Journal of Integer Sequences, Vol. 15 (2012),
Article 12.3.4

Generalized Bivariate Lucas p-Polynomials and Hessenberg Matrices

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Abstract

In this paper, we give some determinantal and permanental representations of generalized bivariate Lucas p-polynomials by using various Hessenberg matrices. The results that we obtained are important since generalized bivariate Lucas p-polynomials are general forms of, for example, bivariate Jacobsthal-Lucas, bivariate Pell-Lucas p-polynomials, Chebyshev polynomials of the first kind, Jacobsthal-Lucas numbers etc.

1 Introduction

The generalized Lucas p-numbers [15] are defined by

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \tag{1}$$

for $n > p+1$, with boundary conditions $L_p(0) = (p+1)$, $L_p(1) = \cdots = L_p(p) = 1$.

The Lucas [8], Pell-Lucas [2] and Chebyshev polynomials of the first kind [17] are defined as follows:

$$\begin{aligned}
l_{n+1}(x) &= xl_n(x) + l_{n-1}(x),\ n \geq 2 \text{ with } l_0(x) = 2,\ l_1(x) = x \\
Q_{n+1}(x) &= 2xQ_n(x) + Q_{n-1}(x),\ n \geq 2 \text{ with } Q_0(x) = 2,\ Q_1(x) = 2x \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x),\ n \geq 2 \text{ with } T_0(x) = 1,\ T_1(x) = x
\end{aligned}$$

respectively.

The generalized bivariate Lucas p-polynomials [16] are defined as follows:

$$L_{p,n}(x,y) = xL_{p,n-1}(x,y) + yL_{p,n-p-1}(x,y)$$

for $n > p$, with boundary conditions $L_{p,0}(x,y) = (p+1)$, $L_{p,n}(x,y) = x^n$, $n = 1, 2, \ldots, p$.

A few terms of $L_{p,n}(x,y)$ for $p = 4$ and $p = 5$ are

$5, x, x^2, x^3, x^4, 5y + x^5, 6xy + x^6, x^7 + 7x^2y, x^8 + 8x^3y, x^9 + 9x^4y, 5y^2 + x^{10} + 10x^5y, \ldots$ and $6, x, x^2, x^3, x^4, x^5, 5y + x^6, 6xy + x^7, x^8 + 7x^2y, x^9 + 8x^3y, \ldots$ respectively.

MacHenry [9] defined generalized Lucas polynomials $(L_{k,n}(t))$ where t_i $(1 \leq i \leq k)$ are constant coefficients of the core polynomial

$$P(x; t_1, t_2, \ldots, t_k) = x^k - t_1x^{k-1} - \cdots - t_k,$$

which is denoted by the vector $t = (t_1, t_2, \ldots, t_k)$.

$G_{k,n}(t_1, t_2, \ldots, t_k)$ is defined by

$$\begin{aligned} G_{k,n}(t) &= 0,\ n < 0 \\ G_{k,0}(t) &= k \\ G_{k,1}(t) &= t_1 \\ G_{k,n+1}(t) &= t_1G_{k,n}(t) + \cdots + t_kG_{k,n-k+1}(t). \end{aligned}$$

MacHenry obtained very useful properties of these polynomials in [10, 11].

Remark 1. [16]Cognate polynomial sequence are as follows

x	**y**	**p**	$L_{p,n}(x,y)$
x	y	1	bivariate Lucas polynomials $L_n(x,y)$
x	1	p	Lucas $p-$polynomials $L_{p,n}(x)$
x	1	1	Lucas polynomials $l_n(x)$
1	1	p	Lucas $p-$numbers $L_p(n)$
1	1	1	Lucas numbers L_n
$2x$	y	p	bivariate Pell-Lucas p-polynomials $L_{p,n}(2x,y)$
$2x$	y	1	bivariate Pell-Lucas polynomials $L_n(2x,y)$
$2x$	1	p	Pell-Lucas p-polynomials $Q_{p,n}(x)$
$2x$	1	1	Pell-Lucas polynomials $Q_n(x)$
2	1	1	Pell-Lucas numbers Q_n
$2x$	-1	1	Chebyshev polynomials of the first kind $T_n(x)$
x	$2y$	p	bivariate Jacobsthal-Lucas p-polynomials $L_{p,n}(x,2y)$
x	$2y$	1	Bivariate Jacobsthal-Lucas polynomials $L_n(x,2y)$
1	$2y$	1	Jacobsthal-Lucas polynomials $j_n(y)$
1	2	1	Jacobsthal-Lucas numbers j_n

Remark 1 shows that $L_{p,n}(x,y)$ is a general form of all sequences and polynomials mentioned in that remark. Therefore, any result obtained from $L_{p,n}(x,y)$ is valid for all sequences and polynomials mentioned there.

Many researchers have studied determinantal and permanental representations of k sequences of generalized order-k Fibonacci and Lucas numbers. For example, Minc [12] defined an $n \times n$ (0,1)-matrix $F(n,k)$, and showed that the permanents of $F(n,k)$ are equal to the generalized order-k Fibonacci numbers. Nalli and Haukkanen [13] defined $h(x)$-Fibonacci and Lucas polynomials and gave determinantal representations of these polynomials. The authors ([6, 7]) defined two $(0,1)$-matrices and showed that the permanents of these matrices are the generalized Fibonacci and Lucas numbers. Öcal et al. [14] gave some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers, and obtained Binet's formula for these sequences. Kılıc and Stakhov [4] gave permanent representation of Fibonacci and Lucas p-numbers. Kılıc and Tasci [5] studied permanents and determinants of Hessenberg matrices. Janjic [3] considers a particular case of upper Hessenberg matrices and gave a determinant representation of a generalized Fibonacci numbers.

In this paper, we give some determinantal and permanental representations of $L_{p,n}(x,y)$ by using various Hessenberg matrices. These results are a general form of determinantal and permanental representations of polynomials and sequences mentioned in Remark 1.

2 The determinantal representations

In this section, we give some determinantal representations of $L_{p,n}(x,y)$ using Hessenberg matrices.

Definition 2. An $n \times n$ matrix $A_n = (a_{ij})$ is called lower Hessenberg matrix if $a_{ij} = 0$ when $j - i > 1$ i.e.,

$$A_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}.$$

Theorem 3. *[1] Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $\det(A_0) = 1$. Then,*

$$\det(A_1) = a_{11}$$

and for $n \geq 2$

$$\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} \left[(-1)^{n-r} a_{n,r} \Big(\prod_{j=r}^{n-1} a_{j,j+1}\Big) \det(A_{r-1}) \right].$$

Theorem 4. *Let $L_{p,n}(x,y)$ be the generalized bivariate Lucas p-polynomials and $W_{p,n} = (w_{ij})$ be an $n \times n$ Hessenberg matrix defined by*

$$w_{ij} = \begin{cases} i, & \text{if } \ i = j-1; \\ x, & \text{if } \ i = j; \\ i^p y, & \text{if } \ p = i - j \ \text{ and } j \neq 1; \\ (p+1) i^p y, & \text{if } \ p = i - j \ \text{ and } j = 1; \\ 0, & \text{otherwise;} \end{cases}$$

that is,

$$W_{p,n} = \begin{bmatrix} x & i & 0 & \cdots & 0 \\ 0 & x & i & \ddots & \vdots \\ \vdots & 0 & x & & 0 \\ (p+1)i^p y & 0 & \vdots & \cdots & \\ 0 & i^p y & 0 & & 0 \\ \vdots & 0 & \ddots & x & i \\ 0 & 0 & \cdots & 0 & x \end{bmatrix}. \tag{2}$$

Then,

$$\det(W_{p,n}) = L_{p,n}(x,y) \tag{3}$$

where $n \geq 1$ *and* $i = \sqrt{-1}$.

Proof. To prove (3), we use mathematical induction on n. The result is true for $n = 1$ by hypothesis.

Assume that it is true for all positive integers less than or equal to n, namely $\det(W_{p,n}) = L_{p,n}(x,y)$. Then, we have

$$\begin{aligned} \det(W_{p,n+1}) &= q_{n+1,n+1}\det(W_{p,n}) + \sum_{r=1}^{n}\left[(-1)^{n+1-r}q_{n+1,r}(\prod_{j=r}^{n}q_{j,j+1})\det(W_{p,r-1})\right] \\ &= x\det(W_{p,n}) + \sum_{r=1}^{n-p}\left[(-1)^{n+1-r}q_{n+1,r}(\prod_{j=r}^{n}q_{j,j+1})\det(W_{p,r-1})\right] \\ &\quad + \sum_{r=n-p+1}^{n}\left[(-1)^{n+1-r}q_{n+1,r}(\prod_{j=r}^{n}q_{j,j+1})\det(W_{p,r-1})\right] \\ &= x\det(W_{p,n}) + \left[(-1)^p(i)^p y\prod_{j=n-p+1}^{n} i\det(W_{p,n-p})\right] \\ &= x\det(W_{p,n}) + [(-1)^p y(i)^p.(i)^p\det(W_{p,n-p})] \\ &= x\det(W_{p,n}) + y\det(W_{p,n-p}) \end{aligned}$$

by using Theorem 3. From the induction hypothesis and the definition of $L_{p,n}(x,y)$ we obtain

$$\det(W_{p,n+1}) = xL_{p,n}(x,y) + yL_{p,n-p}(x,y) = L_{p,n+1}(x,y).$$

Therefore, (3) holds for all positive integers n. □

Example 5. We obtain the 5-th $L_{p,n}(x,y)$ for $p = 4$, by using Theorem 4

$$L_{4,5}(x,y) = \det\begin{bmatrix} x & i & 0 & 0 & 0 \\ 0 & x & i & 0 & 0 \\ 0 & 0 & x & i & 0 \\ 0 & 0 & 0 & x & i \\ 5i^4 y & 0 & 0 & 0 & x \end{bmatrix} = 5y + x^5.$$

Theorem 6. *Let $p \geq 1$ be an integer, $L_{p,n}(x,y)$ be the generalized bivariate Lucas p-polynomials and $M_{p,n} = (m_{ij})$ be an $n \times n$ Hessenberg matrix defined by*

$$m_{ij} = \begin{cases} -1, & \text{if } j = i+1; \\ x, & \text{if } i = j; \\ y, & \text{if } p = i-j \text{ and } j \neq 1; \\ (p+1)y, & \text{if } p = i-j \text{ and } j = 1; \\ 0, & \text{otherwise;} \end{cases}$$

that is,

$$M_{p,n} = \begin{bmatrix} x & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & \cdots & 0 \\ 0 & 0 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (p+1)y & 0 & 0 & \cdots & 0 \\ 0 & y & 0 & \cdots & 0 \\ & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & x \end{bmatrix}. \quad (4)$$

Then,

$$\det(M_{p,n}) = L_{p,n}(x,y).$$

Proof. Since the proof is similar to the proof of Theorem 4, we omit the details. □

3 The permanent representations

Let $A = (a_{i,j})$ be a square matrix of order n over a ring R. The permanent of A is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where S_n denotes the symmetric group on n letters.

Theorem 7. *[14] Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $per(A_0) = 1$. Then, $per(A_1) = a_{11}$ and for $n \geq 2$,*

$$per(A_n) = a_{n,n} per(A_{n-1}) + \sum_{r=1}^{n-1} \left[a_{n,r} \left(\prod_{j=r}^{n-1} a_{j,j+1}\right) per(A_{r-1}) \right].$$

Theorem 8. *Let $p \geq 1$ be an integer, $L_{p,n}(x, y)$ be the generalized bivariate Lucas p-polynomials and $H_{p,n} = (h_{rs})$ be an $n \times n$ lower Hessenberg matrix such that*

$$h_{rs} = \begin{cases} -i, & \text{if } \ s - r = 1 \ ; \\ x, & \text{if } \ r = s \ ; \\ i^p y, & \text{if } \ p = r - s \text{ and } s \neq 1, \ ; \\ (p+1)i^p y, & \text{if } \ p = r - s \text{ and } s = 1; \\ 0, & \text{otherwise;} \end{cases}$$

then

$$per(H_{p,n}) = L_{p,n}(x, y)$$

where $n \geq 1$ and $i = \sqrt{-1}$.

Proof. This is similar to the proof of Theorem 4 using Theorem 7. □

Example 9. We obtain the 6-th $L_{p,n}(x, y)$ for $p = 4$, by using Theorem 8

$$L_{4,6}(x, y) = \text{per} \begin{bmatrix} x & -i & 0 & 0 & 0 & 0 \\ 0 & x & -i & 0 & 0 & 0 \\ 0 & 0 & x & -i & 0 & 0 \\ 0 & 0 & 0 & x & -i & 0 \\ 5y & 0 & 0 & 0 & x & -i \\ 0 & y & 0 & 0 & 0 & x \end{bmatrix} = 6xy + x^6.$$

Theorem 10. *Let $p \geq 1$ be an integer, $L_{p,n}(x, y)$ be the generalized bivariate Lucas p-polynomials and $K_{p,n} = (k_{ij})$ be an $n \times n$ lower Hessenberg matrix such that*

$$k_{ij} = \begin{cases} 1, & \text{if } \ j = i + 1; \\ x, & \text{if } \ i = j; \\ y, & \text{if } \ i - j = p \text{ and } j \neq 1; \\ (p+1)y, & \text{if } \ i - j = p \text{ and } j = 1; \\ 0, & \text{otherwise;} \end{cases}$$

then

$$per(K_{p,n}) = L_{p,n}(x, y).$$

Proof. This is similar to the proof of Theorem 4 by using Theorem 7. □

We note that the theorems given above are still valid for the sequences and polynomials mentioned in Remark 1

Corollary 11. *If we rewrite Theorem 4, Theorem 6, Theorem 8 and Theorem 10 for x, y, p, we obtain the following table.*

For	**x**	**y**	**p**	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{p,n+1}(\mathbf{x},\mathbf{y})$,
for	x	y	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{n}(\mathbf{x},\mathbf{y})$,
for	x	1	p	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{p,n}(\mathbf{x})$,
for	x	1	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = l_n(x)$,
for	1	1	p	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{p}(\mathbf{n})$,
for	1	1	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{n}$,
for	$2x$	y	p	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{p,n}(\mathbf{2x},\mathbf{y})$,
for	$2x$	y	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{n}(\mathbf{2x},\mathbf{y})$,
for	$2x$	1	p	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{Q}_{p,n}(\mathbf{x})$,
for	$2x$	1	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{Q}_{n}(\mathbf{x})$,
for	2	1	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{Q}_{n}$,
for	$2x$	-1	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{T}_{n}(\mathbf{x})$,
for	x	$2y$	p	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{p,n}(\mathbf{x},\mathbf{2y})$,
for	x	$2y$	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{L}_{n}(\mathbf{x},\mathbf{2y})$,
for	1	$2y$	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = \mathbf{j}_{n}(\mathbf{y})$,
for	1	2	1	$\det(W_{p,n}) = \det(M_{p,n}) = per(H_{p,n}) = per(K_{p,n}) = j_n$.

4 Conclusion

In this paper, we have given some determinantal and permanental representations of generalized bivariate Lucas p-polynomials. Our results allow us to derive determinantal and permanantel representations of sequences and polynomials mentioned in Remark 1.

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2010 *Mathematics Subject Classification*: Primary 11B37, Secondary 15A15, 15A51.
Keywords: Generalized bivariate Lucas p-polynomials, determinant, permanent and Hessenberg matrix.

Received November 30 2011; revised version received February 21 2012. Published in *Journal of Integer Sequences*, March 11 2012.

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