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A Note on Cosine Power Sums

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Abstract

Using the multisection series method, we establish formulas for various power sums of cosine functions. As corollaries we derive several combinatorial identities.

1 Introduction

In [2, 3], we presented two open problems concerning the asymptotic behaviour of cosine power sums,

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^p \left(\frac{k\pi}{n} \right) , \tag{1}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Under certain conditions, these cosine power sums can be determined exactly, without approximations [5].

If

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is a finite or convergent infinite series, then for $0 \le r < n$ the sum

$$a_r x^r + a_{r+n} x^{r+n} + a_{r+2n} x^{r+2n} + \cdots$$

is given by the multisection formula (see [1, Ch. 16], [4, Ch. 4, S. 4.3] and [6])

$$\sum_{k \ge 0} a_{r+kn} x^{r+kn} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-kr} f(z^k x) , \tag{2}$$

where $z = e^{\frac{2\pi i}{n}}$ is the nth root of 1.

We shall use the multisection formula to prove:

Theorem 1. *Let n and p be two positive integers. Then*

$$\sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \cos^{2p}\left(\frac{k\pi}{n}\right) = -\frac{1}{2} + \frac{n}{2^{2p+1}} \sum_{k=-\left\lfloor \frac{p}{n} \right\rfloor}^{\left\lfloor \frac{p}{n} \right\rfloor} \binom{2p}{p+kn} .$$

Theorem 2. *Let n and p be two positive integers such that $n \equiv p \pmod 2$. Then*

$$\sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \cos^{p}\left(\frac{k\pi}{n}\right) = -\frac{1}{2} + \frac{n}{2^{p+1}} \sum_{k=-\left[\frac{p}{2n} \right]}^{\left[\frac{p}{2n} \right]} \binom{p}{\frac{p+n}{2}+kn} ,$$

where $[x]$ denotes the nearest integer to x, i.e., $[x] = \left\lfloor x + \frac{1}{2} \right\rfloor$.

Theorem 3. *Let n and p be two positive integers. Then*

$$\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \cos^{2p}\left(\frac{2k-1}{n} \cdot \frac{\pi}{2}\right) = \frac{n}{2^{2p+1}} \sum_{k=-\left\lfloor \frac{p}{n} \right\rfloor}^{\left\lfloor \frac{p}{n} \right\rfloor} (-1)^k \binom{2p}{p+kn} .$$

We note that the formula proved in [5] is the case $p < n$ of Theorem 1. The conjecture published in [3] is solved by Theorem 2. In the final section of this paper we present some relations that can be obtained immediately by the above theorems.

2 Proof of Theorem 1

We apply the multisection formula (2) for

$$r = p - n\left\lfloor \frac{p}{n} \right\rfloor \qquad \text{and} \qquad f(x) = (1+x)^{2p} .$$

Thus, when $x = 1$, we get

$$\sum_{k=-\left\lfloor \frac{p}{n} \right\rfloor}^{\left\lfloor \frac{p}{n} \right\rfloor} \binom{2p}{p+kn} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-k\left(p-n\left\lfloor \frac{p}{n} \right\rfloor\right)} (1+z^k)^{2p} , \tag{3}$$

where $z = e^{\frac{2\pi i}{n}}$. Having

$$\begin{aligned} e^{-ir\varphi}(1+e^{i\varphi})^p &= e^{-ir\varphi}\left(e^{\frac{i\varphi}{2}-\frac{i\varphi}{2}} + e^{\frac{i\varphi}{2}+\frac{i\varphi}{2}}\right)^p \\ &= e^{-ir\varphi} e^{\frac{ip\varphi}{2}} \left(e^{-\frac{i\varphi}{2}} + e^{\frac{i\varphi}{2}}\right)^p \\ &= 2^p \cos^p \frac{\varphi}{2} \cdot e^{i\left(\frac{p}{2}-r\right)\varphi} , \end{aligned} \tag{4}$$

we can write

$$z^{-k\left(p-n\left\lfloor \frac{p}{n} \right\rfloor\right)} (1+z^k)^{2p} = 2^{2p} \cos^{2p}\left(\frac{k\pi}{n}\right) \cdot e^{i2k\left\lfloor \frac{p}{n} \right\rfloor \pi} .$$

So, we may rewrite (3) as

$$\sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} \binom{2p}{p+kn} = \frac{2^{2p}}{n} \sum_{k=0}^{n-1} \cos^{2p}\left(\frac{k\pi}{n}\right) \ , \tag{5}$$

Taking into account that

$$\cos\left(\frac{(n-k)\pi}{n}\right) = -\cos\left(\frac{k\pi}{n}\right) \ , \quad k = 0, 1, \ldots, n \ ,$$

Theorem 1 is proved.

3 Proof of Theorem 2

To prove the theorem we proceed as in Theorem 1. By the multisection formula (2), with

$$x = 1 \ , \quad f(x) = (1+x)^p \quad \text{and} \quad r = \frac{p+n}{2} - n \left\lfloor \frac{p+n}{2n} \right\rfloor \ ,$$

we get

$$\sum_{k=-\lfloor \frac{p+n}{2n} \rfloor}^{\lfloor \frac{p+n}{2n} \rfloor} \binom{p}{\frac{p+n}{2} + nk} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-k\left(\frac{p+n}{2} - n\lfloor \frac{p+n}{2n} \rfloor\right)} (1+z^k)^p \ ,$$

where $z = e^{\frac{2\pi i}{n}}$. By (4), it immediately follows that

$$z^{-k\left(\frac{p+n}{2} - n\lfloor \frac{p+n}{2n} \rfloor\right)} (1+z^k)^p = 2^p \cos^p\left(\frac{k\pi}{n}\right) \cdot e^{ik\pi\left(2\lfloor \frac{p+n}{2n} \rfloor - 1\right)} \ .$$

Thus we can write

$$\sum_{k=-\lfloor \frac{p+n}{2n} \rfloor}^{\lfloor \frac{p+n}{2n} \rfloor} \binom{p}{\frac{p+n}{2} + kn} = \frac{2^p}{n} \sum_{k=0}^{n-1} (-1)^k \cos^p\left(\frac{k\pi}{n}\right) \ . \tag{6}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} (-1)^k \cos^p\left(\frac{k\pi}{n}\right) \\
&= 1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p\left(\frac{k\pi}{n}\right) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-k} \cos^p\left(\frac{(n-k)\pi}{n}\right) \\
&= 1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p\left(\frac{k\pi}{n}\right) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n+p-k} \cos^p\left(\frac{k\pi}{n}\right) \\
&= 1 + 2 \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^p\left(\frac{k\pi}{n}\right) \ .
\end{aligned} \tag{7}$$

According to (3) and (7), Theorem 2 is proved.

4 Proof of Theorem 3

To prove the theorem, we use the formula

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} f(2k-1) = \frac{1}{2}\left(\sum_{k=1}^{n-1} f(k) - \sum_{k=1}^{n-1}(-1)^k f(k)\right) \tag{8}$$

for $f(k) = \cos^{2p}\left(\frac{k\pi}{2n}\right)$. By Theorem 1 (with n replaced by $2n$) and Theorem 2 (with n and p replaced by $2n$ and $2p$, respectively), we have

$$\begin{aligned}
\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} f(2k-1) &= \frac{1}{2}\left(-\frac{1}{2} + \frac{2n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{2n} \rfloor}^{\lfloor \frac{p}{2n} \rfloor} \binom{2p}{p+2kn}\right. \\
&\qquad \left. + \frac{1}{2} - \frac{2n}{2^{2p+1}} \sum_{k=-\lceil \frac{p}{2n} \rceil}^{\lceil \frac{p}{2n} \rceil} \binom{2p}{p+(2k+1)n}\right) \\
&= \frac{n}{2^{2p+1}} \sum_{k=-\lceil \frac{p}{2n} \rceil}^{\lceil \frac{p}{2n} \rceil} \left(\binom{2p}{p+2kn} - \binom{2p}{p+(2k+1)n}\right) \\
&= \frac{n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} (-1)^k \binom{2p}{p+kn}.
\end{aligned}$$

Theorem 3 is proved.

5 Special cases

We begin this section with two immediate consequences of Theorems 1, 2 and 3.

Corollary 4. *Let n be a positive integer. Then*

1. $$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2n}\left(\frac{k\pi}{n}\right) = -\frac{1}{2} + \frac{n}{2^{2n}} + \frac{n}{2^{2n+1}}\binom{2n}{n};$$

2. $$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^{n}\left(\frac{k\pi}{n}\right) = -\frac{1}{2} + \frac{n}{2^{n}};$$

3. $$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \cos^{2n}\left(\frac{2k-1}{n} \cdot \frac{\pi}{2}\right) = -\frac{n}{2^{2n}} + \frac{n}{2^{2n+1}}\binom{2n}{n}.$$

Corollary 5. *Let n and p be two positive integers such that $p < n$. Then*

1. $$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2p}\left(\frac{k\pi}{n}\right) = -\frac{1}{2} + \frac{n}{2^{2p+1}}\binom{2p}{p};$$

2. $$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cos^{p}\left(\frac{k\pi}{n}\right) = -\frac{1}{2}\ , \quad n \equiv p \pmod 2;$$

3. $$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \cos^{2p}\left(\frac{2k-1}{n}\cdot\frac{\pi}{2}\right) = \frac{n}{2^{2p+1}}\binom{2p}{p}.$$

By Theorems 1, 2 and 3, for fixed values of n, we can obtain some combinatorial identities.

Corollary 6. *Let p be a positive integer. Then*

1. $$\sum_{k=0}^{p} \binom{2p}{p-k} = \frac{1}{2}\binom{2p}{p} + 2^{2p-1}\ ;$$

2. $$\sum_{k=0}^{\lfloor p/2 \rfloor} \binom{2p}{p-2k} = \frac{1}{2}\binom{2p}{p} + 2^{2p-2}\ ;$$

3. $$\sum_{k=0}^{\lfloor p/3 \rfloor} \binom{2p}{p-3k} = \frac{1}{2}\binom{2p}{p} + \frac{1+2^{2p-1}}{3}\ ;$$

4. $$\sum_{k=0}^{\lfloor p/4 \rfloor} \binom{2p}{p-4k} = \frac{1}{2}\binom{2p}{p} + 2^{2p-3} + 2^{p-2}\ ;$$

5. $$\sum_{k=0}^{\lfloor p/5 \rfloor} \binom{2p}{p-5k} = \frac{1}{2}\binom{2p}{p} + \frac{(3+\sqrt{5})^p + (3-\sqrt{5})^p + 2^{3p-1}}{5\cdot 2^p}\ ;$$

6. $$\sum_{k=0}^{\lfloor p/6 \rfloor} \binom{2p}{p-6k} = \frac{1}{2}\binom{2p}{p} + \frac{3^p + 2^{2p-1} + 1}{6}\ .$$

Corollary 7. *Let p be a positive integer. Then*

1. $$\sum_{k=1}^{p} \binom{2p-1}{p-k} = 4^{p-1}\ ;$$

2. $$\sum_{k=1}^{\lfloor p/3 \rfloor} \binom{2p-3}{p-3k} = \frac{4^{p-2}-1}{3}\ , \quad p > 1\ ;$$

3. $\displaystyle\sum_{k=1}^{\lfloor p/5 \rfloor} \binom{2p-5}{p-5k} = \frac{4^{p-3}}{5} - \frac{(\sqrt{5}+1)^{2p-5} - (\sqrt{5}-1)^{2p-5}}{5 \cdot 2^{2p-5}}\ , \quad p > 2\ .$

Corollary 8. *Let p be a positive integer. Then*

1. $\displaystyle\sum_{k=0}^{p} (-1)^k \binom{2p}{p-k} = \frac{1}{2}\binom{2p}{p}\ ;$

2. $\displaystyle\sum_{k=0}^{\lfloor p/2 \rfloor} (-1)^k \binom{2p}{p-2k} = \frac{1}{2}\binom{2p}{p} + 2^{p-1}\ ;$

3. $\displaystyle\sum_{k=0}^{\lfloor p/3 \rfloor} (-1)^k \binom{2p}{p-3k} = \frac{1}{2}\binom{2p}{p} + 3^{p-1}\ ;$

4. $\displaystyle\sum_{k=0}^{\lfloor p/4 \rfloor} (-1)^k \binom{2p}{p-4k} = \frac{1}{2}\binom{2p}{p} + \frac{(2+\sqrt{2})^p + (2-\sqrt{2})^p}{4}\ ;$

5. $\displaystyle\sum_{k=0}^{\lfloor p/5 \rfloor} (-1)^k \binom{2p}{p-5k} = \frac{1}{2}\binom{2p}{p} + \frac{(5+\sqrt{5})^p + (5-\sqrt{5})^p}{5 \cdot 2^p}\ .$

Note that Corollary 6 is related in [7] with the sequences A032443, A114121, A007583, A007582, A078789, A085282, Corollary 7 with A000302, A002450, A095931, Corollary 8 with A088218, A005317, A191993, A007052, A081567, respectively.

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