Weighted Gcd-Sum Functions

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Abstract
We investigate weighted gcd-sum functions, including the alternating gcd-sum function and those having as weights the binomial coefficients and values of the Gamma function. We also consider the alternating lcm-sum function.

1 Introduction

The gcd-sum function, called also Pillai’s arithmetical function (OEIS A018804) is defined by

\[ P(n) := \sum_{k=1}^{n} \gcd(k,n) \quad (n \in \mathbb{N} := \{1, 2, \ldots\}). \]  \hfill (1)

The function \( P \) is multiplicative and its arithmetical and analytical properties are determined by the representation

\[ P(n) = \sum_{d|n} d \phi(n/d) \quad (n \in \mathbb{N}), \]  \hfill (2)

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where $\phi$ is Euler’s function. See the survey paper [5]. Note that for every prime power $p^a$ ($a \in \mathbb{N}$),
\begin{equation}
P(p^a) = (a+1)p^a - ap^{a-1}.
\end{equation}

Now let
\begin{equation}
P_{\text{altern}}(n) := \sum_{k=1}^{n} (-1)^{k-1} \gcd(k, n) \quad (n \in \mathbb{N})
\end{equation}
be the alternating gcd-sum function. As far as we know, the function (4) was not considered before.

Furthermore, let
\begin{equation}
P_{\text{binom}}(n) := \sum_{k=1}^{n} \binom{n}{k} \gcd(k, n) \quad (n \in \mathbb{N})
\end{equation}
be the alternating gcd-sum function. As far as we know, the function (4) was not considered before.

More generally, consider the weighted gcd-sum functions defined by
\begin{equation}
P_w(n) := \sum_{k=1}^{n} w(k, n) \gcd(k, n) \quad (n \in \mathbb{N}),
\end{equation}
where the weights are functions $w : \mathbb{N}^2 \to \mathbb{C}$.

In this paper we evaluate the alternating gcd-sum function $P_{\text{altern}}(n)$, deduce a formula for the function $P_{\text{binom}}(n)$ and investigate other special cases of (6). We also give a formula for the alternating lcm-sum function defined by
\begin{equation}
L_{\text{altern}}(n) := \sum_{k=1}^{n} (-1)^{k-1} \lcm[k, n] \quad (n \in \mathbb{N}).
\end{equation}

Similar results can be derived for the weighted versions of certain analogs and generalizations of the gcd-sum function, see [5], but we confine ourselves to the function (6).

\section{General results}

We first give the following simple result.

**Proposition 1.** i) Let $w : \mathbb{N}^2 \to \mathbb{C}$ be an arbitrary function. Then
\begin{equation}
P_w(n) = \sum_{d | n} \phi(d) \sum_{j=1}^{n/d} w(dj, n) \quad (n \in \mathbb{N}).
\end{equation}

ii) Assume that there is a function $g : (0, 1] \to \mathbb{C}$ such that $w(k, n) = g(k/n) \ (1 \leq k \leq n)$ and let $G(n) = \sum_{k=1}^{n} g(k/n) \ (n \in \mathbb{N})$. Then
\begin{equation}
P_w(n) = \sum_{d | n} \phi(d)G(n/d) \quad (n \in \mathbb{N}).
\end{equation}
Proof. i) Using Gauss’ formula \( m = \sum_{d|m} \phi(d) \) for \( m = \gcd(k,n) \), grouping the terms of (6) and denoting \( k = dj \) we obtain at once

\[
P_w(n) := \sum_{k=1}^{n} w(k,n) \sum_{d|\gcd(k,n)} \phi(d) = \sum_{d|n} \phi(d) \sum_{j=1}^{n/d} w(dj,n).
\]

ii) Use (8) and that

\[
\sum_{j=1}^{n/d} w(dj,n) = \sum_{j=1}^{n/d} g(dj/n) = \sum_{j=1}^{n/d} g(j/(n/d)) = G(n/d).
\]

For \( w(k,n) = 1 \) we reobtain formula formula (2). In the next section we investigate other special cases, including those already mentioned in the Introduction.

Remark 2. Consider the function

\[
R_w(n) := \sum_{k=1}^{n} w(k,n) \quad (n \in \mathbb{N}).
\]  

(10)

Then, similar to the proof of i), now with the Möbius \( \mu \) function instead of \( \phi \),

\[
R_w(n) = \sum_{k=1}^{n} w(k,n) \sum_{d|\gcd(k,n)} \mu(d) = \sum_{d|n} \mu(d) \sum_{j=1}^{n/d} w(dj,n).
\]  

(11)

If condition ii) is satisfied, then we have

\[
R_w(n) = \sum_{d|n} \mu(d) G(n/d) \quad (n \in \mathbb{N}).
\]  

(12)

We will also point out some special cases of (11) and (12).

3 Special cases

3.1 Alternating gcd-sum function

Let \( w(k,n) = (-1)^{k-1} \) \((k,n \in \mathbb{N})\). Then we have the function \( P_{\text{altern}}(n) \) defined by (4).

Proposition 3. Let \( n \in \mathbb{N} \) and write \( n = 2^a m \), where \( a \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} \) and \( m \in \mathbb{N} \) is odd. Then

\[
P_{\text{altern}}(n) = \begin{cases} 
  n, & \text{if } n \text{ is odd } (a = 0); \\
  -2^{a-1} a P(m) = -\frac{a}{a+2} P(n), & \text{if } n \text{ is even } (a \geq 1). 
\end{cases}
\]  

(13)
Proof. Use formula (8). Here

\[ S_d(n) := \sum_{j=1}^{n/d} w(dj, n) = \sum_{j=1}^{n/d} (-1)^{\phi(j)} - \sum_{j=1}^{n/d} (-1)^{\phi(j)}. \]

If \( n \) is odd, then every divisor \( d \) of \( n \) is also odd and obtain \( S_d(n) = -\sum_{j=1}^{n/d}(-1)^j = 1 \), where \( n/d \) is odd. Hence, \( P_{\text{altern}}(n) = \sum_{d|n} \phi(d) = n. \)

Now let \( n \) be even and let \( d | n \). For \( d \) odd, \( S_d(n) = -\sum_{j=1}^{n/d}(-1)^j = 0 \), since \( n/d \) is even.

For \( d \) even, \( S_d(n) = -\sum_{j=1}^{n/d}1 = -n/d \). We obtain that

\[ P_{\text{altern}}(n) = -\sum_{d|n, d \text{ even}} \phi(d) \frac{n}{d} = -\sum_{d|n, d \text{ odd}} \phi(d) \frac{n}{d}, \]

where the first sum is \( P(n) \) (cf. (2)), and the second one is

\[ \sum_{d|m, d \text{ even}} \phi(d) \frac{2^{a_m}}{d} = 2^a P(m). \]

Using (3), \( P(n) = P(2^a)P(m) = 2^{a-1}(a + 2)P(m) \) and deduce

\[ P_{\text{altern}}(n) = -P(n) + 2^a P(m) = P(m)(2^a - 2^{a-1}(a + 2)) \]
\[ = -2^{a-1}aP(m) = -\frac{a}{a + 2} P(n). \]

\[ \square \]

Remark 4. More generally, consider the polynomial

\[ f_n(x) := \sum_{k=1}^{n} \gcd(k, n)x^{k-1}, \] (14)

i.e., put \( w(k, n) = x^{k-1} \) (formally). Then \( f_n(1) = P(n) \), \( f_n(-1) = P_{\text{altern}}(n) \) and deduce from Proposition 1,

\[ f_n(x) := (1 - x^n) \sum_{d|n} \frac{\phi(d)x^{d-1}}{1 - x^d}. \] (15)

### 3.2 Logarithms as weights

Let

\[ P_{\text{log}}(n) := \sum_{k=1}^{n} (\log k) \gcd(k, n). \] (16)

**Proposition 5.** For every \( n \in \mathbb{N} \),

\[ P_{\text{log}}(n) = P(n) \log n + \sum_{d|n} \log(d!) \phi(n/d). \] (17)
Proof. Apply formula (8). For \(w(k, n) = \log k\),
\[
\sum_{j=1}^{n/d} w(dj, n) = \sum_{j=1}^{n/d} \log(dj) = \frac{n}{d} \log d + \log \left(\frac{n}{d}\right),
\]

hence
\[
P_{\log}(n) = \sum_{d|n} \phi(d) \left(\frac{n}{d} \log d + \log \left(\frac{n}{d}\right)!\right),
\]

and a short computation leads to (17). \(\square\)

Remark 6. Writing the exponential form of (17),
\[
\prod_{k=1}^{n} k^{\gcd(k, n)} = n^{P(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\phi(n/d)}, \tag{18}
\]

Compare this to the known formula
\[
\prod_{k=1}^{n} k = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}, \tag{19}
\]


3.3 Discrete Fourier transform of the gcd’s

Consider \(w(k, n) = \exp(2\pi ikr/n)\) \((k, n \in \mathbb{N})\), where \(r \in \mathbb{Z}\), and denote
\[
P_{\text{DFT}}^{(r)}(n) := \sum_{k=1}^{n} \exp(2\pi ikr/n) \gcd(k, n), \tag{20}
\]

representing the discrete Fourier transform of the function \(f(k) = \gcd(k, n)\) \((k \in \mathbb{N})\).

Proposition 7. For every \(n \in \mathbb{N}\), \(r \in \mathbb{Z}\),
\[
P_{\text{DFT}}^{(r)}(n) = \sum_{d|\gcd(n, r)} d \phi(n/d). \tag{21}
\]

Proof. Here \(\exp(2\pi ikr/n) = g(k/n)\) with \(g(x) = \exp(2\pi irx)\). Using formula (9) and that
\[
\sum_{k=1}^{n} \exp(2\pi irk/n) = \begin{cases} n, & \text{if } n \mid r; \\ 0, & \text{otherwise}; \end{cases}
\]

we obtain
\[
P_{\text{DFT}}^{(r)}(n) = \sum_{d|n, n/d|r} \phi(d) \frac{n}{d} = \sum_{d|n, d|r} d \phi(n/d). \]
Remark 8. Formula (21) can be written in the form
\[ P^{(r)}_{\text{DFT}}(n) = \sum_{d|n} dc_{n/d}(r), \]
(22)
where \( c_n(k) \) are the Ramanujan sums. Furthermore, (22) can be extended for \( r \)-even functions. See [4], [6, Prop. 2]. Note that in the present treatment we do not need properties of the Ramanujan sums and of \( r \)-even functions.

For \( r = 0 \) (more generally, in case \( n \mid r \) we reobtain from (21) formula (2). For \( r = 1 \) we deduce
\[ \sum_{k=1}^{n} \exp(2\pi ik/n) \gcd(k, n) = \phi(n) \quad (n \in \mathbb{N}), \]
(23)
which gives by writing the real and the imaginary parts, respectively,
\[ \sum_{k=1}^{n} \cos(2\pi k/n) \gcd(k, n) = \phi(n) \quad (n \in \mathbb{N}), \]
(24)
\[ \sum_{k=1}^{n} \sin(2\pi k/n) \gcd(k, n) = 0 \quad (n \in \mathbb{N}), \]
(25)
similar relations being valid for \( \gcd(n, r) = 1 \).

Formulae (23), (24), (25) were pointed out in [4, Ex. 3].

3.4 Binomial coefficients as weights
Let \( w(k, n) = \binom{n}{k} \) \((k, n \in \mathbb{N})\). Then we have the function \( P_{\text{binom}}(n) \) defined by (5).

Proposition 9. For every \( n \in \mathbb{N} \),
\[ P_{\text{binom}}(n) = 2^n \sum_{d|n} \frac{\phi(d)}{d} \sum_{\ell=1}^{d} (-1)^\ell \cos^n(\ell \pi/d) - n. \]
(26)

Proof. Let \( \varepsilon_j^r = \exp(2\pi ij/r) \) \((1 \leq j \leq r)\) denote the \( r \)-th roots of unity. Using the known identity
\[ \sum_{k=0}^{[n/r]} \binom{n}{kr} = \frac{1}{r} \sum_{j=1}^{r} (1 + \varepsilon_j^r)^n = 2^n \frac{1}{r} \sum_{j=1}^{r} \cos^n(j \pi/r) \cos(nj \pi/r) \quad (n, r \in \mathbb{N}), \]
(27)
cf. [1, p. 84], and applying (8) we deduce
\[ P_{\text{binom}}(n) = \sum_{d|n} \phi(d) \sum_{j=1}^{n/d} \binom{n}{dj} = \sum_{d|n} \phi(d) \left( \frac{2^n}{d} \sum_{\ell=1}^{d} \cos^n(\ell \pi/d) \cos(n\ell \pi/d) - 1 \right) \]
\[ = 2^n \sum_{d|n} \frac{\phi(d)}{d} \sum_{\ell=1}^{d} (-1)^\ell \cos^n(\ell \pi/d) - \sum_{d|n} \phi(d), \]
giving (26).
\[ \square \]
Note that (11) and (27) lead to the following formula for the sequence OEIS A056188:

\[ R_{\text{binom}}(n) := \sum_{k=1}^{n} \binom{n}{k} = 2^n \sum_{d|n} \frac{\mu(d)}{d} \sum_{\ell=1}^{d} (-1)^{\ell} \cos^n(\ell \pi/d) \quad (n > 1). \]  

(28)

### 3.5 Weights concerning the Gamma function

Now let

\[ P_{\text{Gamma}}(n) := \sum_{k=1}^{n} \log \Gamma \left( \frac{k}{n} \right) \gcd(k, n), \]  

(29)

where \( \Gamma \) is the Gamma function.

**Proposition 10.** For every \( n \in \mathbb{N} \),

\[ P_{\text{Gamma}}(n) = \frac{\log 2\pi}{2} (P(n) - n) - \frac{1}{2} n \log n + \frac{1}{2} \sum_{d|n} \phi(d) \log d. \]  

(30)

**Proof.** This follows by (9) and by

\[ \prod_{k=1}^{n} \Gamma \left( \frac{k}{n} \right) = (2\pi)^{(n-1)/2} n^{-1/2}, \quad (n \in \mathbb{N}), \]

which is a consequence of Gauss’ multiplication formula.

**Remark 11.** (30) can be written also as

\[ P_{\text{Gamma}}(n) = \frac{\log 2\pi}{2} (P(n) - n) - \frac{1}{2} (\phi * \log)(n), \]  

(31)

where \( * \) deotes the Dirichlet convolution. Note that \( \phi * \log = \mu * \text{id} * \log = \Lambda * \text{id} \), where \( \text{id}(n) = n \ (n \in \mathbb{N}) \) and \( \Lambda \) is the von Mangoldt function.

Writing the exponential form,

\[ \prod_{k=1}^{n} \Gamma \left( \frac{k}{n} \right)^{\gcd(k, n)} = (2\pi)^{(P(n) - n)/2} n^{-n/2} \prod_{d|n} d^{\phi(d)/2}. \]  

(32)

Compare this to the following formula given in [3]:

\[ \prod_{k=1}^{n} \Gamma \left( \frac{k}{n} \right)^{\phi(n)/2} \exp(\Lambda(n)/2) = \begin{cases} (2\pi)^{\phi(n)/2}/\sqrt{n}, & n = p^{a} \text{ (a prime power)}; \\ (2\pi)^{\phi(n)/2}, & \text{otherwise}. \end{cases} \]  

(33)
3.6 Further special cases

It is possible to investigate other special cases, too. As examples we give the next ones with weights regarding, among others, the floor function \(\lfloor . \rfloor\), and the saw-tooth function \(\psi\) defined as \(\psi(x) = x - \lfloor x \rfloor - 1/2\) for \(x \in \mathbb{R} \setminus \mathbb{Z}\) and \(\psi(x) = 0\) for \(x \in \mathbb{Z}\).

**Proposition 12.** For every \(n \in \mathbb{N}\),

\[
P_{\text{id}}(n) := \sum_{k=1}^{n} k \gcd(k, n) = \frac{n}{2} (P(n) + n). \tag{34}
\]

**Proposition 13.** For every \(n \in \mathbb{N}\) and \(\alpha \in \mathbb{R}\),

\[
P_{\text{floor}}(n) := \sum_{k=1}^{n} \left\lfloor \frac{\alpha + k/n}{n} \right\rfloor \gcd(k, n) = \sum_{d|n} \phi(d) \left\lfloor \frac{n\alpha}{d} \right\rfloor. \tag{35}
\]

**Proposition 14.** For every \(n, r \in \mathbb{N}\),

\[
P_{\text{saw-tooth}}^{(r)}(n) := \sum_{k=1}^{n} \psi(kr/n) \gcd(k, n) = 0. \tag{36}
\]

**Proposition 15.** For every \(n \in \mathbb{N}, n > 1\),

\[
P_{\text{sin}}(n) := \sum_{k=1}^{n-1} (\log \sin(k\pi/n)) \gcd(k, n) = (\phi \ast \log)(n) - (\log 2)(P(n) - n). \tag{37}
\]

**Proposition 16.** For every \(n \in \mathbb{N}\) and \(\alpha \in \mathbb{R}\) with \(\alpha + k/n \notin \mathbb{Z}\) (\(1 \leq k \leq n\)),

\[
P_{\text{cot}}(n) := \sum_{k=1}^{n} \cot \pi(\alpha + k/n) \gcd(k, n) = n \sum_{d|n} \frac{\phi(d)}{d} \cot(\pi n\alpha/d). \tag{38}
\]

These follow from Proposition 1 using the following well-known formulae:

\[
\sum_{k=1}^{n} \left\lfloor \frac{\alpha + k/n}{n} \right\rfloor = \lfloor n\alpha \rfloor, \quad (n \in \mathbb{N}), \tag{39}
\]

\[
\sum_{k=1}^{n} \psi(kr/n) = 0 \quad (n, r \in \mathbb{N}), \tag{40}
\]

\[
\prod_{k=1}^{n-1} \sin(k\pi/n) = \frac{n}{2^{n-1}} \quad (n \in \mathbb{N}) \tag{41}
\]

(for \(n = 1\) the empty product is 1),

\[
\sum_{k=1}^{n} \cot \pi(\alpha + k/n) = n \cot \pi n\alpha \quad (n \in \mathbb{N}, \alpha \in \mathbb{R}, \alpha + k/n \notin \mathbb{Z}, 1 \leq k \leq n). \tag{42}
\]
4 The alternating lcm-sum function

Some of the previous results have counterparts for the lcm-sum function (OEIS A051193)

\[ L(n) := \sum_{k=1}^{n} \text{lcm}[k, n] = \frac{n}{2} \left( 1 + \sum_{d|n} d\phi(d) \right) \quad (n \in \mathbb{N}). \tag{43} \]

We consider here the alternating lcm-sum function defined by (7) and then the analog of (18).

Let \( F(n) := \frac{1}{n} \sum_{d|n} d\phi(d) \). Note that \( F(n) = \sum_{k=1}^{n} (\gcd(k, n))^{-1} \) representing the arithmetic mean of the orders of elements in the cyclic group of order \( n \), cf. [5, p. 3]. Furthermore, let \( \beta(n) := (1 \ast \mu \ast \text{id})(n) = \prod_{d|n} (1 - p) \) and let \( h(n) := \prod_{k=1}^{n} k^k \) be the sequence of hyperfactorials (OEIS A002109).

Proposition 17. Let \( n \in \mathbb{N} \). If \( n \) is odd, then

\[
L_{\text{altern}}(n) = \frac{n}{2} \left( 1 + \sum_{d|n} d\mu(d)\tau(n/d) \right) = \frac{n}{2} \left( 1 + \prod_{p^a || n} (a(1 - p) + 1) \right), \tag{44}
\]

where \( \tau \) is the divisor function.

If \( n \) is even of the form \( n = 2^a m \), where \( a \geq 1 \) and \( m \in \mathbb{N} \) is odd, then

\[
L_{\text{altern}}(n) = 2^{a-1} m \left( \frac{2^{2a} - 1}{3} mF(m) - 1 \right) = \frac{n}{2} \left( \frac{2^{2a} - 1}{2^{2a+1} + 1} nF(n) - 1 \right). \tag{45}
\]

Proof. Let \( \text{id}_-1(n) = n^{-1} \) and \( 1(n) = 1 \) \( (n \in \mathbb{N}) \). We have

\[
L_{\text{altern}}(n) = n \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{\gcd(k, n)} = n \sum_{k=1}^{n} (-1)^{k-1} \sum_{d|\gcd(k, n)} (\text{id}_-1 \ast \mu)(d)
\]

\[
= n \sum_{d|n} \beta(d) \sum_{j=1}^{n/d} (-1)^{dj-1} j.
\]

Now using that \( \sum_{k=1}^{n} (-1)^{k-1} k = (-1)^{n-1} \lfloor (n + 1)/2 \rfloor \) \( (n \in \mathbb{N}) \) the given formulae are obtained along the same lines with the proof of Proposition 3.

Proposition 18. For every \( n \in \mathbb{N} \),

\[
\left( \prod_{k=1}^{n} \text{lcm}[k, n] \right)^{1/n} = \prod_{d|n} h(n/d)^{\beta(d)} \left( \prod_{d|n} d^{\beta(d)/d} \right)^{n/2} \left( \prod_{d|n} d^{\beta(d)/d^2} \right)^{n^2/2}. \tag{46}
\]
Proof. Similar to the proofs of above,

\[
\sum_{k=1}^{n} (\log k) \text{lcm}[k, n] = n \sum_{k=1}^{n} (k \log k) \frac{1}{\text{gcd}(k, n)}
\]

\[
= n \sum_{k=1}^{n} (k \log k) \sum_{d \mid \text{gcd}(k, n)} (\text{id}_1 \ast \mu)(d) = n \sum_{d \mid n} (\text{id}_1 \ast \mu)(d) \sum_{j=1}^{n/d} j d \log(jd)
\]

\[
= n \sum_{d \mid n} \beta(d) \log h(n/d) + \frac{n^2}{2} \sum_{d \mid n} \beta(d) \frac{\log d}{d} + \frac{n^3}{2} \sum_{d \mid n} \beta(d) \frac{\log d}{d^2},
\]

equivalent to (46). \qed

References


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