Journal of Integer Sequences, Vol. 15 (2012),
Article 12.1.4

Some New Properties of Balancing Numbers and Square Triangular Numbers

Refik Keskin and Olcay Karaatlı
Department of Mathematics
Faculty of Arts and Science
Sakarya University
54187 Serdivan
Sakarya
Turkey
rkeskin@sakarya.edu.tr
okaraatli@sakarya.edu.tr

Abstract

A number N is a *square* if it can be written as $N = n^2$ for some natural number n; it is a *triangular number* if it can be written as $N = n(n+1)/2$ for some natural number n; and it is a *balancing number* if $8N^2 + 1$ is a square. In this paper, we study some properties of balancing numbers and square triangular numbers.

1 Introduction

A triangular number is a number of the form $T_n = n(n+1)/2$, where n is a natural number. So the first few triangular numbers are $1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots$ (sequence A000217 in [20]). A well known fact about the triangular numbers is that x is a triangular number if and only if $8x + 1$ is a perfect square. Triangular numbers can be thought of as the numbers of dots needed to make a triangle. In a similar way, square numbers can be thought of as the numbers of dots that can be arranged in the shape of a square. The m-th square number is formed using an outher square whose sides have m dots. Let us denote the expression for m-th square number by $S_m = m^2$ [15]. Behera and Panda [1] introduced balancing numbers $m \in \mathbb{Z}^+$ as solutions of the equation

$$1 + 2 + \cdots + (m-1) = (m+1) + (m+2) + \cdots + (m+r), \tag{1}$$

calling $r \in \mathbb{Z}^+$, the balancer corresponding to the balancing number m. For instance $6, 35$, and 204 are balancing numbers with balancers $2, 14$, and 84, respectively. It is clear from (1) that m is a balancing number with balancer r if and only if

$$m^2 = \frac{(m+r)(m+r+1)}{2},$$

which when solved for r gives

$$r = \frac{-(2m+1) + \sqrt{8m^2+1}}{2}. \tag{2}$$

It follows from (2) that m is a balancing number if and only if $8m^2+1$ is a perfect square. Since $8 \times 1^2 + 1 = 9$ is a perfect square, we accept 1 as a balancing number. In what follows, we introduce cobalancing numbers in a way similar to the balancing numbers. By modifying (1), we call $m \in \mathbb{Z}^+$, a cobalancing number if

$$1 + 2 + \cdots + (m-1) + m = (m+1) + (m+2) + \cdots + (m+r) \tag{3}$$

for some $r \in \mathbb{Z}^+$. Here, $r \in \mathbb{Z}^+$ is called a cobalancer corresponding to the cobalancing number m. A few of the cobalancing numbers are $2, 14$, and 84 with cobalancers $6,\ 35$, and 204, respectively. It is clear from (3) that m is a cobalancing number with cobalancer r if and only if

$$m(m+1) = \frac{(m+r)(m+r+1)}{2},$$

which when solved for r gives

$$r = \frac{-(2m+1) + \sqrt{8m^2+8m+1}}{2}. \tag{4}$$

It follows from (4) that m is a cobalancing number if and only if $8m^2+8m+1$ is a perfect square, that is, $m(m+1)$ is a triangular number. Since $8 \times 0^2 + 8 \times 0 + 1 = 1$ is a perfect square, we accept 0 is a cobalancing number [7, 8]. Also since $m(m+1)/2$ is known as a triangular number by the very definition of triangular number, the above discussion means that if m is a cobalancing number, then both $m(m+1)$ and $m(m+1)/2$ are triangular numbers. Panda and Ray [7] proved that every cobalancing number is even. And also they showed that every balancer is a cobalancing number and every cobalancer is a balancing number.

Oblong numbers are numbers of the form $O_n = n(n+1)$, where n is a positive integer. The n-th oblong number represents the number of points in a rectangular array having n columns and $n+1$ rows. The first few oblong numbers are $2, 6, 12, 20, 30, 42, 56, 72, 90, 110, \ldots$ (sequence A002378 in [20]). Since $2+4+6+\cdots+2n = 2(1+2+3+\cdots+n) = 2n(n+1)/2 = n(n+1) = O_n$, the sum of the first n even numbers equals the n-th oblong number. Actually it is clear from the definition of oblong numbers and triangular numbers that an oblong number is twice a triangular number. After the definition of oblong numbers, we can say from (4) that if m is a cobalancing number, then $m(m+1)$ is both an oblong and triangular number. Well then, what about the square triangular numbers? Since triangular numbers are of the form $T_n = n(n+1)/2$ and square numbers are of the form $S_m = m^2$, square triangular numbers are integer solutions of the equation

$$m^2 = \frac{n(n+1)}{2}. \tag{5}$$

Eq.(5) says something about the relation between balancing and square triangular numbers. Behera and Panda [1] proved that a positive integer m is a balancing number if and only if m^2 is a triangular number, that is, $8m^2+1$ is a perfect square. Here, we will get Eq.(5) again using an amusing problem and we will see the interesting relation between balancing and square triangular numbers by means of this problem. In equation (1), if we make the substitution $m+r=n$, then we get $1+2+\cdots+(m-1) = (m+1)+(m+2)+\cdots+n$. Thus this equation gives us a problem as follows.

I live on a street whose houses are numbered in order $1, 2, 3, \ldots, n-1, n$; so the houses at the ends of the street are numbered 1 and n. My own house number is m and of course $0 < m < n$. One day, I add up the house numbers of all the houses to the left of my house; then I do the same for all the houses to the right of my house. I find that the sums are the same. So how can we find m and n [14]? Since $1+2+3+\cdots+m-1 = (m+1)+\cdots+(n-1)+n$, it follows that

$$\frac{(m-1)m}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}.$$

Thus we get $m^2 = n(n+1)/2$. Here, m^2 is both a triangular number and a square number. That is, m^2 is a square triangular number. In Eq.(1), since m is a balancing number, it is easy to see that a balancing number is the square root of a square triangular number. For more information about triangular, square triangular and balancing numbers, one can consult [1, 13, 16, 17, 18].

2 Preliminaries

In this section, we introduce two kinds of sequences named generalized Fibonacci and Lucas sequences (U_n) and (V_n), respectively. Let k and t be two nonzero integers. The generalized Fibonacci sequence is defined by $U_0 = 0$, $U_1 = 1$ and $U_{n+1} = kU_n + tU_{n-1}$ for $n \geq 1$ and generalized Lucas sequence is defined by $V_0 = 2$, $V_1 = k$ and $V_{n+1} = kV_n + tV_{n-1}$ for $n \geq 1$, respectively. Also generalized Fibonacci and Lucas numbers for negative subscript are defined as

$$U_{-n} = \frac{-U_n}{(-t)^n} \text{ and } V_{-n} = \frac{V_n}{(-t)^n} \tag{6}$$

for $n \geqslant 1$. For $k = t = 1$, the sequences (U_n) and (V_n) are called classic Fibonacci and Lucas sequences and they are denoted as (F_n) and (L_n), respectively. The first Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ (sequence A000045 in [20]) and the first Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$ (sequence A000032 in [20]). For $k = 2$ and $t = 1$, the sequences (U_n) and (V_n) are called Pell and Pell-Lucas sequences and they are denoted as (P_n) and (Q_n), respectively. Thus $P_0 = 0$, $P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geqslant 1$ and $Q_0 = 2$, $Q_1 = 2$ and $Q_{n+1} = 2Q_n + Q_{n-1}$ for $n \geqslant 1$. The first few terms of Pell sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$ (sequence A000129 in [20]) and the first few terms of Pell-Lucas sequence are $2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, \ldots$ (sequence A002203 in [20]). Moreover, for $k = 6$ and $t = -1$, we represent (U_n) and (V_n) by (u_n) and (v_n), respectively. Thus $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = 6u_n - u_{n-1}$ and $v_0 = 2$, $v_1 = 6$ and $v_{n+1} = 6v_n - v_{n-1}$ for all $n \geqslant 1$. The first few terms of the sequence (u_n) are $0, 1, 6, 35, 204, \ldots$ (sequence A001109 in [20]) and the first few terms of the sequence (v_n) are $2, 6, 34, 198, 1154, \ldots$ (sequence A003499 in [20]). Furthermore, from the equation (6), it clearly follows that

$$u_{-n} = -u_n \text{ and } v_{-n} = v_n$$

for all $n \geqslant 1$. For more information about generalized Fibonacci and Lucas sequences, one can consult [4, 5, 6, 10, 11, 19]. Now we present some well known theorems and identities regarding the sequences (P_n), (Q_n), (u_n), and (v_n), which will be useful during the proofs of the main theorems and the new properties of the sequence (y_n), where $y_n = (v_n - 2)/4$.

Theorem 1. *Let γ and δ be the roots of the characteristic equation $x^2 - 2x - 1 = 0$. Then we have $P_n = \dfrac{\gamma^n - \delta^n}{2\sqrt{2}}$ and $Q_n = \gamma^n + \delta^n$ for all $n \geq 0$.*

Theorem 2. *Let α and β be the roots of the characteristic equation $x^2 - 6x + 1 = 0$. Then*

$$u_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}} \tag{7}$$

and

$$v_n = \alpha^n + \beta^n$$

for all $n \geq 0$.

The formulas given in the above theorems are known as Binet's formulas. Let B_n denote the $n-$th balancing number. From [8], we know that

$$B_n = \frac{(3+\sqrt{8})^n - (3-\sqrt{8})^n}{2\sqrt{8}}. \tag{8}$$

From Theorems 1 and 2, it is easily seen that $u_n = B_n = P_{2n}/2$ and $v_n = Q_{2n}$ for $n \geq 0$. Moreover, from identities (7) and (8), it is easily seen that $B_n = u_n$ for negative integer n. Then well known identities for (P_n), (Q_n), (B_n) and (v_n) are

$$Q_n^2 - 8P_n^2 = 4(-1)^n, \tag{9}$$

$$v_n^2 - 32B_n^2 = 4, \tag{10}$$

$$B_n^2 - 6B_nB_{n-1} + B_{n-1}^2 = 1, \tag{11}$$

$$Q_n^2 = Q_{2n} + 2(-1)^n, \tag{12}$$

$$B_{2n} = B_n v_n, \tag{13}$$

$$P_{2n} = P_n Q_n, \tag{14}$$

and

$$v_n^2 = v_{2n} + 2. \tag{15}$$

In order to see close relations between balancing numbers and square triangular numbers, we can give the following well known theorem which characterizes all square triangular numbers. We omit the proof of this theorem due to Karaatlı and Keskin [5].

Theorem 3. *A natural number x is a square triangular number if and only if $x = B_n^2$ for some natural number n.*

Since $y_n = (v_n - 2)/4$, it follows that

$$B_n^2 = \frac{v_n^2 - 4}{32} = \frac{1}{2}\frac{(v_n-2)}{4}\left(\frac{(v_n-2)}{4} + 1\right) = \frac{y_n(y_n+1)}{2}.$$

Then, it is seen that $x^2 = \dfrac{y(y+1)}{2}$ for some positive integers x and y if and only if $x = B_n$ and $y = y_n$ for some natural number n. Now we prove the following lemma given in [9].

Lemma 4. *The sequence (y_n) satisfies the recurrence relation $y_{n+1} = 6y_n - y_{n-1} + 2$ for $n \geqslant 1$ where $y_0 = 0$ and $y_1 = 1$.*

Proof. Using the fact that $y_n = \dfrac{v_n - 2}{4}$, we get

$$\begin{aligned} 6y_n - y_{n-1} + 2 &= 6(v_n-2)/4 - (v_{n-1}-2)/4 + 2 \\ &= (6v_n - v_{n-1} - 2)/4 = (v_{n+1}-2)/4 \\ &= y_{n+1}. \end{aligned}$$

□

The first few terms of the sequence (y_n) are $0, 1, 8, 49, 288, \ldots$ (sequence A001108 in [20]). For $n = 1, 2, \ldots$, let b_n be $n-$th cobalancing number and so let (b_n) denote the cobalancing number sequence. Then, the cobalancing numbers satisfy the similar recurrence relation given in Lemma 4. That is, $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 1$ where $b_0 = 0$ and $b_1 = 2$ (see [7, p. 1191]). The first few terms of the cobalancing number sequence are $0, 2, 14, 84, 492, \ldots$ (sequence A053141 in [20]). Moreover, there is a close relation between cobalancing numbers, balancing numbers and the sequence (y_n). In order to see this relation, we can give the following lemma without proof.

Lemma 5. *For every* $n \geqslant 1$, $b_n = y_n + B_n$ *and* $b_n = y_{n+1} - B_{n+1}$.

Lemma 6. *For every* $n \geqslant 1$, $y_{2n} = 8B_n^2$ *and* $y_{2n+1} = 8B_nB_{n+1} + 1$.

Proof. By identities (10) and (15), we get

$$B_n^2 = (v_n^2 - 4)/32 = (v_{2n} - 2)/32 = y_{2n}/8.$$

Thus it follows that $y_{2n} = 8B_n^2$. Also since $y_{n+1} = 6y_n - y_{n-1} + 2$, it is easy to see that $y_n = (y_{n+1} + y_{n-1} - 2)/6$. By using $y_{2n} = 8B_n^2$, we find that

$$y_{2n+1} = (y_{2n+2} + y_{2n} - 2)/6 = (8B_{n+1}^2 + 8B_n^2 - 2)/6 = (8(B_{n+1}^2 + B_n^2) - 2)/6$$

Since $B_{n+1}^2 + B_n^2 = 6B_nB_{n+1} + 1$ by identity (11), it follows that

$$y_{2n+1} = (8(B_{n+1}^2 + B_n^2) - 2)/6 = (8(6B_nB_{n+1} + 1) - 2)/6 = 8B_nB_{n+1} + 1.$$

This completes the proof. □

Now we can give the following theorem. Since its proof is easy, we omit it.

Theorem 7. *If* n *is an odd natural number, then* $y_n = Q_n^2/4$ *and if* n *is an even natural number, then* $y_n = Q_n^2/4 - 1$.

Since $y_{2n+1} = 8B_nB_{n+1} + 1$ and $y_{2n+1} = Q_{2n+1}^2/4$, it follows that B_nB_{n+1} is a triangular number. Moreover, it follows from Lemma 6 that y_n is odd if and only if n is odd and y_n is even if and only if n is even.

3 Main Theorems

In the previous sections, we mentioned the well known elementary properties about triangular, square triangular, balancing and cobalancing numbers. In this chapter, by using the previous theorems, lemmas and identities we prove some new properties concerning balancing numbers and square triangular numbers. The principal question of our interest is whether the product of two balancing numbers greater than 1 is another balancing number. We will show that the answer to this question is negative. Similarly, we will show that the product of two square triangular numbers greater than 1 is not a triangular number. The product of two oblong numbers may be another oblong number and similarly, the product of two triangular numbers may be another one. For a simple example, 2 and 6 are two oblong numbers. The product of them is $2 \times 6 = 12$ and $12 = 3(3+1)$ is another oblong number. Similarly, 3 and 15 are two triangular numbers. The product of 3 and 15 is $3 \times 15 = 45$ and $45 = \dfrac{9(9+1)}{2}$ is another triangular number. Also it is obvious that the product of two consecutive oblong numbers is another oblong:

$$[(x-1)x]\,[x(x+1)] = (x^2-1)x^2.$$

For solving the general problem, we need to solve the Diophantine equation

$$x(x+1)y(y+1) = z(z+1). \tag{16}$$

In [2], Breiteig gave recursion formulae for the solutions x, y, and z satisfying the equation (16). For more information about the product of two oblong numbers, one can consult [2].

The question of when the product of two oblong numbers is another one suggests an analogous question for balancing numbers. When is the product of two balancing numbers another balancing number? Now before giving these properties concerning balancing numbers and square triangular numbers, we present some theorems which will be needed in the proof of the main theorems. Since the following two theorems are given in [19], we omit their proofs.

Theorem 8. *Let $n \in \mathbb{N} \cup \{0\}$ and $m, r \in \mathbb{Z}$. Then*

$$P_{2mn+r} \equiv (-1)^{(m+1)n} P_r \pmod{Q_m}, \tag{17}$$

$$Q_{2mn+r} \equiv (-1)^{(m+1)n} Q_r \pmod{Q_m}, \tag{18}$$

$$P_{2mn+r} \equiv (-1)^{mn} P_r \pmod{P_m}, \tag{19}$$

and

$$Q_{2mn+r} \equiv (-1)^{mn} Q_r \pmod{P_m}. \tag{20}$$

Theorem 9. *Let $n \in \mathbb{N} \cup \{0\}$ and $m, r \in \mathbb{Z}$. Then*

$$B_{2mn+r} \equiv B_r \pmod{B_m}, \tag{21}$$

$$v_{2mn+r} \equiv v_r \pmod{u_m}, \tag{22}$$

$$B_{2mn+r} \equiv (-1)^n B_r \pmod{v_m}, \tag{23}$$

and

$$v_{2mn+r} \equiv (-1)^n v_r \pmod{v_m}. \tag{24}$$

The proofs of the following theorems can be given by using the above two theorems. Also, we can find some of their proofs in [3]. Moreover, some of them are given in [12] without proof.

Theorem 10. *Let $m, n \in \mathbb{N}$ and $m \geqslant 2$. Then $P_m \mid P_n$ if and only if $m \mid n$.*

Theorem 11. *Let $m, n \in \mathbb{N}$ and $m \geqslant 2$. Then $Q_m \mid Q_n$ if and only if $m \mid n$ and $\dfrac{n}{m}$ is an odd integer.*

Theorem 12. *Let $m, n \in \mathbb{N}$ and $m \geqslant 2$. Then $Q_m \mid P_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is an even integer.*

Since $B_n = P_{2n}/2$ and $v_n = Q_{2n}$, the proofs of the following theorems can be given by using the above theorems and identity (23).

Theorem 13. *Let $m, n \in \mathbb{N}$ and $m \geqslant 2$. Then $B_m \mid B_n$ if and only if $m \mid n$.*

Theorem 14. *Let $m, n \in \mathbb{N}$ and $m \geqslant 1$. Then $v_m \mid v_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is an odd integer.*

Theorem 15. *Let $m, n \in \mathbb{N}$ and $m \geqslant 1$. Then $v_m \mid u_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is an even integer.*

The following theorem is a well known theorem (see [8, 12]).

Theorem 16. *Let $m \geqslant 1$ and $n \geqslant 1$. Then $(B_m, B_n) = B_{(m,n)}$.*

Corollary 17. *Let $m \geqslant 1$ and $n \geqslant 1$. Then $(B_m^2, B_n^2) = B_{(m,n)}^2$.*

Theorem 16 says that the greatest common divisor of any two balancing numbers is again a balancing number. As a conclusion of this theorem, Corollary 17 says that the greatest common divisor of any two square triangular numbers is again a square triangular number. Now we will discuss the least common multiple of any two balancing numbers. The least common multiple of any two triangular numbers may be a triangular number. For instance, 15 and 21 are two triangular numbers and $[15, 21] = 105$ is again a triangular number. Note that $15 \nmid 21$. Similarly, the least common multiple of any two oblong numbers may be an oblong number. For a simple example, 6 and 15 are two oblong numbers and $[6, 15] = 30$ is again an oblong number. But this is not true in general for any two balancing numbers. This can be seen from the following theorem.

Theorem 18. *Let $B_n > 1, B_m > 1$ and $B_n < B_m$. Then $[B_n, B_m]$ is a balancing number if and only if $B_n \mid B_m$.*

Proof. Assume that $B_n \mid B_m$. Then $[B_n, B_m] = B_m$ is again a balancing number. Conversely, assume that $B_n > 1, B_m > 1$ and $B_n \nmid B_m$. Then by Theorem 13, $n \nmid m$. Let $d = (m, n)$. Then by Theorem 16, we get $(B_n, B_m) = B_d$. Therefore

$$[B_n, B_m] = \frac{B_n B_m}{(B_n, B_m)} = \frac{B_n B_m}{B_d}. \tag{25}$$

Assume that $[B_n, B_m]$ is a balancing number. Thus $[B_n, B_m] = B_r$ for some natural number r. Then by (25), we have $B_n B_m / B_d = B_r$. That is, $B_n B_m = B_d B_r$. Thus $\frac{B_n}{B_d} B_m = B_r$ and therefore $B_m \mid B_r$. This implies that $r = mt$ for some natural number t by Theorem 13. Assume that t is an odd integer. Then $t = 4q \mp 1$ for some $q \geqslant 1$. Thus $B_r = B_{mt} = B_{4qm \mp m} = B_{2(2qm) \mp m} \equiv B_{\mp m} \pmod{B_{2m}}$ by (21). This shows that $B_r \equiv \mp B_m \pmod{B_{2m}}$. Since $B_{2m} = B_m v_m$ by (13), we see that $\frac{B_n}{B_d} B_m = B_r \equiv \mp B_m \pmod{B_m v_m}$. Then it follows that $\frac{B_n}{B_d} = \mp 1 \ (\text{mod} v_m)$. We assert that $B_n \neq B_d$. On the contrary, assume that $B_n = B_d$. Then $n = d$ and this implies that $n \mid m$, which is impossible since $n \nmid m$. Since $\frac{B_n}{B_d} \neq 1$ and $\frac{B_n}{B_d} \equiv \mp 1 \ (\text{mod } v_m)$, it follows that $v_m \leqslant \frac{B_n}{B_d} \mp 1 \leqslant \frac{B_n}{B_d} + 1 \leqslant B_n + 1$. Since $B_n < B_m$, we get $n < m$. This shows that $v_n < v_m \leqslant B_n + 1$. On the other hand, by identity (10), we get $v_n > 2B_n$. Therefore $2B_n < v_n < B_n + 1$, which implies that $B_n < 1$. But this is a contradiction since $B_n > 1$. Now assume that t is an even integer. Then $t = 2k$ and thus $r = mt = 2mk$. Therefore $\frac{B_n}{B_d} B_m = B_r = B_{2km} = B_{km} v_{km} \geqslant B_m v_m$. This shows that $\frac{B_n}{B_d} \geqslant v_m$ and thus $v_m \leqslant \frac{B_n}{B_d} \leqslant B_n$. Since $n < m$, we get $v_n < v_m \leqslant B_n$. That is, $v_n < B_n$, which is impossible by identity (10). This completes the proof. □

Now as a result of the above theorem, we can give the following corollary which says something about the least common multiple of any two square triangular numbers. The proof of the following corollary is straightforward, using the fact that

$$[a^2, b^2] = \frac{a^2 b^2}{(a^2, b^2)} = \frac{a^2 b^2}{(a, b)^2} = \left(\frac{ab}{(a, b)} \right)^2 = [a, b]^2$$

where a and b are positive integers.

Corollary 19. *Let $B_n > 1, B_m > 1$ and $B_n < B_m$. Then $[B_n^2, B_m^2]$ is a triangular number if and only if $B_n^2 \mid B_m^2$.*

In order to answer the main question which is about the product of two balancing numbers, we give the following theorem. This theorem says something more than the above theorem.

Theorem 20. *Let $n > 1, m > 1$ and $m \geqslant n$. Then there is no integer r such that $B_n B_m = B_r$.*

Proof. Assume that $m > 1, n > 1$ and $B_n B_m = B_r$ for some $r > 1$. Then $B_m \mid B_r$ and therefore $m \mid r$ by Theorem 13. Thus $r = mt$ for some positive integer t. Assume that t is an even integer. Then $t = 2k$ and therefore $r = mt = 2mk$. Thus

$$B_n B_m = B_r = B_{2km} = B_{km} v_{km}$$

by identity (13). This shows that $B_n = \frac{B_{km}}{B_m} v_{km}$ and therefore $v_{km} \mid B_n$. By Theorem 15, we get $km \mid n$ and $n/km = 2s$ for some integer s. Then $n = 2kms$. Since $n = 2kms$ and $r = 2km$, we get $n = rs$. Thus $r \mid n$. On the other hand, since $B_n B_m = B_r$, it follows that $B_n \mid B_r$ and therefore $n \mid r$ by Theorem 13. This implies that $n = r$ and $B_n = B_r$. Since $B_n B_m = B_r$, we get $B_m = 1$, which is a contradiction. Now assume that t is an odd integer. Then $t = 4q \mp 1$ for some positive integer q. Thus $r = mt = 4qm \mp m$ and therefore

$$B_r = B_{4qm \mp m} = B_{2(2qm) \mp m} \equiv B_{\mp m} \ (\text{mod } B_{2m})$$

by (21). This shows that $B_mB_n \equiv \mp B_m \pmod{B_{2m}}$. Since $B_{2m} = B_m v_m$, we get $B_mB_n \equiv \mp B_m \pmod{B_m v_m}$, which implies that $B_n \equiv \mp 1 \pmod{v_m}$. Therefore $v_m \mid B_n \mp 1$ and thus $v_m \leqslant B_n \mp 1$. Since $v_n > 2B_n$ and $m \geq n$, we get $B_n + 1 \geqslant B_n \mp 1 \geqslant v_m \geqslant v_n > 2B_n$. This implies that $B_n + 1 > 2B_n$. Then $B_n < 1$, which is a contradiction. This completes the proof. □

Since square triangular numbers are square of the balancing numbers, the above theorem says that the product of two square triangular numbers greater than one is not a triangular number. Now we can give the following corollary easily.

Corollary 21. *The only positive integer solution of the system of Diophantine equations* $2u^2 = x(x+1)$, $2v^2 = y(y+1)$ *and* $2u^2v^2 = z(z+1)$ *is given by* $(x, y, u, v, z) = (1, 1, 1, 1, 1)$.

The following theorem gives a new property of the sequence (y_n). It is about the product of any two elements of the sequence (y_n) greater than 1.

Theorem 22. *Let* $n > 1$ *and* $m > 1$. *Then there is no integer* r *such that* $y_n y_m = y_r$.

Proof. Assume that $y_n y_m = y_r$. Since y_k is odd if and only if k is odd and y_k is even if and only if k is even, we see that m, n, and r are odd or r and at least one of the numbers n and m are even. Assume that n and r are even. Then $n = 2k$ and $r = 2t$ for some positive integers k and t. By Lemma 6, we have $y_n = y_{2k} = 8B_k^2$ and $y_r = y_{2t} = 8B_t^2$. Therefore

$$y_m = \frac{y_r}{y_n} = \frac{8B_t^2}{8B_k^2} = \left(\frac{B_t}{B_k}\right)^2.$$

If m is even, then $m = 2l$ for some positive integer l. This implies that $y_m = y_{2l} = 8B_l^2$ and therefore $\left(\frac{B_t}{B_u}\right)^2 = 8B_l^2$, which is impossible. So m is odd. Then, by Theorem 7, it follows that $y_m = \frac{Q_m^2}{4}$. Thus, we get $\frac{Q_m}{2} = \frac{B_t}{B_k} = \frac{P_{2t}/2}{P_{2k}/2} = \frac{P_{2t}}{P_{2k}} = \frac{P_r}{P_n}$. This shows that $2P_r = P_nQ_m$. Since n is even, P_n is even. Also, since $P_r = P_n\frac{Q_m}{2}$, we see that $P_n \mid P_r$ and $Q_m \mid P_r$. By Theorem 10 and Theorem 12, we get $r = nu$ and $r = 2ms$ for some natural numbers u and s. Since $2P_r = P_nQ_m$, we have

$$P_nQ_m = 2P_r = 2P_{2ms} = 2P_{ms}Q_{ms} > P_{ms}Q_{ms} > P_{ms}Q_m.$$

Therefore $P_n > P_{ms}$ and this implies that $n > ms$. Then $2n > 2ms$ and thus $2n > r = nu$. This shows that $u < 2$. That is, $u = 1$. Since $u = 1$, we get $r = nu = n$, which is impossible. Assume that m, n and r are odd integers. Since $y_n y_m = y_r$, we get $\frac{Q_n^2}{4}\frac{Q_m^2}{4} = \frac{Q_r^2}{4}$ by Theorem 7. Therefore $Q_nQ_m = 2Q_r$. This shows that $Q_n \mid Q_r$ and $Q_m \mid Q_r$. Then $r = nt$ and $r = mk$ for some odd natural numbers t and k by Theorem 11. Since t is an odd integer, $t = 4q \mp 1$ for some $q \geqslant 1$. Thus

$$Q_r = Q_{nt} = Q_{4qn\mp n} = Q_{2(2qn)\mp n} \equiv \mp Q_n \pmod{Q_{2n}}$$

by the congruence (18). That is, $Q_r \equiv \mp Q_n \pmod{Q_{2n}}$. This implies that $2Q_r \equiv \mp 2Q_n \pmod{Q_{2n}}$ and thus $Q_n Q_m \equiv \mp 2Q_n \pmod{Q_{2n}}$. Since $Q_{2n} = Q_n^2 + 2$, when n is odd, by identity (12), we clearly have that $(Q_n, Q_{2n}) = 2$. Then the congruence

$$Q_n Q_m \equiv \mp 2Q_n \pmod{Q_{2n}}$$

implies that

$$\frac{Q_n}{2} Q_m \equiv \mp 2 \frac{Q_n}{2} \quad \left(\bmod\ \frac{Q_{2n}}{2}\right).$$

This shows that $Q_m \equiv \mp 2 \pmod{Q_{2n}/2}$. Since $m \geq 2$, we get $Q_m > 2$ and therefore $Q_{2n}/2 \leqslant Q_m \mp 2$. Therefore $Q_{2n} \leqslant 2Q_m \mp 4 < 2Q_m + 4$. Similarly, by using $r = mk$, it is seen that $Q_{2m} < 2Q_n + 4$. Then it follows that $Q_{2n} + Q_{2m} < 2Q_m + 2Q_n + 8$. Since n and m are odd integers, $Q_{2n} = Q_n^2 + 2$ and $Q_{2m} = Q_m^2 + 2$ by identity (12). Thus, we get

$$Q_n^2 + 2 + Q_m^2 + 2 < 2Q_m + 2Q_n + 8.$$

Since $Q_n^2 + 2 + Q_m^2 + 2 < 2Q_m + 2Q_n + 8$, it follows that $Q_n^2 + Q_m^2 < 2Q_m + 2Q_n + 4$. This implies that $Q_n^2 - 2Q_n + Q_m^2 - 2Q_m < 4$. Then we get

$$Q_n(Q_n - 2) + Q_m(Q_m - 2) < 4,$$

which implies that $Q_n + Q_m < 4$. This is a contradiction since $m > 1$ and $n > 1$. This completes the proof. □

We easily obtain the following corollary.

Corollary 23. *The only positive integer solution of the system of Diophantine equations* $x(x+1) = 2u^2$, $y(y+1) = 2v^2$, *and* $xy(xy+1) = 2z^2$ *is given by* $(x, y, u, v, z) = (1, 1, 1, 1, 1)$.

Balancing numbers and cobalancing numbers are related to the solutions of some Diophantine equations. Solutions of some of the Diophantine equations are given in [5]. Now we give four of them from [5].

Theorem 24. *All positive integer solutions of the equation* $x^2 = y(y+1)/2$ *are given by* $(x, y) = (B_n, y_n)$ *with* $n \geq 1$.

Theorem 25. *All positive integer solutions of the equation* $(x + y - 1)^2 = 8xy$ *are given by* $(x, y) = (y_n, y_{n+1})$ *with* $n \geq 1$.

Theorem 26. *All positive integer solutions of the equation* $x^2 - 6xy + y^2 - 1 = 0$ *are given by* $(x, y) = (B_n, B_{n+1})$ *with* $n \geq 1$.

Theorem 27. *All positive integer solutions of the equation* $(x + y - 1)^2 = 8xy + 1$ *are given by* $(x, y) = (b_n, b_{n+1})$ *with* $n \geq 1$.

From Theorem 27, it follows that $b_n b_{n+1}$ is a triangular number for every natural number n.

Moreover, we can easily state the following theorems.

Theorem 28. *All positive integer solutions of the equation* $x^2 - y^2 + 2xy + x - y = 0$ *are given by* $(x, y) = (B_n, b_n)$ *with* $n \geq 1$.

Theorem 29. *All positive integer solutions of the equation* $x^2 + 2y^2 - 4xy - x = 0$ *are given by* $(x, y) = (y_n, b_n)$ *with* $n \geq 1$ *or* $(x, y) = (y_n, b_{n-1})$ *with* $n \geq 2$.

4 Concluding Remarks

The sum of two triangular numbers may be a triangular number. For instance, 6 and 15 are triangular numbers and $6 + 15 = 21$ is again a triangular number. Similarly, the sum of two oblong numbers may be another oblong number. For instance, 12 and 30 are oblong numbers and $12 + 30 = 42$ is again an oblong number. But we think that the sum of two square triangular numbers is not a square triangular number. That is, $B_n^2 + B_m^2 = B_r^2$ has no solution if $n \geqslant 1$ and $m \geqslant 1$. We also think that there is no integer r such that $B_n + B_m = B_r$ and $b_n + b_m = b_r$ for $n \geqslant 1$ and $m \geqslant 1$. On the other hand, we think that the product of any two cobalancing numbers greater than 1 is not a cobalancing number. That is, there is no integer r such that $b_n b_m = b_r$ for $n \geqslant 1$ and $m \geqslant 1$. Moreover, we think that there is no solution of the equation $y_n + y_m = y_r$ if $n \geqslant 1$ and $m \geqslant 1$.

5 Acknowledgements

The authors would like to thank the anonymous referee for careful reading and suggestions that improved the clarity of this manuscript.

References

[1] A. Behera and G. K. Panda, On the square roots of triangular numbers, *Fibonacci Quart.* **37** (1999), 98–105.

[2] T. Breiteig, When is the product of two oblong numbers another oblong?, *Math. Mag.* **73** (2000), 120–129.

[3] P. Hilton, J. Pedersen, and L. Somer, On Lucasian numbers, *Fibonacci Quart.* **35** (1997), 43–47.

[4] D. Kalman and R. Mena, The Fibonacci numbers-exposed, *Math. Mag.* **76** (2003), 167–181.

[5] O. Karaatlı and R. Keskin, On some Diophantine equations related to square triangular and balancing numbers, *J. Algebra, Number Theory: Adv. Appl.* **4** (2010), 71–89.

[6] J. B. Muskat, Generalized Fibonacci and Lucas sequences and rootfinding methods, *Math. Comp.* **61** (1993), 365–372.

[7] G. K. Panda and P. K. Ray, Cobalancing numbers and cobalancers, *Int. J. Math. Math. Sci.* **8** (2005), 1189–1200.

[8] G. K. Panda, Some fascinating properties of balancing numbers, in *Proc. Eleventh Internat. Conference on Fibonacci Numbers and Their Applications*, *Cong. Numerantium* **194** (2009), 185–189.

[9] D. C. D. Potter, Triangular square numbers, *Math. Gaz.* **56** (1972), 109–110.

[10] S. Rabinowitz, Algorithmic manipulation of Fibonacci identities, in *Applications of Fibonacci Numbers*, Vol. **6**, Kluwere, 1996, pp. 389–408.

[11] P. Ribenboim, *My Numbers, My Friends*, Springer-Verlag New York, Inc., 2000.

[12] P. Ribenboim, An algorithm to determine the points with integral coordinates in certain elliptic curves, *J. Number Theory*, **74** (1999), 19–38.

[13] M. R. Schroeder, *Number Theory in Science and Communication*, Springer, 2005.

[14] S. A. Shıralı, Fun with triangular numbers, http://www.mathcelebration.com/PDF/TriangleNumPDF.pdf, 2010.

[15] Stanford Math Circle, Sunday, May 9, 2010, Square-triangular numbers, Pell's equation and continued fractions, http://math.stanford.edu/circle/Pell.pdf.

[16] K. B. Subramaniam, A simple computation of square triangular numbers, *Internat. J. Math. Ed. Sci. Tech.* **23** (1992), 790–793.

[17] K. B. Subramaniam, A divisibility property of square triangular numbers, *Internat. J. Math. Ed. Sci. Tech.* **26** (1995), 284–286.

[18] J. J. Tattersall, *Elementary Number Theory in Nine Chapters*, Cambridge University Press, 1999.

[19] Z. Yosma and R. Keskin, Some new identities concerning generalized Fibonacci and Lucas numbers, submitted.

[20] The On-Line Encyclopedia of Integer Sequences, http://www.oeis.org.

2010 *Mathematics Subject Classification*: Primary 11D25; Secondary 11D41, 11D72.
Keywords: triangular number, square triangular number, balancing number, cobalancing number.

(Concerned with sequences A000032, A000045, A000129, A000217, A001108, A001109, A002203, A002378, A003499, and A053141.)

Received June 13 2011; revised version received December 2 2011. Published in *Journal of Integer Sequences*, December 27 2011. Small revision, October 2 2012.

Return to Journal of Integer Sequences home page.