Riordan-Bernstein Polynomials, Hankel Transforms and Somos Sequences

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Abstract

Using the language of Riordan arrays, we define a notion of generalized Bernstein polynomials which are defined as elements of certain Riordan arrays. We characterize the general elements of these arrays, and examine the Hankel transform of the row sums and the first columns of these arrays. We propose conditions under which these Hankel transforms possess the Somos-4 property. We use the generalized Bernstein polynomials to define generalized Bézier curves which can provide a visualization of the effect of the defining Riordan array.

1 Introduction

Given a sequence a_n, we denote by h_n the general term of the sequence with $h_n = |a_{i+j}|_{0 \le i,j \le n}$. The sequence h_n is called the Hankel transform of a_n [21, 22]. A well known example of Hankel transform is that of the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$, where we find that $h_n = 1$ for all n. Hankel determinants occur naturally in many branches of mathematics, from combinatorics [8] to number theory [24] and to mathematical physics [35].

In this note, we shall look at the Hankel transforms of certain sequences of polynomials. In order to define these polynomials, we combine elements of the theory of Riordan arrays [28] with the theory of Bernstein polynomials. Bernstein polynomials can be used in the proof of the Weierstrass approximation theorem [27], for instance, as well as having practical applications in the construction of Bézier curves [2, 26].

The $n+1$ Bernstein polynomials of degree n are the polynomials $B_{n,k}(s) = \binom{n}{k} s^k (1-s)^{n-k}$. We shall call the coefficient array $B_{n,k}(s)$ the (standard) Bernstein array. The fact that this is a lower triangular array inspires us to look at a broader framework, involving the group of Riordan arrays. The Bernstein array, written in matrix form, begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1-s & s & 0 & 0 & 0 & 0 & \cdots \\ (1-s)^2 & 2s(1-s) & s^2 & 0 & 0 & 0 & \cdots \\ (1-s)^3 & 3s(1-s)^2 & 3s^2(1-s) & s^3 & 0 & 0 & \cdots \\ (1-s)^4 & 4s(1-s)^3 & 6s^2(1-s)^2 & 4s^3(1-s) & s^4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $P_i = (x_i, y_i)$ is a sequence of $n+1$ points in the plane ($i = 0 \dots n$), then

$$P(s) = \sum_{k=0}^{n} B_{n,k}(s) P_k$$

defines a smooth curve in the plane, with $P(0) = P_0$ and $P(1) = P_n$. This curve is called the Bernstein Bézier curve defined by the control points P_i. Often the simpler term of Bézier curve is used.

In order to develop a theory of Riordan-array-derived generalized Bernstein polynomials, we first review some notations associated with integer sequences and then we review the concept of Riordan array. We also look at some information pertinent to the calculation of special Hankel transforms. Readers familiar with these notions may skip the next section.

2 Preliminaries on integer sequences, Riordan arrays and Hankel transforms

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_n x^n$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \dots$, A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [17], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

A sequence e_n is called a (α, β) Somos-4 sequence if

$$\alpha e_{n-1} e_{n-3} + \beta e_{n-2}^2 = e_n e_{n-4}.$$

Somos-4 sequences are associated with elliptic curves [15, 19, 29, 34, 36] and Hankel transforms [10, 39]. An example of such a sequence is given in the On-Line Encyclopedia of Integer Sequences as A006720. It begins

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, \dots.$$

This sequence has $\alpha = \beta = 1$ and it is associated with rational points on the cubic

$$y^2 = 4x^3 - 4x + 1.$$

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$. We sometimes write $\bar{f} = \text{Rev} f$.

For a lower triangular matrix $(a_{n,k})_{n,k \ge 0}$ the row sums give the sequence with general term $\sum_{k=0}^{n} a_{n,k}$.

The *Riordan group* [28, 32], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \cdots$ and $f(x) = f_1x + f_2x^2 + \cdots$ where $f_1 \neq 0$ [32]. We often require in addition that $f_1 = 1$, but this is not the case in this note. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

Elements of the form $(g(x), xg(x))$ form a subgroup called the Bell subgroup.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence (expressed as an infinite column vector) with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The so-called *binomial matrix* $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)}.$$

Each Riordan array $(g(x), f(x))$ has bi-variate generating function given by

$$\frac{g(x)}{1-yf(x)}.$$

For instance, the binomial matrix $\mathbf{B}$ has generating function

$$\frac{\frac{1}{1-x}}{1-y\frac{x}{1-x}} = \frac{1}{1-x(1+y)}.$$

Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [30, 31]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ ("Pascal's triangle") is A007318.

There are a number of known ways of calculating Hankel transforms of sequences [9, 20, 21, 25, 35]. One involves the theory of orthogonal polynomials, whereby we seek to represent the sequence under study as moments of a density function. Standard techniques of orthogonal polynomials then allow us to compute the desired Hankel transforms [6, 13]. These techniques are based on the following results (the first is the well-known "Favard's Theorem"), which we essentially reproduce from [21].

Theorem 2. *[21] (Cf. [37], Théorème 9 on p.I-4, or [38], Theorem 50.1). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$ Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ with $\beta_n \neq 0$ for all $n \geq 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \text{for} \quad n \geq 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 3. *[21] (Cf. [37], Proposition 1, (7), on p. V-5, or [38], Theorem 51.1). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some functional L. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \text{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = L(x^k)$ satisfies

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

The Hankel transform of μ_n, which is the sequence with general term $h_n = |\mu_{i+j}|_{0\leq i,j\leq n}$, is then given by

$$h_n = \mu_0^{n+1} \beta_1^n \beta_2^{n-1} \cdots \beta_{n-1}^2 \beta_n.$$

Other methods of proving Hankel transform evaluations include lattice path methods (the Lindström-Gessel-Viennot theorem) [1, 33] and Dodgson condensation (Desnanot-Jacobi adjoint matrix theorem) [1, 7, 14].

The Hankel transform is not an injective mapping. For instance, given a sequence a_n then the sequence $\sum_{k=0}^{n} r^{n-k}\binom{n}{k}a_k$ [22] will also have the same Hankel transform. Similarly we have

Lemma 4. *Let a_n (with $a_0 \neq 0$) have g.f. $f(x)$ and b_n have g.f. $g(x)$ where*

$$g(x) = \frac{f(x)}{1 - sxf(x)}.$$

Then a_n and b_n have the same Hankel transform.

Proof. The proof is a small variation on the proof for the INVERT transform [22]. We have

$$g(x) = f(x) + sxf(x)g(x),$$

which implies that

$$b_n = a_n + s\sum_{k=0}^{n-1} a_{n-1-k}b_k.$$

We then have

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ b_1 & b_2 & b_3 & b_4 & \cdots \\ b_2 & b_3 & b_4 & b_5 & \cdots \\ b_3 & b_4 & b_5 & b_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ sb_0 & 1 & 0 & 0 & \cdots \\ sb_1 & sb_0 & 1 & 0 & \cdots \\ sb_2 & sb_1 & sb_0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & sb_0 & sb_1 & sb_2 & \cdots \\ 0 & 1 & sb_0 & sb_1 & \cdots \\ 0 & 0 & 1 & sb_0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Taking determinants now yields the result, the triangular matrices with 1's on the diagonal having determinants equal to 1. □

When $s = 1$, we obtain the so-called "INVERT" transform. Note that the row sums of elements of the Bell subgroup of Riordan arrays, which are of the form $(g(x), xg(x))$, have a g.f. equal to the INVERT transform of the first column generating function. Explicitly, the row sums have g.f. $\frac{g(x)}{1-xg(x)}$. Thus in this case, the Hankel transform of the row sums sequence is equal to the Hankel transform of the first column sequence.

3 Generalized Bernstein polynomials

We recall that the $n + 1$ Bernstein polynomials of degree n are the polynomials $B_{n,k}(s) = \binom{n}{k}s^k(1-s)^{n-k}$.

Lemma 5. *The Bernstein array $B_{n,k}$ is given by the Riordan array product*

$$\left(\frac{1}{1-(1-s)x}, \frac{sx}{1-(1-s)x}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) \cdot (1, sx) \cdot \left(\frac{1}{1-x}, \frac{x}{1-x}\right)^{-1}. \quad (1)$$

Proof. We note that the matrices $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ and $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)^{-1} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right)$ have (n, k)-th elements given by $\binom{n}{k}$ and $\binom{n}{k}(-1)^{n-k}$ respectively. A straight-forward evaluation of the product on the right-hand side establishes the equation. The details are as follows. First, we have

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right)^{-1} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right).$$

Thus we obtain

$$\begin{aligned}
\left(\frac{1}{1-x},\frac{x}{1-x}\right)\cdot(1,sx)\cdot\left(\frac{1}{1+x},\frac{x}{1+x}\right) &= \left(\frac{1}{1-x},\frac{x}{1-x}\right)\cdot\left(\frac{1}{1+sx},\frac{x}{1+sx}\right)\\
&= \left(\frac{1}{1-x}\frac{1}{1+\frac{sx}{1-x}},\frac{\frac{sx}{1-x}}{1+\frac{sx}{1-x}}\right)\\
&= \left(\frac{1}{1+(s-1)x},\frac{sx}{1+(s-1)x}\right)\\
&= \left(\frac{1}{1-(1-s)x},\frac{sx}{1-(1-s)x}\right).
\end{aligned}$$

We then have, by definition of a Riordan array, that the (n,k)-th term of the Riordan array $\left(\frac{1}{1-(1-s)x},\frac{sx}{1-(1-s)x}\right)$, say $t_{n,k}$, is given by

$$\begin{aligned}
t_{n,k} &= [x^n]\frac{1}{1-(1-s)x}\left(\frac{sx}{1-(1-s)x}\right)^k\\
&= [x^n]\frac{s^kx^k}{(1-(1-s)x)^{k+1}}\\
&= s^k[x^{n-k}](1-(1-s)x)^{-k-1}\\
&= s^k[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k-1}{j}(-(1-s))^jx^j\\
&= s^k[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}(-1)^j(-(1-s))^jx^j\\
&= s^k[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j}{j}(1-s)^jx^j\\
&= s^k\binom{k+n-k}{n-k}(1-s)^{n-k}\\
&= \binom{n}{k}s^k(1-s)^{n-k}.
\end{aligned}$$

□

Now let (g,f) be an arbitrary Riordan array. We define the following notion of generalized Bernstein polynomials. The *generalized Bernstein polynomials defined by the Riordan array* (g,f) are the polynomials $B_{n,k}^{(g,f)}(s)$ defined by the Riordan array

$$\mathbb{B}^{(g,f)} = (g,f)\cdot(1,sx)\cdot(g,f)^{-1}. \tag{2}$$

In other words, $B_{n,k}^{(g,f)}(s)$ is the (n,k)-th element of the triple matrix product (2). When we use these polynomials along with a set of control points to draw a curve, we shall call such a curve a *generalized Bernstein-Bézier curve*. Some examples of such curves are given in what follows.

Example 6. We let $(g, f) = (g, xg)$ with $g(x) = \frac{1}{1+x+x^2}$. We find that $(g, f)^{-1} = (m(x), xm(x))$ where

$$m(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

is the generating function of the Motzkin numbers $M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k$ A001006. With the sequel in mind, we note that the row sums b_n of the inverse matrix $(g, f)^{-1}$ in this case have generating function

$$(m(x), xm(x)) \cdot \frac{1}{1-x} = \frac{m(x)}{1 - xm(x)} = \frac{\sqrt{1 - 2x - 3x^2} + 3x - 1}{2x(1 - 3x)}.$$

We have

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 3^{n-k} C_k.$$

This is A005773$(n + 1)$ (the number of directed animals of size $n + 1$).

We note that the expression $\frac{m(x)}{1-xm(x)}$ corresponds to the so-called "INVERT transform" of $m(x)$.

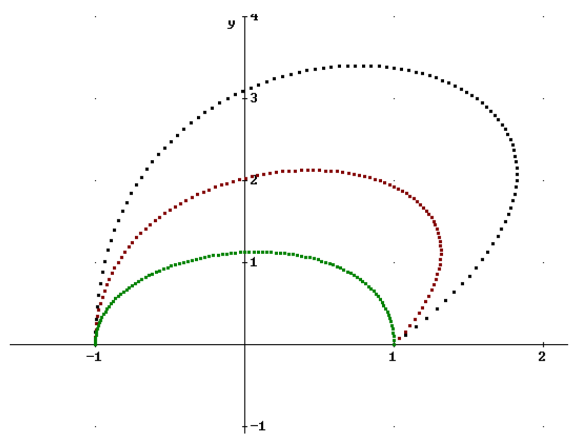

Figure 1: Three generalized Bernstein-Bézier curves

The matrix of polynomials $B_{n,k}^{(g,f)}(s)$ in this case begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ s-1 & s & 0 & 0 & 0 & 0 & \cdots \\ 2s(s-1) & 2s(s-1) & s^2 & 0 & 0 & 0 & \cdots \\ 4s^3 - 6s^2 + s + 1 & 5s^3 - 6s^2 + s & 3s^2 - 3s^2 & s^3 & 0 & 0 & \cdots \\ 9s^4 - 16s^3 + 6s^2 + 2s - 1 & 12s^4 - 20s^3 + 6s^2 + 2s & 9s^4 - 12s^3 + 3s^2 & 4s^4 - 4s^3 & s^4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & s^5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Returning to the general case, we recall that

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right),$$

where $\bar{f}(x)$ is the compositional inverse of $f(x)$ (that is, it is the solution $u(x)$ of the equation $f(u) = x$ for which $u(0) = 0$). Thus we have

$$\begin{aligned}
\mathbb{B}^{(g,f)} &= (g(x), f(x)) \cdot (1, sx) \cdot \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right) \\
&= (g(x), f(x)) \cdot \left(\frac{1}{g(\bar{f}(sx))}, \bar{f}(sx)\right) \\
&= \left(\frac{g(x)}{g(\bar{f}(sf(x)))}, \bar{f}(sf(x))\right).
\end{aligned}$$

The first column of $\mathbb{B}^{(g,f)}$ thus has generating function

$$\frac{g(x)}{g(\bar{f}(sf(x)))} = (g(x), f(x)) \cdot \frac{1}{g(\bar{f}(sx))}. \tag{3}$$

The row sums of $\mathbb{B}^{(g,f)}$ are seen to have generating function

$$\left(\frac{g(x)}{g(\bar{f}(sf(x)))}, \bar{f}(sf(x))\right) \cdot \frac{1}{1-x} = \frac{g(x)}{g(\bar{f}(sf(x)))} \frac{1}{1-\bar{f}(sf(x))}.$$

Proposition 7. *Let the elements of the Riordan array $(g(x), f(x))$ be denoted by $d_{n,k}$, and let the first column of $(g(x), f(x))^{-1}$ have elements a_n. Then the elements of the first column of $\mathbb{B}^{(g,f)}$ are given by*

$$\sum_{k=0}^{n} d_{n,k} a_k s^k.$$

Proof. The a_n have generating function $\frac{1}{g(\bar{f}(x))}$. Thus $\frac{1}{g(\bar{f}(sx))}$ generates the sequence $s^n a_n$. Since $(g(x), f(x))$ has general term $d_{n,k}$, the result follows from the expression in Eq. (3). □

Thus the first column elements of $\mathbb{B}^{(g,f)}$, which we denote by $\tilde{a}_n(s)$,

$$\tilde{a}_n(s) = \sum_{k=0}^{n} d_{n,k} a_k s^k$$

are polynomials in s of degree at most n.

In similar manner, we have

Proposition 8. *Let b_n denote the row sums of the inverse matrix $(g(x), f(x))^{-1}$. Then the row sums of the Bernstein array $\mathbb{B}^{(g,f)}$ are given by the expression*

$$\sum_{k=0}^{n} d_{n,k} b_k s^k.$$

Proof. The row sums of $\mathbb{B}$ have generating function given by

$$\begin{aligned}\mathbb{B}\cdot\frac{1}{1-x} &= (g(x),f(x))\cdot\left(\frac{1}{g(\bar{f}(sx))},\bar{f}(sx)\right)\frac{1}{1-x}\\ &= (g(x),f(x))\cdot\frac{1}{g(\bar{f}(sx))}\frac{1}{1-\bar{f}(sx)}.\end{aligned}$$

The result follows by observing that the sequence b_n has g.f. given by $\frac{1}{g(\bar{f}(x))}\frac{1}{1-\bar{f}x}$. □

We conclude that the row sum elements of $\mathbb{B}^{(g,f)}$,

$$\tilde{b}_n(s)=\sum_{k=0}^{n} d_{n,k}b_k s^k$$

are polynomials in s of degree at most n.

Proposition 9. *When $g(x)=\frac{1}{1-x}$, the row sum sequence of $\mathbb{B}^{(g,f)}$ is the sequence of all 1s: $1,1,1,\ldots$.*

Proof. By the above, the row sums of $\mathbb{B}$ are generated in this case by

$$\begin{aligned}(g(x),f(x))\cdot\frac{1}{g(\bar{f}(sx))}\frac{1}{1-\bar{f}(sx)} &= \left(\frac{1}{1-x},f(x)\right)\cdot(1-\bar{f}(sx))\frac{1}{1-\bar{f}(sx)}\\ &= \left(\frac{1}{1-x},f(x)\right)\cdot 1\\ &= \frac{1}{1-x}.\end{aligned}$$

□

We note that this is true independently of the nature of f.

Lemma 10. *When $(g(x),f(x))=(g(x),xg(x))$ is an element of the Bell subgroup of the Riordan group, the sequences a_n and b_n have the same Hankel transform.*

Proof. The inverse of a Bell matrix is again a Bell matrix, so that b_n is the INVERT transform of a_n. □

Proposition 11. *If $(g,f)=(g,xg)$ is an element of the Bell subgroup, then $\tilde{a}_n(s)$ and $\tilde{b}_n(s)$ have the same Hankel transforms.*

Proof. In this case, we have $(g,f)^{-1}=\left(\frac{\bar{f}}{x},\bar{f}\right)$. Then

$$\begin{aligned}\mathbb{B} &= (g,f)\cdot(1,sx)\cdot\left(\frac{\bar{f}}{x},\bar{f}\right)\\ &= (g,f)\cdot\left(\frac{\bar{f}(sx)}{sx},\bar{f}(sx)\right)\\ &= \left(g(x)\frac{\bar{f}(sf(x))}{sf(x)},\bar{f}(sf(x))\right)\\ &= \left(\frac{\bar{f}(sf(x))}{sx},\bar{f}(sf(x))\right).\end{aligned}$$

Thus the g.f. for $\tilde{a}_n(s)$ is given by $G = \frac{\bar{f}(sf(x))}{sx}$. The g.f. of $\tilde{b}_n(s)$ is given by

$$\mathbb{B} \cdot \frac{1}{1-x} = \left(\frac{\bar{f}(sf(x))}{sx}, \bar{f}(sf(x))\right) \cdot \frac{1}{1-x} = \frac{G}{1-sxG}.$$

But for any g.f. G (with $G(0) \neq 0$, which is the case here), G and $\frac{G}{1-sxG}$ generate sequences with the same Hankel transform. □

Example 12. We consider the first column of the matrix $\mathbb{B}$ defined by $\left(\frac{1}{1+x+x^2}, \frac{x}{1+x+x^2}\right)$ (this array is A104562). The elements of this sequence begin

$$1, s-1, 2s^2-2s, 4s^3-6s^2+s+1, 9s^4-16s^3+6s^2+2s-1, \ldots$$

We have, in fact,

$$\tilde{a}_n(s) = \sum_{k=0}^{n} d_{n,k} M_k s^k,$$

where M_n is the n-th Motzkin number and

$$d_{n,k} = \sum_{j=0}^{n} (-1)^{\frac{n-j}{2}} \binom{\frac{n+j}{2}}{j} \frac{1+(-1)^{n-j}}{2} (-1)^{j-k} \binom{j}{k}.$$

The Hankel transform of $\tilde{a}_n(s)$ begins

$$1, s^2-1, s^3(s^3-3s+2), s^4(s^8-5s^6+4s^5+4s^3-5s^2+1),$$
$$s^8(s^{12}-8s^{10}+4s^9+18s^8-57s^6+54s^5+6s^4-28s^3+9s^2+2s-1), \ldots$$

We can express this sequence of polynomials in s of degree $n(n+1)$ as

$$h_n(s) = (s-1)^{\lfloor \frac{(n+1)^2}{4} \rfloor} s^{\lfloor \frac{n^2}{2} \rfloor} P_n(s),$$

where $P_n(s)$ is a polynomial in s of degree $\lfloor \frac{(n+1)^2}{4} \rfloor$. We can conjecture that this sequence is a

$$(s^4(s-1)^2, s^4(s-1)(2s^2-s-1))$$

Somos-4 sequence.

By the result above, the Hankel transform of $\tilde{b}_n(s)$ is the same as that of $\tilde{a}(s)$. We have in this case that

$$\tilde{b}_n(s) = \sum_{k=0}^{n} d_{n,k} b_k s^k,$$

where

$$b_n = \sum_{k=0}^{n} (-1)^k 3^{n-k} \binom{n}{k} C_k.$$

Example 13. We consider $(g, f) = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$. The (n, k)-th element of this array is $\binom{n+k}{2k}(-1)^{n-k}$. The first column elements of the inverse $(g, f)^{-1}$ are the Catalan numbers C_n, and the row sums of the inverse are the central binomial numbers $\binom{2n}{n}$ A000984. Thus we have

$$\tilde{a}_n(s) = \sum_{k=0}^{n} \binom{n+k}{2k} C_k (-1)^{n-k} s^k,$$

and

$$\tilde{b}_n(s) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} (-1)^{n-k} s^k.$$

The g.f. of $\tilde{a}_n(s)$ is

$$\frac{1 + x - \sqrt{1 + 2x(1 - 2s) + x^2}}{2sx},$$

from which we deduce (via the Stieltjes-Perron transform [11, 18]) the moment representation

$$\tilde{a}_n(s) = \frac{1}{2\pi} \int_{2s-1-2\sqrt{s(s-1)}}^{2s-1+2\sqrt{s(s-1)}} x^n \frac{\sqrt{-x^2 + 2x(2s - 1) - 1}}{sx}\, dx + \frac{0^n}{s}, \quad (s \neq 0).$$

The g.f. of $\tilde{b}_n(s)$ reduces to

$$\frac{1}{\sqrt{1 - 2x(2s - 1) + x^2}}.$$

In this instance, the polynomials $\tilde{b}_n(s)$ coincide with the shifted Legendre polynomials [5]

$$\tilde{b}_n(s) = P_n(2s - 1).$$

The Hankel transforms of the polynomials $\tilde{a}_n(s)$ and $\tilde{b}_n(s)$ have been studied by several authors [5, 12]. They are $(s(s-1))^{\binom{n+1}{2}}$ and $2^n(s(s-1))^{\binom{n+1}{2}}$, respectively. An immediate calculation shows that these sequences are both $(s^3(s-1)^3, 0)$ Somos-4 sequences. We have the following continued fraction expressions for the g.f. of $\tilde{a}_n(s)$,

$$\cfrac{1}{1 - (s-1)x - \cfrac{s(s-1)x^2}{1 - (2s-1)x - \cfrac{s(s-1)x^2}{1 - (2s-1)x - \cdots}}},$$

and for the g.f. of $\tilde{b}_n(s)$,

$$\cfrac{1}{1 - (2s-1)x - \cfrac{2s(s-1)x^2}{1 - (2s-1)x - \cfrac{s(s-1)x^2}{1 - (2s-1)x - \cdots}}},$$

from which we can deduce the form of their Hankel transforms.

Example 14. We consider the Pascal-like triangle $(g, f) = \left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$. This is called the Delannoy triangle A008288. This triangle [3] begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 5 & 5 & 1 & 0 & 0 & \cdots \\ 1 & 7 & 13 & 7 & 1 & 0 & \cdots \\ 1 & 9 & 25 & 25 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The general term of this matrix is given by

$$\sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{n-k-j} = \sum_{j=0}^{k} \binom{k}{j} \binom{n-k}{n-k-j} 2^j.$$

The first column of $(g, f)^{-1}$ has generating function $1 - xS(-x)$, where $S(x)$ is the g.f. of the large Schroeder numbers $S_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k$ A006318. This sequence can be expressed as

$$\tilde{S}_n = (-1)^n (0^n + \sum_{k=0}^{n-1} \binom{n-1+k}{2k} C_k).$$

We therefore have

$$\tilde{a}_n(s) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{n-k-j} \tilde{S}_k s^k.$$

The Hankel transform of this polynomial sequence is given by a product of the form

$$h_n(s) = s^{\binom{n+1}{2}} (s-1)^{\lfloor \frac{(n+1)^2}{4} \rfloor} 2^{\sum_{k=0}^{n} \lfloor \frac{5k}{7} \rfloor} P_n(s),$$

where $P_n(s)$ is a polynomial in s of degree $\lfloor \frac{n^2}{4} \rfloor$. Numerical evidence suggests that $h_n(s)$ is a $(36s^4(s-1)^2, -8s^4(7s^4 - 18s^3 + 13s^2 - 2))$ Somos-4 sequence.

4 Some Somos-4 conjectures

Motivated by examples from the last section we continue to explore links between generalized Bernstein arrays and Somos-4 sequences. In this section, we shall posit two conjectures concerning different families of Riordan arrays (g, f), the Hankel transform of the first column elements of $\mathbb{B}^{(g,f)}$, and Somos-4 sequences. We are currently not in a position to prove these conjectures. The resolution of similar conjectures is an active area of research [10].

We consider for example the Bernstein array $\mathbb{B}$ generated by the Riordan array

$$(g(x), f(x)) = (1 - x, x(1 - x)),$$

with general element $\binom{k+1}{n-k}(-1)^{n-k}$ whose inverse array $(g(x), f(x))^{-1} = (c(x), xc(x))$ is often called the Catalan array A033184 [23]. As we have seen, the generating function for the first column elements

$$\tilde{a}_n(s) = \sum_{k=0}^{n} \binom{k+1}{n-k}(-1)^{n-k} C_k s^k$$

of $\mathbb{B}$ is given by

$$\frac{g(x)}{g(\bar{f}(sf(x)))} = \frac{1-\sqrt{1-4sx(1-x)}}{2sx},$$

from which we deduce (via the Stieltjes-Perron transform) the moment representation

$$\tilde{a}_n(s) = \frac{1}{2\pi}\int_{2(s-\sqrt{s(s-1)})}^{2(s+\sqrt{s(s-1)})} x^n \frac{\sqrt{4s(x-1)-x^2}}{2sx}\,dx + \frac{0^n}{\sqrt{s}}, \quad (s \neq 0).$$

We have

$$h_n(s) = (s-1)^{\binom{n+1}{2}} s^{\lfloor \frac{n^2}{2} \rfloor} \sum_{k=0}^{n} \binom{n+1}{2k+\frac{1+(-1)^n}{2}} s^k,$$

where

$$\sum_{k=0}^{n} \binom{n+1}{2k+\frac{1+(-1)^n}{2}} s^k = [x^n]\frac{1+(s+1)x-(s-1)x^2-(s-1)^2x^2}{1-2(s+1)x^2+(s-1)^2x^4}.$$

We can then conjecture that the Hankel transform of $\tilde{a}_n$ is a $(4s^4(s-1)^2, -s^4(s-1)^3(3s+1))$ Somos-4 sequence.

We next consider the Bernstein array generated by the Pascal-like [2] Riordan array

$$(g, f) = \left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right).$$

The first column of this array can be shown to have generating function

$$\frac{1+2r-(2r-s+1)x+rsx^2}{2(1-x)^2}$$
$$-\frac{\sqrt{1+(2rs+s-1)x+((s-1)^2-2rs(1-2r))x^2-2rs(2r-s+1)x^3+r^2s^2x^4}}{2(1-x)^2}.$$

The quartic within the square root suggests that the Hankel transform of this sequence could be a candidate for a Somos-4 sequence. Numerical evidence suggests the following.

Conjecture 15. *The Hankel transform of the first column elements of the generalized Bernstein array generated by* $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ *is a*

$$(r^2s^4(s-1)^2(r+1)^2(2r+1)^2, -s^4(s-1)^2(r(r+1))^3(r(r+1)(3s^2-2s-1)+s^2))$$

Somos-4 sequence.

We can generalize this as follows.

Conjecture 16. *The Hankel transform of the first column elements of the generalized Bernstein array generated by* $\left(\frac{1}{1+ax}, \frac{x(1+bx)}{1+ax}\right)$ *is a*

$$(b^2s^4(s-1)^2(a-b)^2(a-2b)^2, -s^4b^3(s-1)^2(a-b)^3(a^2s^2+(b-ab)(3s^2-2s-1))$$

Somos-4 sequence.

We next consider the generalized Bernstein array defined by the Riordan array

$$(g(x), f(x)) = \left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right).$$

Thus we let $h_n(s)$ denote the Hankel transform of $\tilde{a}_n(s)$. This sequence has g.f. given by

$$\frac{1+a(1-s)x+bx^2-\sqrt{1+2a(1-s)x+(a^2(1-s)^2+2b(1-2s^2))x^2+2a(1-s)bx^3+b^2x^4}}{2bs^2x^2}.$$

For the case $a=0$, this reduces to

$$\frac{1+bx^2-\sqrt{1+2b(1-2s^2)x^2+b^2x^4}}{2bs^2x^2}.$$

In this case, we can conjecture that

$$h_n(s) = b^{\binom{n+1}{2}}s^{\lfloor \frac{n^2}{2} \rfloor}(s^2-1)^{\lfloor \frac{(n+1)^2}{4} \rfloor}.$$

In the general case, we have

Conjecture 17. *The Hankel transform of the first column elements of the generalized Bernstein array generated by* $\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)$ *is a*

$$(a^2b^2s^4(s-1)^2, -b^3s^4(s-1)^2(s^2(a^2-b)-2bs-b))$$

Somos-4 sequence.

When $a=0$, it is easy to verify this conjecture by direct evaluation. We must show that

$$h_n(s) = b^{\binom{n+1}{2}}s^{\lfloor \frac{n^2}{2} \rfloor}(s^2-1)^{\lfloor \frac{(n+1)^2}{4} \rfloor}$$

is a $(0, b^4s^4(s^2-1)^2)$ Somos-4 sequence. Thus we must show that

$$h_{n+4}(s) = b^4s^4(s^2-1)^2\frac{h_{n+2}(s)^2}{h_n(s)},$$

that is, we wish to show that

$$b^{\binom{n+5}{2}}s^{\lfloor \frac{(n+4)^2}{2} \rfloor}(s^2-1)^{\lfloor \frac{(n+5)^2}{4} \rfloor} = \frac{b^4s^4(s^2-1)^2b^{2\binom{n+3}{2}}s^{2\lfloor \frac{(n+2)^2}{2} \rfloor}(s^2-1)^{2\lfloor \frac{(n+3)^2}{4} \rfloor}}{b^{\binom{n+1}{2}}s^{\lfloor \frac{n^2}{2} \rfloor}(s^2-1)^{\lfloor \frac{(n+1)^2}{4} \rfloor}}.$$

Now this is so, since

$$\lfloor \frac{(n+5)^2}{4} \rfloor = 2 + 2\lfloor \frac{(n+3)^2}{4} \rfloor - \lfloor \frac{(n+1)^2}{4} \rfloor,$$

$$\lfloor \frac{(n+4)^2}{2} \rfloor = 2\lfloor \frac{(n+2)^2}{2} \rfloor - \lfloor \frac{n^2}{2} \rfloor + 4,$$

and

$$\binom{n+5}{2} = 4 + 2\binom{n+3}{2} - \binom{n+1}{2}.$$

Returning to the general case, we note that since in this case the defining array is a member of the Bell subgroup, the same conjecture holds for the Hankel transform of $\tilde{b}_n(s)$.

5 Drawing generalized Bernstein-Bézier curves

Although Riordan arrays are algebraic entities, it is interesting to use the mechanism of generalized Bernstein-Bezier curves to provide a geometric visualization of their effect. To do this, we select a common set of control points to produce a curve that in some way reflects properties of the array. For our example, we select the set of control points $\langle(-1,0),(-1,1),(1,1),(1,0)\rangle$.

Example 18. Figure 1 shows points on three generalized Bernstein-Bézier curves, corresponding to the Riordan array

$$\left(\frac{1}{1-x}, \frac{1}{1-rx-rx^2}\right),$$

for $r = 1, 2, 3$, respectively, with the lowest curve corresponding to $r = 1$. The control points are $\langle(-1,0),(-1,1),(1,1),(1,0)\rangle$.

When $r = 2$, the $\mathbb{B}$ matrix begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -s+1 & s & 0 & 0 & 0 & 0 & \cdots \\ 2s^2-3s+1 & 3s-3s^2 & s^2 & 0 & 0 & 0 & \cdots \\ -2s^3+10s^2-9s+1 & 6s^3-15s^2+9s & 5s^2-5s^3 & s^3 & 0 & 0 & \cdots \\ -4s^4-14s^3+42s^2-25s+1 & -4s^4+42s^2-63s^2+25s & 14s^4-35s^3+21s^2 & 7s^3-7s^4 & s^4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & s^5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

and hence the x coordinate function is given by

$$x(s) = (-2s^3+10s^2-9s+1)(-1)+(6s^3-15s^2+9s)(-1)+(5s^2-5s^3)(1)+s^3(1) = -8s^3+10s^2-1,$$

while the y-coordinate function is given by

$$y(s) = (-2s^3+10s^2-9s+1)(0)+(6s^3-15s^2+9s)(1)+(5s^2-5s^3)(1)+s^3(0) = s^3-10s^2+9s.$$

Example 19. We take the case of $(g, f) = \left(\frac{1+x}{1-x}, \frac{x}{\sqrt{1-4x^2}}\right)$. The corresponding Bernstein matrix $\mathbb{B}$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2s+2 & s & 0 & 0 & 0 & 0 & \cdots \\ 2s^2-4s+2 & 2s-2s^2 & s^2 & 0 & 0 & 0 & \cdots \\ 2s^3+4s^2-8s+2 & 4s-4s^2 & 2s^2-2s^3 & s^3 & 0 & 0 & \cdots \\ -6s^4+4s^3+12s^2-12s+2 & 6s^4-12s^2+6s & -2s^4-4s^3+6s^2 & 2s^3-2s^4 & s^4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & s^5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We note that $\frac{1+x}{1-x}$ generates the sequence $1, 2, 2, 2, \ldots$ apparent in the first column of $\mathbb{B}$. Using the same control points as before, we obtain the following coordinate functions:

$$(x(s), y(s)) = (-3s^3 + 2s^2 + 4s - 2, -2s^3 - 2s^2 + 4s).$$

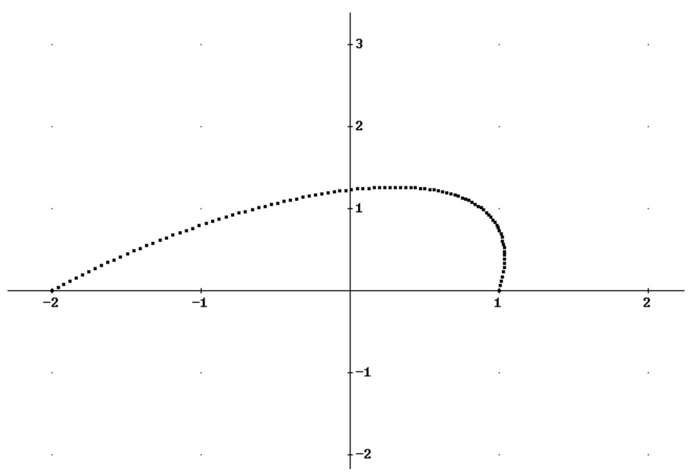

Figure 2: Curve for $\left(\frac{1+x}{1-x}, \frac{x}{\sqrt{1-4x^2}}\right)$

Example 20. We take the case of $(g, f) = \left(\frac{1}{1-x}, \frac{x(1+x)}{(1-2x)^2}\right)$. We leave it as an exercise for the reader to verify that

$$(x(s), y(s)) = (-20s^3 + 22s^2 - 1, 33s^3 - 55s^2 + 22s).$$

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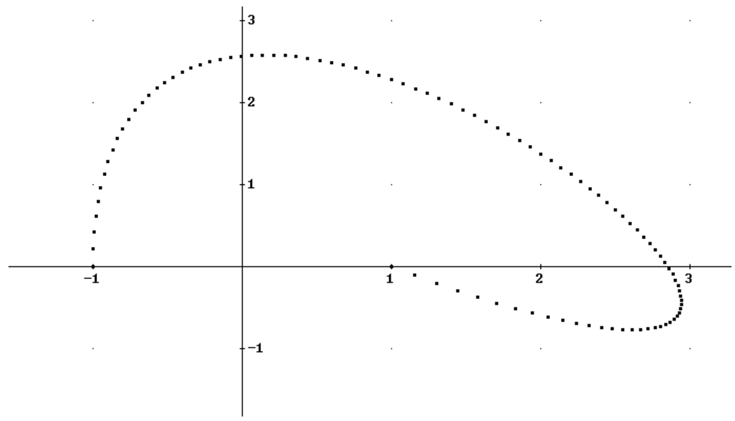

Figure 3: Curve for $\left(\frac{1}{1-x}, \frac{x(1+x)}{(1-2x)^2}\right)$

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