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The Arithmetic Derivative and Antiderivative

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Abstract

The notion of the arithmetic derivative, a function sending each prime to 1 and satisfying the Leibnitz rule, is extended to the case of complex numbers with rational real and imaginary parts. Some constraints on the solutions to some arithmetic differential equations are found. The homogeneous arithmetic differential equation of the k-th order is studied. The factorization structure of the antiderivatives of natural numbers is presented. Arithmetic partial derivatives are defined and some arithmetic partial differential equations are solved.

1 Introduction: Basic concepts

The goal of this paper is: (i) to review what is known about the arithmetic derivative [1, 2, 3], (ii) to define some related new concepts, and (iii) to prove some new results, mostly related to the conjectures formulated in [1, 3].

1.1 Definition of the arithmetic derivative

Barbeau [1] defined the arithmetic derivative as the function $D : \mathbb{Z} \to \mathbb{Z}$, defined by the rules:

$D(1) = D(0) = 0$

$D(p) = 1$ for any prime $p \in \mathbb{P} := \{2, 3, 5, 7, \ldots, p_i, \ldots\}$.

$D(ab) = D(a)b + aD(b)$ for any $a, b \in \mathbb{N}$ (the Leibnitz rule)

$D(-n) = -D(n)$ for any $n \in \mathbb{N}$.

Ufnarovski and Åhlander [3] extended D to rational numbers by the rule:

$$D(\frac{a}{b}) = (\frac{a}{b})' = \frac{a'b - ab'}{b^2}$$

They also defined D for real numbers of the form $x = \prod_{i=1}^{k} p_i^{x_i}$, where p_i are different primes and $x_i \in \mathbb{Q}$, by the additional rule:

$$D(x) = x' = x \sum_{i=1}^{k} \frac{x_i}{p_i}.$$

They showed that the definition of D can be extended to all real numbers and to the arbitrary unique factorization domain.

I extend the definition of the arithmetic derivative to the numbers from the unit circle in the complex plane $\mathbb{E} = \{e^{i\varphi}, \varphi \in [0, 2\pi]\}$ and to the Gaussian integers $\mathbb{Z}[i] = \{a+bi, a, b \in \mathbb{Z}\}$.

It is not possible to extend the definition of the arithmetic derivative to the Gaussian integers by the demand $D(q) = q' = 0$ for all irreducible Gaussian integers q (the analog of primes). If one wants the Leibnitz rule to still remain valid, one easily gets a contradiction: $2 = (1+i)(1-i)$, hence $D(2) = 1 \cdot (1-i) + (1+i) \cdot 1 = 2$, but $D(2) = 1$ because 2 is a prime.

However, one can extend D to $\mathbb{Q}[i] = \{a + bi, a, b \in \mathbb{Q}\}$ using the polar decomposition of complex numbers as follows:

Proposition 1. *i) For any $\varphi = (m/n)\pi$, where $m, n \in \mathbb{Q}$ and $n \neq 0$, the equation $D(e^{i\varphi}) = 0$ holds.*

ii) Let $\mathbb{E} = \{z \in \mathbb{C}; |z| = 1\}$ be the unit circle in the complex plane. There are uncountably many functions $D : \mathbb{E} \to \mathbb{E}$ satisfying the condition $D(zw) = D(z)w + zD(w)$ for all $z, w \in \mathbb{E}$.

iii) If one defines the arithmetic derivative on $\mathbb{E}$ as follows: $D(e^{i\varphi}) = 0$ for all $e^{i\varphi} \in \mathbb{E}$, then the definition of the arithmetic derivative can be uniquely extended to $\mathbb{Q}[i]$ so that the Leibnitz rule and the quotient rule remain valid.

Proof. i) Take any $\varphi = (m/n)\pi$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. Then $e^{in\varphi} = \pm 1$, hence $D(e^{in\varphi}) = 0$. The Leibnitz rule implies $D(e^{in\varphi}) = D(e^{i\varphi})(e^{i\varphi})^{n-1}$ and because $(e^{i\varphi})^{n-1} \neq 0$ it must be that $D(e^{i\varphi}) = 0$.

ii) Sketch of the proof: One can easily see that there is a 1-1 correspondence between the functions $D : \mathbb{E} \to \mathbb{C}$ satisfying the Leibnitz rule $D(zw) = D(z)W + zD(w)$ and the additive functions $L : \mathbb{R} \to \mathbb{C}$ with a period of 2π (the correspondence is given by the formulas $L(s) := \frac{D(e^{is})}{e^{is}}$ and $D(e^{is}) := L(s)e^{is}$).

It is also easy to see that for any additive function (satisfying Cauchy's functional equation $L(s+t) = L(s) + L(t)$) it is: $L(qx) = q \cdot L(x)$ for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. Because $\mathbb{R}$ and $\mathbb{C}$ are linear spaces over $\mathbb{Q}$, this means that every additive function $L : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{Q}$-linear transformation. Additive functions from $\mathbb{R}$ into $\mathbb{C}$ with a period of 2π are therefore in a 1-1 correspondence with $\mathbb{Q}$-linear transformations $L : \mathbb{R} \to \mathbb{C}$ such that $L(2\pi) = 0$.

A linear transformation is uniquely determined with its values on any of the bases of its domain. Let B be a base of the space $\mathbb{R}$ such that $\pi \in B$ (its existence is guaranteed by Zorn's lemma). Thus the $\mathbb{Q}$-linear transformations $L : \mathbb{R} \to \mathbb{C}$ such that $L(2\pi) = L(\pi) = 0$ are in a 1-1 correspondence with the set of all functions $L : B \to \mathbb{C}$ such that $L(\pi) = 0$.

The base B is obviously not countable, hence the same also holds for the set of all functions $L : B \setminus \{\pi\} \to \mathbb{C}$. By the chain of proven 1-1 correspondences, this also holds for the set of all functions $D : \mathbb{E} \to \mathbb{E}$ satisfying the Leibnitz rule.

iii) If one writes such numbers in their polar form $a+bi = re^{i\varphi}$, then $D(re^{i\varphi}) = D(r)e^{i\varphi} + rD(e^{i\varphi}) = D(r)e^{i\varphi} = D((a^2+b^2)^{1/2})e^{i\varphi}$ because $D(e^{i\varphi}) = 0$ and $r = (a^2+b^2)^{1/2}$. The Leibnitz rule is still valid because for any $z = re^{i\varphi}$ and $w = se^{i\psi}$ it is: $D(zw) = D((re^{i\varphi})(se^{i\psi})) = D(rs)e^{i\varphi+\psi} = (D(r)s + r(D(s))e^{i\varphi+\psi} = D(r)e^{i\varphi}se^{i\psi} + re^{i\varphi}D(s)e^{i\psi} = D(z)w + zD(w)$. The quotient rule is also still valid because $\frac{D(z)w-zD(w)}{w^2} = \frac{D(r)e^{i\varphi}se^{i\psi}-re^{i\varphi}D(s)e^{i\psi}}{s^2e^{i2\varphi}} = \frac{(D(r)s-rD(s))e^{i\varphi}}{s^2e^{i\psi}} = D(\frac{r}{s})e^{i(\varphi-\psi)} = D(\frac{r}{s}e^{i(\varphi-\psi)}) = D(\frac{re^{i\varphi}}{se^{i\psi}}) = D(\frac{z}{w})$. □

Stay [2] generalized the concept of the arithmetic derivative still further, practically for any number, using advanced techniques such as exponential quantum calculus. In this paper I focus on the arithmetic derivative as a function defined for natural numbers and rational numbers.

1.2 Higher derivatives and the logarithmic derivative

Higher derivatives $n^{(k)}$ are defined inductively: $n^{(2)} = D^2(n) = D(D(n)) = n'' = (n')'$, $n^{(k+1)} = D^{k+1}(n) = D(D^k(n))$. Many conjectures on the arithmetic derivative focus on the behavior of sequences $(n, n', n'', \ldots)$.

Ufnarovski and Åhlander [3] conjectured that for each $n \in \mathbb{N}$ exactly one of the following can happen: either $n^{(k)} = 0$ for sufficiently large k, or $lim_{k\to\infty} = \infty$, or $n = p^p$ for some prime p (in this case $n^{(k)} = n$ for each $k \in \mathbb{N}$). They introduced the function L, called the *logarithmic derivative*, satisfying the condition

$$L(x) = \frac{x'}{x} = \frac{D(x)}{x}.$$

For any $x = \prod_{i=1}^{k} p_i^{x_i}$, where p_i are different primes and $x_i \in \mathbb{Q}$, L satisfies the condition

$$L(x) = \sum_{i=1}^{k} \frac{x_i}{p_i}.$$

For every prime p and every $m, n \in \mathbb{N}$ the following formulas hold: $L(p) = \frac{1}{p}$, $L(p^{\frac{m}{n}}) = \frac{m}{np}$, $L(1) = 0$. From the definition of the logarithmic derivative also follow the formulas $L(-x) = L(x), L(0) = \infty$ and $D(x) = L(x) \cdot x$ [3], hence $D^2(x) = D(L(x))x + L(x)D(x)$. I also use the notation $L(x) = x^*$.

The logarithmic derivative is an additive function: $L(xy) = L(x)+L(y)$ for any $x, y \in \mathbb{Q}$. Consequently, using a table of values $L(p) = \frac{1}{p}$ (computed to sufficient decimal places!) and the formula $D(x) = L(x) \cdot x$, it is easy to find $D(n)$ for $n \in \mathbb{N}$ having all its prime factors in the table. For example, $D(5 \cdot 11^3) = L(5 \cdot 11^3) \cdot 5 \cdot 11^3$ and because $L(5 \cdot 11^3) = L(5) + 3L(11) = 0.2000 + 3 \cdot 0.0909 = 0.4727$, one can calculate $D(5 \cdot 11^3) = \lceil 0.4727 \cdot 6655 \rceil = \lceil 3145.8185 \rceil = 3146$.

P_N	2	3	5	7	11	13	17
$\frac{1}{P_N}$	0.5000	0.3333	0.2000	0.1429	0.090	0.0769	0.0588

Table 1: Values of the logarithmic derivative for the first seven primes

2 Brief review of known results and conjectures

Barbeau [1] proved that if n is not a prime or unity then $D(n) \geq 2\sqrt{n}$ with equality only if $n = p^2, p \in \mathbb{P}$. He showed that, for integers possessing a proper divisor of the form $p^p, p \in \mathbb{P}$, $\lim_{k\to\infty} D^k(n) = \infty$.

Ufnarovski and Åhlander [3] translated some famous conjectures in number theory (e.g., the Goldbach conjecture, the Prime twins conjecture) into conjectures about the arithmetic derivative. They formulated many other conjectures, mostly related to (arithmetic) differential equations; for example that the equation $x' = 1$ has only primes as positive rational solutions (but it has a negative rational solution $x = -\frac{5}{4}$).

They also conjectured that the equation $n'' = n$ has no other solutions than $n = p^p, p \in \mathbb{P}$ in natural numbers and that there are some rational numbers without antiderivatives (or "integrals"). I study these conjectures in Section 3.

2.1 Arithmetic differential equations and integrals

Most of the known results by Ufnarovski and Åhlander [3] focus on arithmetic differential equations of the first and second order. For example:

The only solutions to the equation $n' = n$ in natural numbers are $n = p^p$, where p is any prime.

The nonzero solutions to $x' = 0$ are the rational numbers of the form: $x = \pm \prod p_i^{\alpha_i p_i}$, where $p_1, \ldots, p_k$ are different primes and $\{\alpha_1, \ldots, \alpha_n\}$ is a set of integers such that $\Sigma_{i=1}^{k} \alpha_i = 0$.

The rational solutions to differential equations $x' = x\alpha$ for all rational numbers $\alpha = a/b$, where $\gcd(a, b) = 1$ and $b > 0$, are all of the form $x = x_0 y$, where x_0 is a nonzero particular solution and y is any rational solution of the equation $y' = 0$.

Let $I(a)$ denote all the solutions of differential equation $n' = a$ for $n \in \mathbb{N}$ and let $i(a)$ denote the number of such solutions, called the integrals of n. Because $n' \geq 2\sqrt{n}$ if n is not a prime or unity [1], the solutions satisfy $n \leq \frac{a^2}{4}$, hence $i(a) < \infty$ for any $a \in \mathbb{N}$. Because $\gcd(n, n') = 1$ if and only if n is square-free, all integrals of primes are products of different primes: $p_1 \cdots p_k$. It will be seen (Corollary 21) that there are primes without integrals (e.g., primes 2, 3, 17). If $D(n)$ were known for each $n \in \mathbb{N}$, one would know which natural numbers are primes (because the equation $n' = 1$ has only primes as solutions in natural numbers). Ufnarovski and Åhlander [3] gave a list of all $a \leq 1000$ having no integral, a list of those numbers $a \leq 100$ having more than one integral, and a list of those $a \leq 100$ for which $i(a) = 1$. It will be seen (Corollary 25) that something can also be said about the possible factorization structure of the integrals of a given natural number.

3 New results

Ufnarovski and Åhlander [3] solved the equation $x' = \alpha x$ in rational numbers for every rational number α. Nonetheless it is interesting to know which natural numbers solve this equation when $\alpha = m$ is a natural number.

Proposition 2. *Let $m \in \mathbb{N}$. The solutions to the equation $x' = mx$ in natural numbers x are exactly the numbers of the form $x = p_1^{p_1 n_1} \cdots p_k^{p_k n_k}$, where $n_1 + \cdots + n_k = m$.*

Proof. Let $x = p_1^{e_1} \cdots p_k^{e_k}$ be the factorization of x. The condition $x' = L(x)x = mx$ implies $L(x)(p_1 \cdots p_k) = (\frac{e_1}{p_1} + \cdots + \frac{e_k}{p_k})(p_1 \cdots p_k) = m(p_1 \cdots p_k)$. Because the right side of this equation is divisible by any of the primes $p_1, \ldots, p_k$, the left side must also be. This is possible only if for every prime p_i the corresponding exponent e_i is the multiple of this prime $e_i = n_i p_i$. The derivative of such a number $x = p_1^{p_1 n_1} \cdots p_k^{p_k n_k}$ is $D(x) = D(p_1^{p_1 n_1} \cdots p_k^{p_k n_k}) = (n_1 + \cdots + n_k)x$ (by the Leibnitz rule and because $D(p^{p_i n_i}) = n_i p^{p_i n_i}$) and this is equal to mx if and only if $n_1 + \cdots + n_k = m$. □

3.1 The homogeneous differential equation of the k'th order

What are the solutions x of the differential equation

$$a_k x^{(k)} + a_{k-1} x^{(k-1)} + \cdots + a_2 x^{(2)} + a_1 x' + a_0 x = 0$$

with rational coefficients a_i? In order to answer this question I introduce the concept of a logarithmic class.

Definition 3. *Let A be any chosen subring of the ring of complex numbers for which the arithmetic derivative is defined. The logarithmic class $N_{r,A}$ of the number r consists of all numbers $x \in A$ with the same logarithmic derivative: $N_{r,A} = \{x \in A; x^* = L(x) = r\}$.*

Remark 4. If it is made perfectly clear which A is being worked with the shorter notation N_r can be used. For the purpose of this article, let A be the set of all complex numbers with rational real and imaginary parts: $A = \mathbb{Q}[i] = \{a + bi, a, b \in \mathbb{Q}\}$.

Example 5. Besides the number 0 and the complex numbers $z = e^{i\varphi} \in A$ on the unit circle in the Gaussian plane the class N_0 also contains rational solutions $x = \pm \prod p_i^{\alpha_i p_i}$, where $p_1, \ldots, p_k$ are different primes and $\{\alpha_1, \ldots, \alpha_n\}$ is a set of integers such that $\Sigma_{i=1}^k \alpha_i = 0$; for instance, $x = \frac{4}{27}$ [3]. All $x \in N_0$ solve the equation $x' = 0$.

The class $N_{1,\mathbb{Q}}$ contains all numbers p^p, where p is any prime because $D(p^p) = p^p$. The class $N_{-1,\mathbb{Q}}$ contains all numbers p^{-p}, where p is any prime because $D(p^{-p}) = -p^{-p}$. Do $N_{1,\mathbb{Q}}$ and $N_{-1,\mathbb{Q}}$ also contain other rational solutions? Yes:

Proposition 6. *If $(\frac{a}{b})' = \pm\frac{a}{b} \in \mathbb{Q}$ and if a and b have no common factors greater than 1, then $a = p_1^{p_1 n_1} \cdots p_k^{p_k n_k}$, $b = q_1^{q_1 s_1} \cdots q_k^{q_k n_s}$, and $a^* - b^* = \pm 1$, where n_i, m_j, k, l are nonnegative integers and a^*, b^* logarithmic derivatives. All such numbers are solutions because: $(\frac{a}{b})' = (\frac{a}{b})^*(\frac{a}{b}) = \pm(\frac{a}{b})$.*

Proof. If $(\frac{a}{b})' = \pm\frac{a'b - b'a}{b^2} = \pm\frac{a}{b}$ then $a'b - b'a = \pm ab$, hence $\gcd(a, b) = 1$ implies $a' = ma$ and $b' = nb$, where $m = a^*$ and $n = b^*$ are natural numbers. Hence (by Proposition 2) $a = p_1^{p_1 n_1} \cdots p_k^{p_k n_k}$, where $n_1 + \cdots + n_k = m$, and $b = q_1^{q_1 s_1} \cdots q_k^{q_k n_s}$, where $s_1 + \cdots + s_k = n$. Because $(\frac{a}{b})' = \pm\frac{a}{b}$ implies $(\frac{a}{b})^* = \pm 1$, it must be $a^* - b^* = \pm 1$. □

Proposition 7. *i) The derivative D sends logarithmic classes into logarithmic classes as follows: $D(N_r) \subseteq N_{r^*+r}$. Consequently $D(N_r) \subseteq N_r$ if and only if $r^* = 0$ and therefore $r' = 0$.*

ii) If $r^* + r = 0$ and $r \neq 0$ then $D(N_r) = N_0 - \{0\} \neq N_0$. So in this case $D(N_r) \supseteq N_{r^*+r}$ is not true.

iii) If $r^* + r \neq 0$ then $D(N_r) = N_{r^*+r}$.

Proof. If $r = 0$ then $N_r = N_0 = N_{r^*+r}$. So let us now assume that $r \neq 0$.

i) $D(N_r) \subseteq N_{r^*+r}$ is true because $x \in N_r$ implies $x' = rx$, hence $(x')' = r'x + rx' = \frac{r'x'}{r} + rx' = (\frac{r'}{r} + r)x' = (r^* + r)x'$.

ii) It is possible indeed that $r^* + r = 0$ while $r \neq 0$ (an example is $r = \frac{1}{5}, r' = -\frac{1}{5^2}, r^* = -\frac{1}{5}$). If $r^* + r = 0$ then $0 \in N_{r^*+r}$ because $0' = 0$. Now suppose there is an $x \in N_r$ such that $D(x) = 0$. Wowever, this implies $x \in N_0$, hence $r = 0$ and there is a contradiction with the assumption $r \neq 0$. This means that in this case it is not true $D(N_r) \supseteq N_{r^*+r} = N_0$ because there is no $x \in N_r$ such that $D(x) = 0$.

Now let it be proved that for each nonzero $y \in N_{r^*+r}$ there is an $x \in N_r$ such that $D(x) = y$. If there is any such $x \in N_r$, then it must be $y = x' = rx$, hence there is at most one such x and it is defined by the formula $x = \frac{y}{r}$. Now suppose the derivative of this x is not equal to y. This assumption leads to a contradiction because $D(\frac{y}{r}) = \frac{D(y)r - yD(r)}{r^2} = \frac{(r^*+r)yr - yr^*r}{r^2} = \frac{yr^2}{r^2} \neq y$ implies $1 \neq 1$. Because $D(x) = y = rx$, this x is indeed a member of N_r. Thus it has been proved that $D(N_r) \supseteq N_0 - \{0\}$.

Because it is already known that i) is true and because it has been shown that there is no $x \in N_r$ such that $D(x) = 0$, the equation $D(N_r) = N_0 - \{0\}$ holds. Moreover, because $0 \in N_0$, it is also true that N_0 is a proper subset of $N_0 - \{0\}$.

iii) If $r^* + r \neq 0$ then $r \neq 0$, and N_{r^*+r} contains only nonzero elements because $0 \in N_0$. Now it is possible to repeat the reasoning as in ii) and for any $y \in N_{r^*+r}$ one finds an $x \in N_r$ such that $y = D(x)$. Hence $D(N_r) \supseteq N_{r^*+r}$ and this together with i) implies $D(N_r) = N_{r^*+r}$. □

Remark 8. It is already known that there are many rational solutions to the equation $r' = 0$; for example, $r = 1, -1, 0$, hence: $D(N_1) \subseteq N_1, D(N_{-1}) \subseteq N_{-1}$, $D(N_0) \subseteq N_0$.

Proposition 9. *All the derivatives $x^{(k)}$ of any number x can be expressed as functions of the logarithmic derivative $L(x) = x^*$ as follows: $x^{(k)} = f_k(x^*)x$ where $f_1(x^*) = x^*$ and $f_{k+1}(x^*) = ((f_k(x^*))' + f_k(x^*))x^*$. Thus $x' = x^*x, x^{(2)} = ((x^*)' + (x^*)^2)x$ etc.*

Proof. This is true for $k = 1$. Suppose $x^{(k)} = f_k(x^*)x$. Then by the Leibnitz rule: $x^{(k+1)} = (f_k(x^*))'x + f_k(x^*)x' = ((f_k(x^*))' + f_k(x^*)x^*)x$. □

Proposition 10. *Any homogeneous differential equation $f(x) \equiv a_k x^{(k)} + a_{k-1}x^{(k-1)} + \cdots + a_2x^{(2)} + a_1x' + a_0x = 0$ reduces to an equation $g(x^*) = 0$. Consequently, if the set of nontrivial solutions to any homogeneous differential equation $f(x) = 0$ is not empty, then it consists of some classes N_r.*

Proof. Let $r = x^* = \frac{x'}{x}$. Because $x^{(k)} = f_k(x^*)x$, one can divide the differential equation by x and get an equation of the form $g(r) = 0$. Thus $f(x) = 0$ if and only if $g(x^*) = 0$. Hence whether $f(x) = 0$ or not depends only on the logarithmic class $N_r = N_{x^*}$. Note also that the degree of the polynomial g is $k - 1$, one less than the degree of the polynomial f. □

Example 11. The nonzero solutions to the equation $x'' - x = 0$ implying $((x^*)' + (x^*)^2)x - x = 0$ satisfy the equation $((x^*)' + (x^*)^2 - 1 = 0$ or $(x^*)' = 1 - (x^*)^2$. Thus the nonzero solutions to $x'' - x = 0$ exist if and only if the nonhomogeneous equation $r' = 1 - r^2$ can be solved.

Proposition 12. *For every $r \in \mathbb{Q}$ the class N_r is not empty because it contains the numbers p^{pr}, where p is any prime.*

Proof. $L(p^{pr}) = r$ for any prime p, hence $p^{pr} \in N_r$. $\square$

Thus in solving the homogenous differential equation $f(x) = 0$ one can always search for the solutions of the form $x = p^{pr}$. Of course, other solutions are also possible.

Remark 13. Numbers p^{pr} behave in the first derivative just like the exponential function e^{xr} whose derivative is re^{xr}. However, for the higher derivatives the analogy is no longer valid because $(p^p r)'' = (r' + r^2)p^{pr}$, while $(e^{xr})'' = r^2 e^{xr}$. This is one of the reasons why solving an arithmetic differential equation is harder than solving the analogous problem for functions.

Proposition 14. *i) In addition to the trivial solutions $x \in N_0$ the equation $x'' = 0$ also has solutions $x \in N_{1/p}$, where p is any prime.*

ii) All solutions $x \in N_r$ of $x'' = 0$ satisfy the condition: $r^* + r = 0$ (hence $r \in N_{-r}$ and $r' = r^* r = -r^2$) and all such x are solutions to $x'' = 0$. If $r = \frac{a}{b} \in \mathbb{Q}$ then $\frac{a}{b} = b^* - a^* = (\frac{b}{a})^*$, hence $(\frac{b}{a})' = 1$.

Proof. i) If $x' = \frac{1}{p}$ then $x'' = -\frac{1}{p^2}x + \frac{1}{p}\frac{1}{p}x = 0 = (x')^* x'$, hence $(x')^* = 0$.

ii) Any solutions $x \in N_r$ of $x'' = 0$ satisfy the condition $D^2(N_r) \subseteq D(N_{r^*+r}) = D_0$, hence $r^* + r = 0$. Conversely, $r^* + r = 0$ implies $D^2(N_r) \subseteq D(N_{r^*+r}) = D(N_0) = \{0\}$, hence $x'' = 0$. If $r = \frac{a}{b}$ then $r' = \frac{a'b - ab'}{b^2} = \frac{a'b-ab'}{ab}\frac{a}{b}$, hence $(\frac{a}{b})^* = \frac{a'b-ab'}{ab} = -r = -\frac{a}{b}$, therefore $\frac{a'}{a} - \frac{b'}{b} = -\frac{a}{b}$ and $\frac{a}{b} = b^* - a^* = (\frac{b}{a})^*$. Hence $(\frac{b}{a})' = \frac{b}{a}^* \frac{b}{a} = \frac{a}{b}\frac{b}{a} = 1$. $\square$

Example 15. Ufnarovski and Åhlander [3] observed that if $x = -\frac{5}{4}$ then $x' = 1$ (hence $x'' = 0$). Then $r = x^* = -\frac{4}{5}$, $r' = \frac{-4\cdot 5 - (-4)\cdot 1}{5^2} = -\frac{16}{25} = \frac{4}{5}\frac{-4}{5}$, hence $r^* = \frac{4}{5}$ and $r^* + r = \frac{4}{5} + (-\frac{4}{5}) = 0$. Hence all $x \in N_{-\frac{4}{5}}$ solve $x'' = 0$.

3.2 The graph of derivatives of natural numbers

Of interest is the structure of the infinite directed graph G_D, whose vertices correspond to natural numbers n and whose arcs $n \to D(n)$ connect the number and its derivative. The corresponding dynamic system: $n \to D(n)$ has two obvious attractors: 0 and ∞. There are numbers n with an increasing sequence of derivatives $n < n' < n'' < \cdots < n^{(k)} < n^{(k+1)} < \cdots$ (e.g., $n = p^{pk}, p \in \mathbb{P}$), so there are paths of infinite length in the graph G_D.

Ufnarovski and Åhlander [3] conjectured that the equation $n^{(k)} = n$ has only trivial solutions p^p, where $p \in \mathbb{P}$, satisfying $n' = n$. If this is true then the only cycles in G_D are the loops in these fixed points. They have shown that if $m' = n$ and $n' = m$ then m and n must be square-free numbers: $n = \prod_{i=1}^{k} p_i$ and $m = \prod_{j=1}^{l} q_j$, where all p_i are distinct from all q_i. I present further constraints (Propositions 16, 17, 18, 19) on the structure of possible solutions of the equation $n'' = n$ in natural numbers.

However, the equation $x'' = x$ has non-trivial rational solutions of the form $x = p^{-p}$ where $p \in \mathbb{P}$ and they also satisfy the equation $x' = -x$.

Let us first show that any eventual nontrivial solutions m, n (different from $m = n = p^p$ where $p \in \mathbb{P}$) of the system $n' = m$, $m' = n$ in natural numbers cannot be just a product of two primes; at least one of the numbers $m, n \in \mathbb{N}$ solving such a system must have at least three different prime factors.

Proposition 16. *The system $n' = m$, $m' = n$ has no solutions in natural numbers of the form $m = p_1p_2$ and $n = q_1q_2$, where $p_1, p_2, q_1, q_2 \in \mathbb{P}$.*

Proof. Because p_1, p_2, q_1, q_2 must be different primes, it is not possible for both m and n to have the factor 2. Therefore one can assume that p_1 and p_2 are both odd. Then $m' = p_1 + p_2 = n = 2q_2$ and $n' = 2 + q_2 = m = p_1p_2$, hence $2q_2 = 2p_1p_2 - 4 = p_1 + p_2$ and $p_1 = \frac{p_2+4}{2p_2-1} = 1 + \frac{5-p_2}{2p_2-1}$, and this implies $p_2 \le 5$ because p_1 must be a natural number. Moreover, it was assumed that p_2 is odd. However, for $p_2 = 3$ one would get $p_1 = 1 + \frac{2}{5} = \frac{7}{5}$ and for $p_2 = 5$ one would get $p_1 = 1$. $\square$

A computer search showed that there are no natural solutions to $x'' = x$ such that $x < 10000$. This result can be improved, at least if the smaller of the numbers m and n is odd.

Proposition 17. *Let $n' = m$, $m' = n$ and let $n = 2j - 1 < m$. Then there are at least nine primes in the factorization of n and $n > 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$.*

Proof. It is known that m and n must be products of different primes $n = \prod_{i=1}^{k} p_i$ and $m = \prod_{j=1}^{l} q_j$, where all p_i are distinct from all q_i. Because $n < m = n' = L(n)n$ we have $L(n) = \sum_{i=1}^{k} \frac{1}{p_i} > 1$. Because n is odd, all p_i are greater than 2. The sum of the reciprocals of the first eight odd primes is: $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} = 0.9987\cdots < 1$. Thus $k \ge 9$ and $n > 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$. $\square$

If $n = 2j < m$ then the sum $\frac{1}{2} + \frac{1}{3} + \frac{1}{5}$ is already greater than 1 and one does not get any such estimate.

Another possible approach to the system $n' = m, m' = n$ in natural numbers is based on the comparison of a square-free number and its derivative; this comparison will imply some inequalities for the smallest primes and biggest primes in the factorizations of n and m.

Proposition 18. *Let $n = p_1 \cdots p_r$, where $p_1 < \cdots < p_r$, and let $m = q_1 \cdots q_s$ where $q_1 < \cdots < q_s$ be square-free numbers such that $n' = m, m' = n$. Then: $p_1q_s < rs \le (\frac{r+s}{2})^2$, $q_1p_r < rs \le (\frac{r+s}{2})^2$, $p_1 < r$, $q_1 < s$. As a consequence n and m must together have at least 34 prime factors p_i and q_j, thus: $r + s \ge 34$. Hence at least one of m and n has at least 17 prime factors and is not smaller than the product of the first 17 primes: $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 27 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59$. If $\min\{p_1, q_1\} \ge 3$, then $r + s \ge 57$, hence at least one of m and n has at least 29 prime factors. If $\min\{p_1, q_1\} \ge 5$, then $r + s \ge 110$, hence at least one of m and n has at least 55 prime factors.*

Proof. Let $n = p_1 \cdots p_r$ and $m = q_1 \cdots q_s$ be square-free numbers such that $n' = m, m' = n$. Then $0 < r\frac{n}{p_r} < n' < r\frac{n}{p_1}$ and $0 < s\frac{m}{q_s} < m' < s\frac{m}{q_1}$. Hence: $p_1q_s < rs$ and $q_1p_r < rs$. Because $r > 1, s > 1$ we have $p_r > r$ and $q_s > s$, hence $p_1 < r$ and $q_1 < s$. Let $r + s = N$. Then $rs \le (\frac{r+s}{2})^2 = (\frac{N}{2})^2$. Thus $2q_s \le p_1q_s < (\frac{N}{2})^2$ and $2p_r \le q_1p_r < (\frac{N}{2})^2$, hence $2\max\{p_r, q_s\} < \frac{N^2}{4}$.

Let P_i denote the i-th prime (thus $P_1 = 2$, $P_2 = 3$, $P_3 = 5$, etc.) Because all the primes $p_1, \ldots, p_r, q_1, \ldots, q_s$ are distinct, we have $\max\{p_r, q_s\} \ge P_{r+s} = P_N$. Thus $2P_N = 2P_{r+s} \le 2\max\{p_r, q_s\} < rs \le \frac{N^2}{4}$.

It is possible to directly check that $P_N \ge \frac{N^2}{8}$ if $N \le 33$.

N	1	2	3	4	5	6	7	8	9	10	11	12
P_N	2	3	5	7	11	13	17	19	23	27	31	37
N^2	1	4	9	16	25	36	49	64	81	100	121	144

N	13	14	15	16	17	18	19	20	21	22	23
P_N	41	43	47	53	59	61	67	71	73	79	83
N^2	169	196	225	256	289	324	361	400	441	484	529

N	24	25	26	27	28	29	30	31	32	33	34
P_N	89	97	101	103	107	109	113	127	131	137	139
N^2	576	625	676	729	784	841	900	961	1024	1089	1156

Table 2: The first 34 primes p_N and squares N^2

Hence it must be that $N \geq 34$. If $\min\{p_1, q_1\} \geq 3$, one can get a much better estimate from the condition: $p_N < \frac{N^2}{12}$, which is first fulfilled when $N \geq 57$. If $\min\{p_1, q_1\} \geq 5$, then $p_N < \frac{N^2}{20}$, which is first fulfilled when $N \geq 110$. □

Proposition 19. *Suppose $n' = m$ and $n' = m$ and n, m are both odd. Let n have r_1 primes $p_i \equiv 1 \pmod 4$ and s_1 primes $q_j \equiv -1 \pmod 4$ and let m have r_2 primes $p_i \equiv 1 \pmod 4$ and s_2 primes $q_j \equiv -1 \pmod 4$. Then $(r_1 - s_1)(r_2 - s_2) \equiv 1 \pmod 4$.*

Proof. Barbeau [1, pp. 121–122] proved that if $n = p_1 \cdot p_2 \dots p_r \cdot q_1 \cdot p_2 \dots q_s$, where $p_i \equiv 1 \pmod 4$, $q_j \equiv -1 \pmod 4$ are primes, not necessarily distinct, then $D(n) \equiv (-1)^s(r - s) \pmod 4$. Hence $m = n' = (-1)^{s_1}(r_1 - s_1)$ and $n = m' = (-1)^{s_2}(r_2 - s_2)$. Because m and n are odd, $r_1 - s_1$ and $r_2 - s_2$ are not even. However, $n \equiv (-1)^{s_1} \pmod 4$ and $m \equiv (-1)^{s_2} \pmod 4$. Hence $mn \equiv (-1)^{s_1}(-1)^{s_2} \equiv (-1)^{s_1}(-1)^{s_2}(r_1 - s_1)(r_2 - s_2) \pmod 4$. Consequently $(r_1 - s_1)(r_2 - s_2) \equiv 1 \pmod 4$. □

3.3 Integrals of natural and rational numbers

Any a such that $a' = b$ is called an integral of b. The set of all such integrals is denoted as $I(b)$. The same number can have different integrals: $25' = (5^2)' = 2 \cdot 5 = 10$ and $21' = 3 \cdot 7 = 3 + 7 = 10$. Because $p' = 1$ for any prime, $I(1) = \mathbb{P}$. It is shown that 1 is the only natural number with infinitely many integrals among the natural numbers.

Proposition 20. *i) Let $b < 2 \cdot 3 \cdot 5 \cdots P_n$, where P_n is the n-th consecutive prime and let $a' = b$, where $a \in \mathbb{N}$. Then $a = p_1^{n_1} \cdots p_m^{n_m}$, where $m \leq n$ (hence a is divisible by at most n primes p_i).*

ii) If $a' = b > 1$, where $a \in \mathbb{N}$, then $a \leq \max(2 \cdot 3 \cdot 5 \cdots P_n, b^{bn})$. Consequently every $b > 1$ has at most a finite number of integrals $a \in \mathbb{N}$.

Proof. For each natural number b there is a $n \in \mathbb{N}$ such that $b < 2 \cdot 3 \cdot 5 \cdots P_n$. If $a = \prod_{i=1}^{m} p_i^{n_i}$ has more than n different prime factors p_i, then each summand of $a' = (\sum_{i=1}^{m} \frac{n_i}{p_i}) \cdot b$ has at least n prime factors, and the smallest of them is not smaller than $2 \cdot 3 \cdot 5 \cdots P_n$, therefore in that case $a' > 2 \cdot 3 \cdot 5 \cdots P_n > b$, so it cannot be $a' = b$. Thus $a = p_1^{n_1} \cdots p_m^{n_m}$ and $m \leq n$.

ii) If a contains a factor p^p, then $b = a' \geq a$, hence $a \leq b < 2 \cdot 3 \cdot 5 \cdots P_n$. The other possibility is that all exponents of $a = p_1^{n_1} \cdots p_m^{n_m}$ are smaller than their primes: $n_i < p_i$. It is necessary to consider two cases:

If $m = 1$ then $a = p_1^{n_1}$ and $a' = n_1 p_1^{n_1 - 1} = \frac{n_1 a}{q_1} = b$. Now $b > 1$ implies $n_1 \geq 2$ thus p_1 divides b, hence $p_1 \leq b$ and $a = \frac{b p_1}{n_1} \leq b q_1 \leq b^2$.

If $m \geq 2$ then $a = p_1^{n_1} \cdots p_m^{n_m} < p_1^{p_1} \cdots p_m^{p_m} < b^{p_1 + \cdots + p_m} < b^{bn}$ because $p_i < a' = b$ for each p_i.

Hence $a \leq \max(2 \cdot 3 \cdot 5 \cdots P_n, b^{bn})$ and $I(b) \bigcap \mathbb{N}$ is finite for any $b > 1$. $\square$

Corollary 21. *If $a' = p$ and p is a prime, then $a = p_1 \cdots p_m$ and all $p_i < p$. There are some primes without integrals $a \in \mathbb{N}$; for example 2,3,17.*

Proof. If $a = p^2 c$, then $a' = p \cdot (2 + pc')$ is not a prime. If $a = p_1 \cdots p_m$, then $p = a' > p_i$. Because $2 < 2 \cdot 3$ and $3 < 2 \cdot 3$, the only candidates for integrals of 2 and 3 are numbers of the form $a = p \cdot q$, where $p, q \in \mathbb{P}$. However, then $(p \cdot q)' = p + q > 5$, hence 2 and 3 can have no integrals $a \in \mathbb{N}$. Ufnarovski and Åhlander [3] found their integrals in rational numbers: $(-\frac{21}{16})' = 2$, $(-\frac{13}{4})' = 3$. Any integral of 17 has at most 3 different prime factors because $17 < 2 \cdot 3 \cdot 5$. It cannot have only two different prime factors because if $a = pq$ then $a' = p + q$ and the sum of any two primes is not 17. However, if $a = pqr$, then $(pqr)' = pq + pr + qr \geq 2 \cdot 3 + 2 \cdot 5 + 3 \cdot 5 = 31 > 17$. Thus there is no integral of 17. $\square$

The numbers $s \in \mathbb{N}$ not divisible by any square c^2 where $c > 1$ are called square-free numbers. They can be either products of different primes or equal to 1. The set of square-free numbers is denoted $\mathbb{S}$.

Definition 22. *Let $a = p_1^{n_1} \cdots p_m^{n_m}$, where $p_i \in \mathbb{P}$. The number n_i is called the exponent of a prime p_i in a. Let us define the following functions of a:*

$s(a)$ is the greatest square-free divisor of a, such that $\gcd(s(a), \frac{a}{s(a)}) = 1$,

$p(a) = p_1 \cdots p_m$ is the product of all prime factors of a,

$f(a) = \frac{a}{p(a)} = \frac{a}{p_1 \cdots p_m}$, $r(a) = a \cdot p(a) = a \cdot (p_1 \cdots p_m)$,

$h_{\max}(a) = \max\{n_1, \ldots, n_m\}$, $h_{\min}(a) = min\{n_1, \ldots, n_m\}$.

For $a = 1$ we define $s(a) = p(a) = f(a) = r(a) = h_{\max}(a) = h_{\min}(a) = 1$.

From this definition it follows that if $a \neq s(a)$ then $h_{min}(\frac{a}{p(a)}) \geq 2$.

Proposition 23. *i) Let $a \in \mathbb{N}$. Then $a' = f(a)p(a)a^*$, $p(a)a^* \in \mathbb{N}$ and $a = r(f(a))s(a)$.*

ii) If $a \in \mathbb{P}$ then $p(a)a^* = 1$ and $f(a) = a'$. If $a \in \mathbb{N}$ is not a prime then $p(a)a^* > 1$ and $f(a) < a'$.

iii) If $a = p_1^{n_1} \cdots p_m^{n_m}$ and n_i is not divisible by p_i for all $i \in \{1, 2, \ldots, m\}$ then $\gcd(f(a), p(a)a^*) = 1$ and $\gcd(a, a') = f(a)$.

iv) If $h_{\min}(a) \geq 2$, then $s(a) = 1$, hence $a = r(f(a))$. If $h_{\max}(a) = 1$ then $p(a) = a$ and $f(a) = 1$, hence $r(f(a)) = r(1) = 1$.

Proof. i) If $a = 1$ then $a' = a^* = 0$ implies $a' = f(a)p(a)a^*$. If $a = p_1^{n_1} \cdots p_m^{n_m}$ then $a' = aa^* = (\frac{a}{p(a)})p(a)a^*$ and $\frac{a}{p(a)} = f(a) = p_1^{n_1 - 1} \cdots p_m^{n_m - 1}$. Obviously $p(a)a^* = (p_1 \cdots p_m)(\frac{n_1}{p_1} +$

$\cdots + \frac{n_m}{p_m}) \in \mathbb{N}$. Because $f(a)$ is divisible by exactly those primes p_i for which p_i^2 divides a, it is $r(f(a)) = \frac{a}{s(a)}$. Hence $a = r(f(a))s(a)$.

ii) If $a \in \mathbb{P}$ then $p(a) = a$, $a' = 1$, $a^* = \frac{1}{a}$, $p(a)a^* = a\frac{1}{a} = 1$ and $f(a) = 1$. Now suppose $a \in \mathbb{N}$ is not a prime. Then either $a = p_1^{n_1}$, where $n_1 > 1$, or $a = p_1^{n_1} \cdots p_m^{n_m}$, where $m \geq 2$. In the first case $p(a) = p_1$, $f(a) = p_1^{n_1 - 1}$, $a' = n_1 p_1^{n_1-1}$, $a^* = \frac{n_1}{p_1}$, hence $p(a)a^* = p_1 \frac{n_1}{p_1} = n_1 > 1$. In the second case, $p(a)a^* = (p_1 \cdots p_m)(\frac{n_1}{p_1} + \cdots + \frac{n_m}{p_m})$ is a sum of at least two natural numbers, hence $p(a)a^* > 1$. Therefore in both cases $f(a) = \frac{a'}{p(a)a^*} < a'$.

iii) Now $p(a)a^* = (p_1 \cdots p_m)(\frac{n_1}{p_1} + \cdots + \frac{n_m}{p_m})$ is not divisible by any of the primes p_i dividing a because n_i is not divisible by p_i. Hence $p(a)a^* = q_1^{u_1} \cdots q_t^{u_t}$ and $a' = p_1^{n_1-1} \cdots p_m^{n_m-1} q_1^{u_1} \cdots q_t^{u_t}$, where all the primes $p_1, \cdots, p_m, q_1, \cdots, q_t$ are distinct. Hence $\gcd(f(a), p(a)a^*) = 1$ and $\gcd(a, a') = f(a)$.

iv) If $h_{\min}(a) \geq 2$, then $p(a) = p(f(a))$, hence $r(f(a)) = a$. If $h_{\max}(a) = 1$ then $p(a) = a$ and $f(a) = 1$, hence $r(f(a)) = 1$. $\square$

Definition 24. If $a = p_1^{n_1} \cdots p_m^{n_m}$ where all $n_i < p_i$ then a is in the set $\mathbb{L}$ whose elements are called "low" numbers.

Now it is possible to describe the factorization and consequently obtain some bounds of the integrals of "low" numbers.

Corollary 25. *i) If $b \in \mathbb{L}$ then $I(b) \bigcap \mathbb{N} \subset \mathbb{L}$. Moreover, every $a \in I(b) \bigcap \mathbb{N}$ is of the form $a = r(c)s(a)$, where $b = cd$, $c < b$, $\gcd(c, d) = 1$, and $b = a' = r'(c)s(a) + r(c)s'(a)$. Therefore:*

i.a) If $s(a) = a$ then $r(c) = 1$ hence $c = 1$. Conversely, if $c = 1$ then $s(a) = a$.

i.b) If $s(a) = 1$ then $a = r(c)$.

i.c) If $r(c) \neq 1$ then $s(a) \leq \frac{b - r(c)}{r'(c)} \leq \frac{b}{c}$, hence $a \leq r(b) - r(r(c)) \leq r(b) - 1$.

i.d) If $r(c) \neq 1$ and $s(a)$ is a prime then $s(a) = \frac{b - r(c)}{r'(c)}$.

i.e) If $r(c) \neq 1$ and $s(a) = p_1 p_2$ then $\frac{b - r(c)}{r'(c) + r(c)} \leq s(a) \leq \frac{b - 5r(c)}{r'(c)}$.

ii) For every $b = p_1^{n_1} \cdots p_k^{n_k} \in \mathbb{L}$ there are at most $2^k - 1$ different divisors c of b such that there is an integral a of b of the form $a = r(c)s(a)$.

iii) If $b = p_1^{n_1} \cdots p_k^{n_k}$ and $n_i = p_i - 1$ for all i, then any integral $a \in I(b) \bigcap \mathbb{N}$ is a square-free number: $a = s(a)$.

Proof. i) If $a' = b$ and a contains a factor $p_i^{p_i}$ then $a' = b$ is also divisible by $p_i^{p_i}$, too, hence b is not in $\mathbb{L}$. Thus, because $b \in \mathbb{L}$, it must be $a \in \mathbb{L}$. Hence $f(a)$ divides $a' = b$, $f(a) < a'$, and $r(f(a)) = \frac{a}{s(a)}$. Thus one writes $c = f(a)$, one really can get all the integrals of $b \in \mathbb{L}$ in the described form.

i.a) and i.b) are obvious.

i.c) If $r(c) \neq 1$ then $r'(c) \neq 0$ and $s(a) = b - \frac{r(c)s'(a)}{r'(c)} \leq \frac{b - r(c)}{r'(c)} = \frac{b}{(cp(c))'} = \frac{b}{c'p(c) + cp'(c)} \leq \frac{b}{c}$. Hence $a = r(c)s(a) \leq c \cdot p(c)\frac{b - r(c)}{c} \leq p(b)b - p(c)r(c) = r(b) - r(r(c)) \leq r(b) - 1$ because $p(c) = p(r(c))$.

i.d) If $r(c) \neq 1$ and $s(a)$ is a prime, then $s'(a) = 1$ and $s(a) = \frac{b - r(c)}{r'(c)}$.

i.e) If $r(c) \neq 1$ and $s(a) = p_1p_2$ then $s'(a) = p_1 + p_2 \leq 2 + 3 = 5$ hence $s(a) = \frac{b-5r(c)}{r'(c)}$. The inequality $\frac{b-r(c)}{r'(c)+r(c)} \leq s(a)$ follows from the fact that $p_1 + p_2 \leq p_1p_2$ for any two primes p_1 and p_2 implying $a' = (r(c)p_1p_2)' = r'(c)p_1p_2 + r(c)(p_1 + p_2) \leq (r'(c) + r(c))p_1p_2$.

ii) Because each factor $p_i^{n_i}$ of $b = cd = p_1^{n_1} \cdots p_k^{n_k}$ belongs either to $c = f(a)$ or $d = \frac{b}{f(a)}$, and because $c < a'$, there are at most $2^k - 1$ possible factors c dividing $b = a'$ such that $a = r(c)s(a)$. However, different integrals of b may correspond to the same c, as in the example $(3 \cdot 13)' = (5 \cdot 11)' = 16$, in which both integrals $3 \cdot 13$ and $5 \cdot 11$ correspond to $c = 1$.

iii) If $p_i = n_i - 1$ for all i, then $r(c) = 1$ or $r(c)$ contains at least one factor $p_i^{p_i}$, but then this is a contradiction because in that case $a = b'$ would also be divisible by $p_i^{p_i}$. Hence $r(c) = 1$ and $a = s(a)$. $\square$

Example 26. Let $b = 5^2 11$. There are three possible factorizations $b = cd$ such that $\gcd(c, d) = 1$ and $1 \leq c < b$, corresponding to numbers $c_1 = 1$, $c_2 = 5^2$ and $c_3 = 11$. Thus the candidates for $a \in I(b) \bigcap \mathbb{N}$ are:

square-free numbers $a = r(1)s(a) = s(a)2 \cdot 5^2 11$, not divisible by 5 or 11,

numbers of the form: $a = r(5^2)s(a) = 5^3 s(a)$, where $s(a) \leq \frac{b}{r'(c)} = \frac{5^2 11}{3 \cdot 5^2} = \frac{11}{3} \leq 4$, hence $s(a) \in \{1, 2, 3\}$, and

numbers of the form $a = r(11)s(a) = 11^2 s(a)$, where $s(a) \leq \frac{b}{r'(c)} = \frac{5^2 11}{2 \cdot 11} = \frac{5^2}{2}$, hence $s(a) \in \{1, 2, 3, 5, 6, 7, 10, 11\}$.

It can easily be seen that $s(a) > 1$ because $s(a) = 1$ implies $r'(c) = b$, but it is $1' = 0 \neq b$, $(5^3)' = 3 \cdot 5^2 \neq b$ and $(11^2)' = 2 \cdot 11 \neq b$. Thus in the case $c_2 = 5^2$ it must be $s(a) \in \{2, 3\}$ and in the case of $c_3 = 11$ it is seen that $s(a) \in \{2, 3, 5, 6, 7, 10, 11\}$ is either a prime or a product of two primes.

Hence in the case $a = 5^3 s(a)$ one can use the formula for the prime $s(a) = \frac{b-r(c)}{r'(c)} = \frac{5^2 11 - 5^3}{3 \cdot 5^2} = \frac{11-5}{3} = 2$ and one can verify directly that $(2 \cdot 5^3)' = 5^3 + 2 \cdot 3 \cdot 5^2 = 5^2(5 + 6) = b$.

Now let $a = 11^2 s(a)$. If $s(a)$ is a prime one gets $s(a) = \frac{b-r(c)}{r'(c)} = \frac{5^2 11 - 11^2}{2 \cdot 11)} = \frac{25-11}{2} = 7$ and one can check directly that $(11^2 \cdot 7)' = 2 \cdot 11 \cdot 7 + 11^2 = 11(14 + 11) = b$. It can easily be seen that $s(a)$ cannot be a product of two primes because that would imply a contradiction: $0 \leq s(a) \leq \frac{b-5r(c)}{r'(c)} = \frac{5^2 11 - 5 \cdot 11^2}{2 \cdot 11} < 0$.

Square-free integrals corresponding to $c_1 = 1$ are more difficult to find. It is necessary to check the derivatives of all square-free numbers $a \leq \frac{b^2}{4}$ and compare them with b.

A similar estimate as in Proposition 17 can be made about positive solutions to the equation $(\frac{a}{b})' = \frac{2}{p}$, where $p \in \mathbb{P}$.

Definition 27. Let P_{i+1} denote the i-th odd prime. For any $d \in \mathbb{R}$ let $O_d = 1 \cdot \prod_{i=1}^{m} P_{i+1}$ denote the product of the first m odd primes such that the sum of their reciprocals, denoted $R(d)$, is not smaller than d.

Thus $R(d) = \sum_{i=1}^{m} \frac{1}{P_{i+1}} \geq d$. Because the series $\sum_{i=1}^{\infty} \frac{1}{P_{i+1}}$ diverges, O_d is well defined for any real d.

Example 28. If $d \leq \frac{1}{3}$ then $O_d = P_2 = 3$. If $\frac{1}{3} \leq d \leq \frac{1}{3} + \frac{1}{5}$ then $O_d = P_2P_3 = 3 \cdot 5 = 15$. If $\frac{1}{3} + \frac{1}{5} \leq d \leq \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$ then $O_d = P_2P_3P_4 = 3 \cdot 5 \cdot 7 = 105$.

Proposition 29. *Let $(\frac{a}{b})' = \frac{2}{p}$, where $p \in \mathbb{P}$, $p > 2$, $\gcd(a,b) = 1$, $\frac{a}{b} > 0$. Then $a = p_1 \cdots p_m$ is an odd square-free number with $m \geq 9$ prime factors $p_i \in \mathbb{P}$ and b is of the form $b = q_1^{q_1 n_1} \dots q_s^{q_s n_s}$ where $q_i \in \mathbb{P}$ and $n_i \in \mathbb{N}$. Moreover, $L(b) \in \mathbb{N}$, $L(a) > L(b) \geq 1$ and $a' > a > O_1 \geq 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$.*

Proof. $(\frac{a}{b})' = \frac{2}{p}$ implies $\frac{a'b-ab'}{b^2} = \frac{2}{p}$, hence $b = pc$, where $c \in \mathbb{N}$. Therefore $a'pc - ab' = 2c^2p$ and because a and b have no common prime factors, we have $b' = kc$. Hence $a'pc - akc = 2c^2p$ and by dividing this equation by c one gets $a'p - ak = 2cp$, hence $k = pd$ because p divides b so it cannot divide a. This implies $a'p - apd = 2cp$, hence $a' - ad = 2c$, and $b' = pdc = bd$. Thus $d = L(b) = b^*$. By Proposition 2, $b = q_1^{q_1 n_1} \cdots q_s^{q_s n_s}$, where $n_1 + \cdots + n_s = d$.

Because $\frac{a}{b} > 0$, it can be assumed $a > 0$ and $b > 0$. It must be $a > 1$ because $a = 1$ implies $a' = 0$ and $-d = 2c$, implying a contradiction: $0 = 2c + d > 0$.

From the equation $a' - ad = 2c$ follows $\gcd(a, a') = 1$, because any such common prime factor different from 2 would also divide c and b and this would contradict $\gcd(a,b) = 1$, and if 2 divides a and a' then it must be $a = 4e$, hence $a' = 4(e + e')$ and $4(e' + e - 4ed) = 2c$ would imply that 2 also divides b.

Now it can be seen that a cannot be divisible by 2. In that case 2 would also divide a' or p. However, this is impossible because $\gcd(a', a) = 1$ and $p > 2$.

Because $\gcd(a, a') = 1$, a must be a square-free number. Because $a > 1$, it must be: $a = p_1 \cdots p_m$, and because $a' - ad = 2c > 0$ it is $a' > ad \geq a$, hence $L(a) = \frac{a'}{a} = \sum_{i=1}^{m} \frac{1}{p_i} > d \geq 1$. Therefore $a > O_d \geq O_1 \geq 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$ and $m \geq 9$, because the sum of the reciprocals of the first eight odd primes is less than 1: $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} = 0.9987 \dots < 1$. □

Ufnarovski and Åhlander [3] conjectured that some rational numbers have no integrals. The above proposition shows that the integrals of $\frac{2}{3}$ are not easy to find, if they exist at all. The same holds for any $\frac{2}{p}, p \in \mathbb{P}$.

In Proposition 7.iii) I showed that if $y \in N_{y^*} = N_{r+r^*}$ and if $r + r^* \neq 0$ then there is exactly one $x \in N_r$ such that $x' = y$ and this is $x = \frac{y}{r}$. Hence to find a rational integral x of any nonzero rational number y one just has to find a rational number $r = x^*$ such that $r + r^* = y^*$. In other words, one has to find a logarithmic class N_r such that $D(N_r) = N_{y^*}$. The equation $r + r^* = y^*$ translates into $r^2 - y^* r + r' = 0$. Perhaps for some rational numbers y this equation cannot be satisfied by any rational number r, hence such y cannot have a rational integral x. Expressing r as a function of its derivative r', one gets at most two different solutions $r_{1,2} = \frac{y^* \pm \sqrt{(y^*)^2 - 4r'}}{2}$. Now $r \in \mathbb{Q}$ implies $(y^*)^2 - 4r' \geq 0$ and $r' \leq \frac{(y^*)^2}{4}$. Because $x \in \mathbb{Q}$ implies $r = x^* \in \mathbb{Q}$ the expression $(y^*)^2 - 4r' = q^2$ must be a square of a rational number q.

Example 30. If $y = \frac{2}{3}$ then $y^* = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. If there is a positive rational number $x = \frac{a}{b} \in N_r$ such that $x' = y = \frac{2}{3}$ then $r = x^* = (\frac{a}{b})^* = \frac{y}{x} = \frac{2}{3x} > 0$ satisfies the equation $r + r^* = y^* = \frac{1}{6}$. This equation translates into $r^2 - \frac{1}{6}r + r' = 0$. For each r' one gets at most two different solutions $r_{1,2} = \frac{y^* \pm \sqrt{(y^*)^2 - 4r'}}{2} = \frac{\frac{1}{6} \pm \sqrt{\frac{1}{36} - 4r'}}{2} = \frac{1}{12} \pm \sqrt{\frac{1}{144} - r'}$. Hence $r' \leq \frac{(y^*)^2}{4} = \frac{1}{144}$.

It is known from Proposition 29 that $r = a^* - b^* = \sum_{i=1}^{m} \frac{1}{p_i} - d > 0$, where $d \in \mathbb{N}$. Thus $x = \frac{y}{r} > 0$. Now $r' = (\frac{2}{3x})' = \frac{3x-6}{9x^2} = \frac{x-2}{3x^2} = \frac{x-2}{3x^2} \frac{3x}{2} \frac{2}{3x}$, hence $r^* = \frac{x-2}{2x}$. If $x \leq 2$ then $r^* \leq 0$

and $r = y^* - r^* \le y^* = \frac{1}{6}$ leads to a contradiction: $x = \frac{y}{r} \ge \frac{y}{y^*} = \frac{\frac{2}{3}}{\frac{1}{6}} = 4$. Thus it must be $x > 2$, hence $r^* > 0$. Now $0 < r = y^* - r^* < y^* = \frac{1}{6}$ implies $x = \frac{y}{r} > \frac{y}{y^*} = \frac{\frac{2}{3}}{\frac{1}{6}} = 4$, hence $a > 4b$. Thus there is no positive rational number x such that $x \le 4$ and $x' = \frac{2}{3}$.

3.4 Partial derivatives and partial differential equations

Definition 31. *Let $a = \prod_{i=1}^{k} p_i^{x_i}$ be the factorization of $a \in \mathbb{N}$ into primes. The partial derivative $\frac{\partial a}{\partial pi} = D_{p_i}(a)$ is defined as $\frac{\partial a}{\partial p_i} = a\frac{x_i}{p_i}$. If $p \in \mathbb{P}$ is not in the factorization of a, then $\frac{\partial a}{\partial p_i} = 0$.*

From this definition it immediately follows that $D(a) = a\sum_{i=1}^{k} \frac{\partial a}{\partial p_i}$. One can also define higher partial derivatives; for example: $\frac{\partial^2 a}{\partial p_i \partial p_j} = D^2_{p_i p_j} = \frac{\partial(\partial a/\partial p_j)}{\partial p_i}$, $\frac{\partial^2 a}{\partial p_i \partial p_j} = D^2_{p_i^2} = \frac{\partial(\partial a/\partial p_i)}{\partial p_i}$, etc., and study partial differential equations.

Example 32. $D_2(2^3 \cdot 5^4) = 3 \cdot 2^2 \cdot 5^4$, $D_5(3 \cdot 2^2 5^4) = 3 \cdot 2^2 \cdot 4 \cdot 5^3 = 3 \cdot 2^4 \cdot 5^3$
$D_5(2^3 \cdot 5^4) = 2^3 \cdot 4 \cdot 5^3 = 2^5 \cdot 5^3$ and $D_2(2^5 \cdot 5^3) = 5 \cdot 2^4 \cdot 5^3$. Thus the order of applying D_p and D_q is important: $D_p D_q$ is not always equal to $D_q D_p$.

Proposition 33. *If $a = p^e q^f$ and $\gcd(e, q) = 1$ and $\gcd(f, p) = 1$, then $D_p D_q(a) = D_q D_p(a)$.*

Proof. In that case it is $D_p D_q(p^e q^f) = D_p(p^e f q^{f-1}) = e p^{e-1} f q^{f-1}$ and $D_q D_p(p^e q^f) = D_q(e p^{e-1} q^f) = e p^{e-1} f q^{f-1}$. ☐

Proposition 34. *Let $n = p^e q^f c$, where p, q are primes not dividing c. Then $D_p(p^e q^f c) = D_q(p^e q^f c)$ if and only if $e = kp$ and $f = kq$, where $k \in \mathbb{N}$.*

Proof. From $D_p(p^e q^f c) = e p^{e-1} q^f c = f p^e q^{f-1} c = D_q(p^e q^f c)$ follows $eq = bp$, hence $e = kp$ and $f = kq$. Then $D_p(p^{kp} q^{kq} c) = k(p^{kp} q^{kq} c) = D_q(p^e q^f c)$. ☐

Proposition 35. *i) The only solutions to the partial differential equation $D^2_{p^2}(n) = n$ in natural numbers are $n = p^{pk} c$, where $\gcd(c, p) = 1$.*

ii) If $n = p^e d$ where $\gcd(p, d) = 1$ and $e \ge p$ then $D^2_{p^2}(n) \ge n$.

iii) The only solutions to the partial differential equation $(D^2_{p^2} + D^2_{q^2})(n) = n$ in natural numbers are $n = p^p c$ and $n = q^q c$, where $\gcd(pq, c) = 1$.

Proof. i) Let $n = p^e c$, where $e \in \mathbb{N} \bigcup \{0\}$ and $c \in \mathbb{N}$ is not divisible by p.

If $e = 0$ then $D_p(n) = 0$, hence $D^2_{p^2}(n) = 0$. If $e = kp$ and $k \in \mathbb{N}$ then $D_p(n) = D_p(p^{pk} c) = pkp^{pk-1} c = p^{pk} c = n$ hence $D^2_{p^2}(n) = n$. If $e = 1$ then $D_p(n) = c$, hence $D^2_{p^2}(n) = D_p(c) = 0$.

ii) If $e = kp$ and $k \in \mathbb{N}$ then $D^2_{p^2}(n) = n$, as is already known from i). If $\gcd(e, p) = 1$ and $e > p$ then $D_p(n) = D_p(p^e c) = e p^{e-1} c$, hence $D^2_{p^2}(n) = D_p(e p^{e-1} c) = e(e-1) p^{e-2} c \ge (p+1) p p^{e-2} c > p^e c = n$.

iii). One can write $n = p^e q^f c$, where $\{e, f\} \subseteq \mathbb{N} \bigcup \{0\}$ and $c \in \mathbb{N}$ is not divisible by p or q. So it is necessary to study the equation $(D^2_{p^2} + D^2_{q^2})(p^e q^f c) = p^e q^f c$. If $f = 1$ then $D^2_{q^2}(p^e c) = 0$ and one gets the equation $D^2_p(n) = n$ whose only solution is $n = p^p c$, as is known from i). Likewise if $e = 1$ one gets $n = q^q c$.

In the case of $e \geq 2$ and $f \geq 2$, one can use the following argument:

Obviously $\gcd(qc, p) = 1$ implies that $D^2_{p^2}(n)$ is divisible by $q^f c$. Likewise $\gcd(pc, q) = 1$ implies that $D^2_{q^2}(n)$ is divisible by $p^e c$. So one can write $D^2_{p^2}(n) = aq^f c$, $D^2_{q^2}(n) = bp^e c$, where $a, b \in \mathbb{N} \bigcup \{0\}$. Suppose $a \neq 0$ and $b \neq 0$. Hence one gets the equation $aq^f c + bp^e c = p^e q^f c$. This equation can be solved only if a is divisible by p^e and if b is divisible by q^f. Suppose $a \neq 0$ and $b \neq 0$. Then $D^2_{p^2}(n) \geq n$ and $D^2_{q^2}(n) \geq n$, hence $D^2_{p^2}(n) + D^2_{q^2}(n) \geq 2n$ and the equality is not possible. Hence it must be either $a = 0$ or $b = 0$. However, this is not possible if $e \geq 2$ and $f \geq 2$. $\square$

4 Concluding remarks

I have proved some new results about the arithmetic derivative and integral. I have defined arithmetic partial derivatives and solved some arithmetic partial differential equations. I have shown that, for any solution to the system $m' = n, n' = m$ in natural numbers, at least one of the numbers m and n is not smaller than the number $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59$. The arithmetic derivative can be defined on sequences of numbers as follows: $D(a_1, a_2, \ldots, a_n, \ldots) = (D(a_1), D(a_2), \ldots, D(a_n), \ldots))$. Thus taking any integer sequence $(a) = (a_1, a_2, \ldots, a_n, \ldots)$ one can get an infinite family of derived sequences $D(a), D^2(a), \ldots, D^k(a), \ldots$. I believe many other useful applications of the arithmetic derivative will be discovered.

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