and } |t| < \infty;$$
$$p = q+1 \text{ and } |t| < 1;$$
$$p = q+1,\ \ |t| = 1 \text{ and } \operatorname{Re}\left\{ \sum_{m=1}^{q} b_m - \sum_{m=1}^{p} a_m \right\} > 0,\ b_m, a_m \notin \{0, -1, -2, -3, \ldots\},$$

and $(a)_r$ is Pochhammer's symbol defined by $(a)_r = a(a+1)(a+2)\cdots(a+r-1)$, $r \in \mathbb{N}$, $(a)_0 = 1$. The beta function

$$B(s,t) = \int_0^1 z^{s-1}(1-z)^{t-1}\,dz = \frac{\Gamma(s)\,\Gamma(t)}{\Gamma(s+t)}\ \ (\operatorname{Re}(s) > 0;\ \operatorname{Re}(t) > 0),$$

and the gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\,dt \quad (\operatorname{Re}(z) > 0).$$

The Riemann zeta function

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z} \quad (\operatorname{Re}(z) > 1)$$

and the generalized harmonic numbers in power α are defined as

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^{\alpha}}.$$

The n^{th} harmonic number, for $\alpha = 1$,

$$H_n^{(1)} = H_n = \int_0^1 \frac{1-t^n}{1-t}dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n} \frac{1}{r} - \log(n) \right) = -\psi(1) \doteq 0.5772156649\ldots$$

The polygamma functions $\psi^{(k)}(z)$, are defined by

$$\begin{aligned}\psi^{(k)}(z) &:= \frac{d^{k+1}}{dz^{k+1}} \log\Gamma(z) = \frac{d^k}{dz^k}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) \\ &= -\int_0^1 \frac{[\log(t)]^k\ \ t^{z-1}}{1-t}dt,\ (k \in \mathbb{N}_0)\end{aligned} \tag{3}$$

and $\psi^{(0)}(z) = \psi(z)$, denotes the psi, or digamma function, defined by

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We also recall the relation, for $m = 1, 2, 3, \ldots$

$$H_{z-1}^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!} \psi^{(m)}(z). \tag{4}$$

The evaluation of series with harmonic numbers, or series which sum to harmonic numbers dates back to the time of Euler. Since then many other results have been obtained. The sum

$$\sum_{k=1}^{n} \frac{(-1)^{k+1} \binom{n}{k}}{k} = H_n^{(1)}$$

is well known and appears in a number of problems related to random allocations and theory of records. Adamchik [1] for example obtained

$$\sum_{k=1}^{n} \frac{H_k^{(1)}}{k} = \left(H_n^{(1)}\right)^2 + H_n^{(1)} = 2 \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \frac{1}{r^2}. \tag{5}$$

Flajolet and Sedgewick [6] also got results of the type

$$6 \sum_{k=1}^{n} \frac{(-1)^{k+1} \binom{n}{k}}{k^3} = \left(H_n^{(1)}\right)^3 + 3H_n^{(1)} H_n^{(2)} + 2H_n^{(3)}. \tag{6}$$

There are many papers dealing with identities of harmonic numbers see, for example, [4], [13], [14] and [16].

Binomial coefficients play an important role in many areas of mathematics, including number theory, statistics and probability. The binomial coefficient is defined as

$$\binom{z}{w} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$$

for z and w non-negative integers, where $\Gamma(x)$ is the gamma function.

The representation of sums in closed form can in some cases be achieved through a variety of different methods, including, integral representations, transform techniques, Riordan arrays and the $W-Z$ method. The interested reader is referred to the works of [2], [3], [8], [9], [10] and [11].

The next lemma deals with the derivatives of binomial coefficients (see [12]).

Lemma 2. *Let $j \geq 0$, $n > 0$ and let $Q(n,j) = \binom{n+j}{j}^{-1}$ be an analytic function in j. Then we have,*

$$Q^{(1)}(n,j) = \frac{dQ}{dj} = \begin{cases} -Q(n,j)P(n,j), & (j \in \mathbb{N}), \\ -H_n^{(1)} & (j = 0), \end{cases}$$

where

$$P(n,j) := \sum_{r=1}^{n} \frac{1}{r+j} = \psi(j+1+n) - \psi(j+1).$$

We also have

$$\begin{aligned} Q^{(\lambda)}(n,j) &= \frac{d^{\lambda}Q}{dj^{\lambda}} \\ &= -\sum_{\rho=0}^{\lambda-1} \binom{\lambda-1}{\rho} Q^{(\rho)}(n,j) P^{(\lambda-1-\rho)}(n,j) \quad (\lambda \in \mathbb{N}\setminus\{1\}), \end{aligned} \tag{7}$$

where $P^{(0)}(n,j) = \sum_{r=1}^{n} \frac{1}{r+j}$ $(n \in \mathbb{N})$, and $Q^{(0)}(n,j) = Q(n,j)$, and

$$\begin{aligned} P^{(i)}(n,j) = \frac{d^{i}P}{dj^{i}} &= \frac{d^{i}}{dj^{i}} \left(\sum_{r=1}^{n} \frac{1}{r+j} \right) \\ &= (-1)^{i} i! \sum_{r=1}^{n} \frac{1}{(r+j)^{i+1}} \\ &= (-1)^{i} i! \left[\zeta(i+1, j+1) - \zeta(i+1, j+1+n) \right] \quad (i \in \mathbb{N}), \end{aligned}$$

$$\zeta(q,b) = \sum_{k=0}^{\infty} \frac{1}{(k+b)^{q}} \quad \left(\operatorname{Re}(q) > 1; b \notin \mathbb{Z}_0^{-} \right)$$

is the generalized (or Hurwitz) zeta function and $\mathbb{Z}_0^{-}$ denotes the set of non-positive integers.

Now we list some particular cases of Lemma 2,

$$Q^{(1)}(n,j) = -\binom{n+j}{j}^{-1} \sum_{r=1}^{n} \frac{1}{r+j},$$

$$\begin{aligned} Q^{(2)}(n,j) &= \binom{n+j}{j}^{-1} \left[\left(\sum_{r=1}^{n} \frac{1}{r+j} \right)^{2} + \sum_{r=1}^{n} \frac{1}{(r+j)^{2}} \right] \\ &= \binom{n+j}{j}^{-1} \left[\sum_{r=1}^{n} \sum_{s=1}^{r} \frac{2}{(r+j)(s+j)} \right], \end{aligned}$$

$$\begin{aligned} &Q^{(3)}(n,j) \\ &= -\binom{n+j}{j}^{-1} \left[\left(\sum_{r=1}^{n} \frac{1}{r+j} \right)^{3} + 2\sum_{r=1}^{n} \frac{1}{(r+j)^{3}} + 3\sum_{r=1}^{n} \frac{1}{(r+j)^{2}} \sum_{r=1}^{n} \frac{1}{r+j} \right] \end{aligned}$$

and

$$Q^{(4)}(n,j)$$

$$= \binom{n+j}{j}^{-1}\left[6\sum_{r=1}^{n}\frac{1}{(r+j)^2}\left(\sum_{r=1}^{n}\frac{1}{r+j}\right)^2 + 8\sum_{r=1}^{n}\frac{1}{(r+j)^3}\sum_{r=1}^{n}\frac{1}{r+j} + 3\left(\sum_{r=1}^{n}\frac{1}{(r+j)^2}\right)^2 + \left(\sum_{r=1}^{n}\frac{1}{r+j}\right)^4 + 6\sum_{r=1}^{n}\frac{1}{(r+j)^4}\right].$$

In the special case when $j=0$ we may write

$$Q^{(1)}(n,0) = -H_n^{(1)}, \tag{8}$$

$$Q^{(2)}(n,0) = \left(H_n^{(1)}\right)^2 + H_n^{(2)}, \tag{9}$$

$$Q^{(3)}(n,0) = -\left(H_n^{(1)}\right)^3 - 3H_n^{(1)}H_n^{(2)} - 2H_n^{(3)} \tag{10}$$

and

$$Q^{(4)}(n,0) = \left(H_n^{(1)}\right)^4 + 6\left(H_n^{(1)}\right)^2 H_n^{(2)} + 8H_n^{(1)}H_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)}. \tag{11}$$

Lemma 1 is restated and proved in the following theorem.

2 First class of identities

Theorem 3. *Let p be a positive integer. Then we get*

$$\sum_{n=1}^{p}\frac{(-1)^{n+1}\binom{p}{n}H_n^{(1)}}{n} = \int_0^1 \frac{\log(1-x)}{x}\left((1-x)^p - 1\right)dx \tag{12}$$

$$= H_p^{(2)}. \tag{13}$$

Proof. We can write, for $t\in\mathbb{R}$,

$$\sum_{n=1}^{p}\frac{t^n\binom{p}{n}}{n\binom{n+j}{j}} = \sum_{n=1}^{p}\frac{t^n\binom{p}{n}\Gamma(n)\Gamma(j+1)}{\Gamma(n+j+1)} = \sum_{n=1}^{p}t^n\binom{p}{n}B(n,j+1)$$

where $B(\cdot,\cdot)$ is the classical beta function. Now

$$\sum_{n=1}^{p}\frac{t^n\binom{p}{n}}{n\binom{n+j}{j}} = \sum_{n=1}^{p}t^n\binom{p}{n}B(n,j+1) = \sum_{n=1}^{p}t^n\binom{p}{n}\int_0^1 x^{n-1}(1-x)^j\,dx.$$

Differentiating the last resulting equation with respect to j and putting $t=-1$, we have

$$\sum_{n=1}^{p}\frac{(-1)^{n+1}\binom{p}{n}}{n\binom{n+j}{j}}\sum_{r=1}^{n}\frac{1}{r+j} = \int_0^1\frac{\log(1-x)}{x}(1-x)^j\sum_{n=1}^{p}(-x)^n\binom{p}{n}dx,$$

which, upon setting $j=0$, yields

$$\begin{aligned}\sum_{n=1}^{p} \frac{(-1)^{n+1}\binom{p}{n}\ H_n^{(1)}}{n} &= \int_0^1 \frac{\log(1-x)}{x}\left((1-x)^p-1\right)dx\\ &= \zeta\left(2\right)-\psi'\left(p+1\right)\\ &= H_p^{(2)},\end{aligned}$$

by the relation (4). $\square$

By recalling the inversion formula (see [7]):

$$g\left(n\right)=\sum_k (-1)^k \binom{n}{k} f\left(k\right) \Leftrightarrow f\left(n\right)=\sum_k (-1)^k \binom{n}{k} g\left(k\right),$$

we find from the identity (13):

$$\sum_{n=1}^{p} (-1)^{n+1} \binom{p}{n}\ H_n^{(2)} = \frac{H_p^{(1)}}{p}.$$

Theorem 3 can be generalized as follows.

Theorem 4. *We have*

$$\begin{aligned}\sum_{n=1}^{p} \frac{(-1)^{n+1}\binom{p}{n}\ Q^{(q)}\left(n,0\right)}{n} &= \int_0^1 \frac{\log^q(1-x)}{x}\left((1-x)^p-1\right)dx\\ &= (-1)^{q+1}\, q!H_p^{(q+1)} \qquad (p,q\in\mathbb{N})\end{aligned}$$

and by inversion

$$\sum_{n=1}^{p} (-1)^n \binom{p}{n}\ H_n^{(q+1)} = \frac{(-1)^q\ Q^{(q)}\left(p,0\right)}{p\ q!}$$

where $Q^{(q)}\left(\cdot,\cdot\right)$ is defined by (7).

Proof. The proof follows the same pattern as given in Theorem 3, by differentiating with respect to j, q times, and utilizing (3). $\square$

Example 5. For $q=1$, we obtain the result (13). For $q=4$ and using (11) we get

$$\sum_{n=1}^{p} \frac{(-1)^n\binom{p}{n}\left(\left(H_n^{(1)}\right)^4+6\left(H_n^{(1)}\right)^2 H_n^{(2)}+8H_n^{(1)}H_n^{(3)}+3\left(H_n^{(2)}\right)^2+6H_n^{(4)}\right)}{n} = 24H_p^{(5)},$$

and

$$\sum_{n=1}^{p} (-1)^n \binom{p}{n}\ H_n^{(5)} = \frac{\left(\left(H_p^{(1)}\right)^4+6\left(H_p^{(1)}\right)^2 H_p^{(2)}+8H_p^{(1)}H_p^{(3)}+3\left(H_p^{(2)}\right)^2+6H_p^{(4)}\right)}{24p}.$$

3 Second class of infinite sum identities

In this section we investigate another class of infinite sums involving harmonic numbers. In particular we give closed form representations for the class of sums $\sum_{n\geq 1}\frac{Q^{(q)}(n+b,0)}{n(n+b)}$. The next lemma gives an alternative representation for $Q^{(q)}(b,z)$.

Lemma 6. *Let b and q be positive real integers and $z\in\mathbb{R}^{+}\cup\{0\}$. Also let $Q^{(q)}(b,z)=\frac{d^q}{dz^q}=\binom{b+z}{z}^{-1}$ be an analytic function in z. Then we have*

$$\begin{aligned}
&(-1)^{q-1}\,q!\sum_{r=0}^{b}(-1)^{r}\binom{b}{r}\frac{r}{(r+z)^{q+1}}\\
&=Q^{(q)}(b,z) \qquad (14)\\
&=\frac{(-1)^{q}\,q!\;b}{(z+1)^{q+1}}\quad {}_{q+2}F_{q+1}\left[\left.\begin{array}{c}\overbrace{z+1,\ldots,z+1}^{(q+1)-terms},1-b\\ \underbrace{z+2,\ldots,z+2}_{(q+1)-terms}\end{array}\right|1\right]. \qquad (15)
\end{aligned}$$

Proof. We first notice that

$$\begin{aligned}
\sum_{r=0}^{b}(-1)^{r}\binom{b}{r}\frac{1}{r+z}&=\sum_{r=0}^{b}(-1)^{r}\binom{b}{r}\int_{0}^{1}x^{r+z-1}dx\\
&=\int_{0}^{1}x^{z-1}\sum_{r=0}^{b}(-1)^{r}\binom{b}{r}x^{r}\;dx\\
&=\int_{0}^{1}x^{z-1}(1-x)^{b}\;dx=B(z,b+1)\\
&=\frac{1}{z\binom{b+z}{b}}
\end{aligned}$$

and so

$$z\sum_{r=0}^{b}(-1)^{r}\binom{b}{r}\frac{1}{r+z}=\frac{1}{\binom{b+z}{b}},$$

which is a known result (see [5] and [7]). If we let

$$F(b,z)=\sum_{r=0}^{b}(-1)^{r}\binom{b}{r}\frac{1}{r+z},$$

then

$$zF(b,z)=Q(b,z).$$

Differentiating q-times with respect to z, we have

$$q\;F^{(q-1)}(b,z)+z\;F^{(q)}(b,z)=Q^{(q)}(b,z)$$

or

$$(-1)^{q-1}\, q! \sum_{r=0}^{b} (-1)^r \binom{b}{r} \frac{r}{(r+z)^{q+1}} = Q^{(q)}(b,z).$$

Chu [4] has some special cases for $q=1$. The hypergeometric function (15) is defined by (2). When $z=0$, then

$$(-1)^{q-1}\, q! \sum_{r=1}^{b} (-1)^r \binom{b}{r} \frac{1}{r^q} = Q^{(q)}(b,0) \tag{16}$$

$$= (-1)^q\, q!\; b\; {}_{q+2}F_{q+1}\left[\left.\begin{array}{c} \overbrace{1,\ldots,1}^{(q+1)-\text{terms}},1-b \\ \underbrace{2,\ldots,2}_{(q+1)-\text{terms}} \end{array}\right| 1\right],$$

we note that (8), (9), (10) and (11) are special cases of $Q^{(q)}(b,0)$ for $q=1,2,3,4$. □

The following two lemmas will be useful in the proof of the subsequent theorem.

Lemma 7. *Let $b>0$ be a real positive number. Then we have*

$$-\frac{H_b}{b} = \int_0^1 x^{b-1}\ \log(1-x)\ dx.$$

Proof. By symmetry

$$\begin{aligned}\int_0^1 x^{b-1}\ \log(1-x)\ dx &= \int_0^1 (1-x)^{b-1}\ \log x\ dx \\ &= \int_0^1 \sum_{r=0}^{b-1} (-1)^r \binom{b-1}{r} x^r\ \log x\ dx \\ &= -\sum_{r=0}^{b-1} (-1)^r \binom{b-1}{r} \frac{1}{(r+1)^2} \\ &= \frac{1}{b}\sum_{r=1}^{b} (-1)^r \binom{b}{r}\frac{1}{r} = -\frac{H_b}{b},\end{aligned}$$

where the last equality follows from (16). □

Lemma 8. *Let b be a positive real number. Then we have*

$$\sum_{n=1}^{\infty} \frac{H_{n+b}}{n\,(n+b)} = \frac{1}{b} Q^{(2)}(b,0) = \frac{1}{b}\left(\left(H_b^{(1)}\right)^2 + H_b^{(2)}\right) \tag{17}$$

Proof. From Lemma 7

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_{n+b}}{n\left(n+b\right)} &= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \left(1-x\right)^{n+b-1} \log x \; dx, \\
&= \int_0^1 x^{b-1} \log\left(1-x\right) \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right) \; dx \\
&= \int_0^1 x^{b-1} \log^2\left(1-x\right) \; dx \\
&= \int_0^1 \left(1-x\right)^{b-1} \log^2 x \; dx \\
&= \int_0^1 \sum_{r=0}^{b-1} \left(-1\right)^r \binom{b-1}{r} x^r \log^2 x \; dx \\
&= \sum_{r=0}^{b-1} \left(-1\right)^r \binom{b-1}{r} \frac{2}{\left(r+1\right)^3} \\
&= -\frac{2}{b} \sum_{r=1}^{b} \left(-1\right)^r \binom{b}{r} \frac{1}{r^2} \\
&= \frac{1}{b} Q^{(2)}(b,0) = \frac{1}{b}\left(\left(H_b^{(1)}\right)^2 + H_b^{(2)}\right),
\end{aligned}$$

from Lemma 6 and (5). □

The following theorem can now be stated.

Theorem 9. *Let b and q be positive real integers. Then we have*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{Q^{(q)}(n+b,0)}{n(n+b)} &= \frac{1}{b} Q^{(q+1)}(b,0) \\
&= \left(-1\right)^{q-1} \; q! \;\; {}_{q+2}F_{q+1}\left[\left.\begin{matrix} \overbrace{1,\ldots,1}^{(q+1)-terms}, 1-b \\ \underbrace{2,\ldots,2}_{(q+1)-terms} \end{matrix}\right| 1\right]
\end{aligned}$$

where $Q^{(q)}(b,0)$ is defined by (16).

Proof. By the induction principle, the case $q=1$ holds, from Lemma 8. Now

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{Q^{(q)}(n+b,0)}{(n+b)}\right) &= \sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 (1-x)^{n+b-1}\log^q x\ dx \\
&= \sum_{n=1}^{\infty}\frac{x^n}{n}\int_0^1 x^{b-1}\log^q(1-x)\ dx, \text{ by symmetry,} \\
&= \int_0^1 x^{b-1}\log^{q+1}(1-x)\ dx \\
&= \int_0^1 (1-x)^{b-1}\log^{q+1} x\ dx \\
&= \int_0^1 \sum_{r=0}^{b-1}(-1)^r\binom{b-1}{r}x^r\log^{q+1}x\ dx \\
&= (-1)^q(q+1)!\sum_{r=0}^{b-1}(-1)^r\binom{b-1}{r}\frac{1}{(r+1)^{q+2}},
\end{aligned}$$

and by Lemma 6 we get

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{Q^{(q)}(n+b,0)}{(n+b)}\right) = \frac{(-1)^q(q+1)!}{b}\sum_{r=1}^{b}(-1)^r\binom{b}{r}\frac{1}{r^{q+1}} = \frac{1}{b}Q^{(q+1)}(b,0).$$

This completes the proof of Theorem 9. □

Example 10. For the case $q=2$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\left(H_{n+b}^{(1)}\right)^2+H_{n+b}^{(2)}}{n(n+b)} &= \frac{1}{b}\left(\left(H_b^{(1)}\right)^3+3H_b^{(1)}H_b^{(2)}+2H_b^{(3)}\right) \\
&= \frac{2}{b}\sum_{r=1}^{b}(-1)^{r+1}\binom{b}{r}\frac{1}{r^3}
\end{aligned}$$

this result also follows from (6). For $q=3$

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\left(H_{n+b}^{(1)}\right)^3+3H_{n+b}^{(1)}H_{n+b}^{(2)}+2H_{n+b}^{(3)}}{n(n+b)} \\
&= \frac{1}{b}\left(\left(H_b^{(1)}\right)^4+6\left(H_b^{(1)}\right)^2H_b^{(2)}+8H_b^{(1)}H_b^{(3)}+3\left(H_b^{(2)}\right)^2+6H_b^{(4)}\right) \\
&= \frac{6}{b}\sum_{r=1}^{b}(-1)^{r+1}\binom{b}{r}\frac{1}{r^4} = 6\ {}_5F_4\left[\left.\begin{matrix}1,1,1,1,1-b\\2,2,2,2\end{matrix}\right|1\right].
\end{aligned}$$

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