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Remodified Bessel Functions via Coincidences and Near Coincidences

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Abstract

By considering a particular probabilistic scenario associated with coincidences, we are led to a family of functions akin to the modified Bessel function of the first kind. These are in turn solutions to a certain family of linear differential equations possessing structural similarities to the modified Bessel differential equation. The Stirling number triangle of the second kind arises quite naturally from these differential equations, as do more complicated, yet related, truncated number triangles, none of which appear in Sloane's *On-Line Encyclopedia of Integer Sequences*.

1 Introduction

Let $X \sim \mathrm{Po}(\lambda)$ denote a discrete random variable having the Poisson distribution [3] with parameter λ. The mass function of X is given by

$$\mathrm{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, k \geq 0.$$

Furthermore, suppose that, for some $n \in \mathbb{N}$, the random variables $X_1, X_2, \ldots, X_n$ are independently and identically distributed as X. We shall term the event

$$C_n = \bigcup_{k=0}^{\infty} \{X_1 = X_2 = \cdots = X_n = k\}$$

a *coincidence*. In order to place such an event into some of context, we might imagine n call centers each receiving an average of λ telephone calls per hour. In this situation a coincidence is said to occur when, in a given hour, all of the centers receive exactly the same number of calls.

Next, let $r \in \{1, 2, \ldots, n\}$ and S_r denote the set $\{1, 2, \ldots, n\} \backslash \{r\}$. We define the event A_r by

$$A_r = \{X_{a_1} = X_{a_2} = \cdots = X_{a_{n-1}} = k\},$$

where $\{a_1, a_2, \ldots, a_{n-1}\} = S_r$. The event

$$N_n = \bigcup_{k=0}^{\infty} \bigcup_{r=1}^{n} \{A_r \cap \{X_r = k+1\}\}$$

is known as a *near coincidence*. Returning to the call-center scenario, a near coincidence is said to occur when, in a given hour, all but one of the centers receive exactly the same number of calls, with the remaining center receiving exactly one more call than all of the others.

In this paper we show how probabilities associated with coincidences, near coincidences and beyond, give rise to functions which may be regarded as extended versions of certain Bessel functions. Via the linear differential equations these functions satisfy, this leads first to the triangle of Stirling numbers of the second kind and then on to rather more complicated, yet related, truncated number triangles.

2 A connection with Bessel functions and Stirling numbers of the second kind

Bessel functions arise as solutions to certain linear differential equations. They come in several varieties, and we will be concerned here with a particular Bessel function that appears, amongst other things, in connection with special relativity [4, 5] and the Skellam distribution [10].

Definition 1. The *modified Bessel function of the first kind* [9],

$$I_m(x) = \sum_{k=0}^{\infty} \frac{x^{2k+m}}{2^{2k+m} k! \Gamma(k+m+1)},$$

is one of the solutions to the *modified Bessel differential equation* [8] given by

$$x^2 y'' + xy' - \left(m^2 + x^2\right) y = 0,$$

where $\Gamma(x)$ is the *gamma function* [2, 6].

The probability of a coincidence occurring in any given hour is given by

$$\begin{aligned} \mathrm{P}(C_n) &= \sum_{k=0}^{\infty} \left(\frac{e^{-\lambda}\lambda^k}{k!}\right)^n \\ &= e^{-n\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{nk}}{(k!)^n}. \end{aligned}$$

Then, noting that

$$\begin{aligned} e^{-2\lambda} I_0(2\lambda) &= e^{-2\lambda} \sum_{k=0}^{\infty} \frac{(2\lambda)^{2k}}{2^{2k} k! \Gamma(k+1)} \\ &= e^{-2\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(k!)^2} \\ &= \mathrm{P}(C_2), \end{aligned}$$

we are led first to extend the definition of $I_0(x)$ as follows:

Definition 2. Let $n \in \mathbb{N}$. Then

$$I_0(n, x) = \sum_{k=0}^{\infty} \frac{x^{nk}}{n^{nk} (k!)^n}.$$

The function $I_0(n, x)$ is related to the probability $\mathrm{P}(C_n)$ by way of

$$\begin{aligned} \mathrm{P}(C_n) &= e^{-n\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{nk}}{(k!)^n} \\ &= e^{-n\lambda} I_0(n, n\lambda). \end{aligned}$$

It is clear that $I_0(1, x) = e^x$ and $I_0(2, x) = I_0(x)$. Note also that $y = I_0(1, x)$ and $y = I_0(2, x)$ satisfy $y' - y = 0$ and $xy'' + y' - xy = 0$, respectively. We now find a third-order linear differential equation having $I_0(3, x)$ as a solution.

Result 3. *The function $y = I_0(3, x)$ is a solution to*

$$x^2 y''' + 3xy'' + y' - x^2 y = 0.$$

Proof. Let $y = I_0(3, x)$. We have

$$y' = \sum_{k=1}^{\infty} \frac{x^{3k-1}}{3^{3k-1} (k!)^2 (k-1)!}$$

and

$$y'' = \sum_{k=1}^{\infty} \frac{(3k-1) x^{3k-2}}{3^{3k-1} (k!)^2 (k-1)!},$$

so that

$$x(y' + xy'') = \sum_{k=1}^{\infty} \frac{x^{3k}}{3^{3k-2} k! ((k-1)!)^2},$$

and hence

$$\begin{aligned} (x(y' + xy''))' &= \sum_{k=1}^{\infty} \frac{x^{3k-1}}{3^{3(k-1)} ((k-1)!)^3} \\ &= \sum_{k=0}^{\infty} \frac{x^{3k+2}}{3^{3k} (k!)^3} \\ &= x^2 y. \end{aligned}$$

It follows from this that $y = I_0(3, x)$ satisfies

$$x^2 y''' + 3xy'' + y' - x^2 y = 0,$$

as required. $\square$

Adopting a method similar to that used in Result 3, we may show that $y = I_0(4, x)$ is a solution to

$$\left(x\left(x^2 y''' + 3xy'' + y'\right)\right)' - x^3 y = 0,$$

and so on. It is in fact the case that $I_0(n, x)$ satisfies the nth-order linear differential equation

$$\sum_{k=1}^{n} S(n,k) x^{k-1} y^{(k)} - x^{n-1} y = 0, \tag{1}$$

where $S(n, k)$ is a Stirling number of the second kind, enumerating the partitions of n distinct objects into exactly k non-empty parts, and $y^{(k)}$ denotes the kth derivative of y with respect to x. The number triangle associated with $S(n, k)$ appears in Sloane's *On-Line Encyclopedia of Integer Sequences* [7] as sequence A008277. It is straightforward to prove (1) by using induction in conjunction with the well-known result

$$S(n,k) = kS(n-1,k) + S(n-1,k-1), \tag{2}$$

which may be found in [1] and [6].

3 Near coincidences

Let us now consider the probability of the occurrence of a near coincidence, assuming that $n \geq 2$. We have

$$\begin{aligned}
\mathrm{P}(N_n) &= \binom{n}{1} \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!} \left(\frac{e^{-\lambda}\lambda^k}{k!}\right)^{n-1} \\
&= ne^{-n\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{nk+1}}{(k!)^n (k+1)}.
\end{aligned}$$

This leads us to generalize $I_0(n, x)$ as follows:

Definition 4. For $n \geq m + 1$,

$$I_m(n,x) = \sum_{k=0}^{\infty} \frac{x^{nk+m}}{n^{nk+m} (k!)^n (k+1)^m}.$$

Note then that $\mathrm{P}(N_n) = ne^{-n\lambda} I_1(n, n\lambda)$. In this section we will indeed consider the special case $m = 1$, which is the one associated with near coincidences.

Result 5. *The function $y = I_1(2, x)$ satisfies the differential equation*

$$x^3 y''' + 2x^2 y'' - xy' + y - x^2 \left(xy' + y\right) = 0.$$

Proof. First,

$$\begin{aligned}
(xI_1(2,x))' &= \sum_{k=0}^{\infty} \frac{2(k+1)x^{2k+1}}{2^{2k+1}(k!)^2(k+1)} \\
&= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k}(k!)^2} \\
&= xI_0(2,x).
\end{aligned}$$

Therefore

$$\begin{aligned}
I_0(2,x) &= \frac{1}{x}\left(xI_1'(2,x) + I_1(2,x)\right) \\
&= I_1'(2,x) + \frac{1}{x}I_1(2,x),
\end{aligned}$$

and so, since $y = I_0(2,x)$ is a solution to $xy'' + y' - xy = 0$, we may obtain

$$x\left(I_1'(2,x) + \frac{1}{x}I_1(2,x)\right)'' + \left(I_1'(2,x) + \frac{1}{x}I_1(2,x)\right)' - x\left(I_1'(2,x) + \frac{1}{x}I_1(2,x)\right) = 0.$$

From this it follows that $y = I_1(2,x)$ does in fact satisfy

$$x^3y''' + 2x^2y'' - xy' + y - x^2\left(xy' + y\right) = 0.$$

$\square$

Taking things further,

$$\begin{aligned}
\left(x^2I_1(3,x)\right)' &= \sum_{k=0}^{\infty} \frac{3(k+1)x^{3k+2}}{3^{3k+1}(k!)^3(k+1)} \\
&= \sum_{k=0}^{\infty} \frac{x^{3k+2}}{3^{3k}(k!)^3} \\
&= x^2I_0(3,x),
\end{aligned}$$

from which we may obtain, using Result 3, that $y = I_1(3,x)$ is a solution to

$$x^4y'''' + 5x^3y''' + x^2y'' + 2xy' - 2y - x^3\left(xy' + 2y\right) = 0.$$

More generally,

$$\begin{aligned}
\left(x^{n-1}I_1(n,x)\right)' &= \sum_{k=0}^{\infty} \frac{n(k+1)x^{nk+n-1}}{n^{nk+1}(k!)^n(k+1)} \\
&= \sum_{k=0}^{\infty} \frac{x^{nk+n-1}}{n^{nk}(k!)^n} \\
&= x^{n-1}I_0(n,x),
\end{aligned}$$

giving

$$I_0(n,x) = I_1'(n,x) + \frac{n-1}{x}I_1(n,x). \tag{3}$$

Result 6. *The function $y = I_1(n,x)$ is a solution to*

$$\sum_{k=1}^{n} f(n,k)x^{k+1}y^{(k+1)} + (n-1)(-1)^{n+1}(xy'-y) - x^n(xy'+(n-1)y) = 0, \tag{4}$$

where

$$f(n,k) = S(n,k) + \frac{n-1}{(k+1)!}\sum_{j=k+1}^{n}(-1)^{j-k+1}j!S(n,j), \tag{5}$$

noting that the sum on the far right is defined to be zero when $k \geq n$.

Proof. Starting with (3) and proceeding by induction gives

$$I_0^{(k)}(n,x) = I_1^{(k+1)}(n,x) + k!(n-1)\sum_{j=0}^{k}\frac{(-1)^{k-j}I_1^{(j)}(n,x)}{x^{k-j+1}j!}.$$

From this we obtain, using (1) and induction once more, the general result that $I_1(n,x)$ satisfies the differential equation

$$\begin{aligned}\sum_{k=1}^{n} f(n,k)x^{k+1}y^{(k+1)} + (n-1)\sum_{k=1}^{n}(-1)^{k+1}k!S(n,k)(xy'-y)&\\ - x^n(xy'+(n-1)y) = 0,&\end{aligned} \tag{6}$$

where $f(n,k)$ is as given in the statement of the result.

Since, by definition,

$$\sum_{k=1}^{n} S(n,k)(x)_k = x^n,$$

where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ denotes the *falling factorial*, on setting $x=-1$ it follows that

$$\begin{aligned}(-1)^n &= \sum_{k=1}^{n} S(n,k)(-1)_k\\ &= \sum_{k=1}^{n} S(n,k)k!(-1)^k.\end{aligned}$$

Therefore (6) may be simplified somewhat to give the desired result. □

Since in Definition 4 we require $n \geq m+1$, the number triangle for $f(n,k)$ is, unlike that for $S(n,k)$, truncated. Its first few rows may be seen in Table 1 of Section 5.

4 Further coincidences

Next, let $r, s \in \{1, 2, \ldots, n\}$ such that $r \neq s$, and let $Q_{r,s}$ denote the set $\{1, 2, \ldots, n\} \backslash \{r, s\}$. We define $B_{r,s}$ by

$$B_{r,s} = \{X_{a_1} = X_{a_2} = \cdots = X_{a_{n-2}} = k\},$$

where $\{a_1, a_2, \ldots, a_{n-2}\} = Q_{r,s}$, and consider the event

$$M_n = \bigcup_{k=0}^{\infty} \bigcup_{r,s} \{B_{r,s} \cap \{X_r = X_s = k+1\}\},$$

where the inner union is over all possible pairs (r, s) such that $r, s \in \{1, 2, \ldots, n\}$ and $r \neq s$. It is clear that

$$\begin{aligned} \mathrm{P}(M_n) &= \binom{n}{2} \sum_{k=0}^{\infty} \left(\frac{e^{-\lambda} \lambda^{k+1}}{(k+1)!} \right)^2 \left(\frac{e^{-\lambda} \lambda^k}{k!} \right)^{n-2} \\ &= e^{-n\lambda} \binom{n}{2} \sum_{k=0}^{\infty} \frac{\lambda^{nk+2}}{(k!)^n (k+1)^2} \\ &= e^{-n\lambda} \binom{n}{2} I_2(n, n\lambda). \end{aligned}$$

For this new scenario we now obtain, in correspondence to Result 6, a family of linear differential equations and the associated truncated number triangle.

First,

$$\begin{aligned} \left(x \left(x^{n-2} I_2(n, x) \right)' \right)' &= \left(x \sum_{k=0}^{\infty} \frac{n(k+1) x^{nk+n-1}}{n^{nk+2} (k!)^n (k+1)^2} \right)' \\ &= \left(\sum_{k=0}^{\infty} \frac{x^{nk+n}}{n^{nk+1} (k!)^n (k+1)} \right)' \\ &= \sum_{k=0}^{\infty} \frac{n(k+1) x^{nk+n-1}}{n^{nk+1} (k!)^n (k+1)} \\ &= \sum_{k=0}^{\infty} \frac{x^{nk+n-1}}{n^{nk} (k!)^n} \\ &= x^{n-1} I_0(n, x), \end{aligned}$$

from which we have

$$I_0(n, x) = I_2''(n, x) + \frac{2n-3}{x} I_2'(n, x) + \frac{(n-2)^2}{x^2} I_2(n, x). \tag{7}$$

It follows from this that

$$I_0^{(k)}(n,x) = I_2^{(k+2)}(n,x) + k!(2n-3)\sum_{j=0}^{k}\frac{(-1)^{k-j}I_2^{(j+1)}(n,x)}{x^{k-j+1}j!} + k!(n-2)^2\sum_{j=0}^{k}\frac{(-1)^{k-j}(k-j+1)I_2^{(j)}(n,x)}{x^{k-j+2}j!}.$$

Then, to obtain a linear differential equation satisfied by $I_2(n,x)$, we may use the fact that $I_0(n,x)$ is a solution to (1) to give

$$\sum_{k=1}^{n} g(n,k)x^{k+2}y^{(k+2)} + (n-2)^2\sum_{k=1}^{n} k!S(n,k)\sum_{j=0}^{2}\frac{(-1)^{k+j}}{j!}x^j y^{(j)}(k-j+1) + (2n-3)(-1)^{n+1}\left(x^2y'' - xy'\right) - x^n\left(x^2y'' + (2n-3)xy' + (n-2)^2y\right) = 0, \quad (8)$$

where

$$g(n,k) = S(n,k) + \frac{2n-3}{(k+1)!}\sum_{j=k+1}^{n}(-1)^{j-k-1}j!S(n,j) + \frac{(n-2)^2}{(k+2)!}\sum_{j=k+2}^{n}(-1)^{j-k}j!S(n,j)(j-k-1), \quad (9)$$

and the first and second sums on the right are defined to be zero when $k \geq n$ and $k \geq n-1$, respectively. The truncated number triangle for $g(n,k)$ is shown in Table 2 of Section 5.

The sequence of numbers along the mth diagonal of the triangle of Stirling numbers of the second kind is given by $\{S(n+m,n) : n = 1,2,3,\ldots\}$, where the convention is that the upper-most diagonal, consisting solely of 1s, is the zeroth diagonal. The sequences for $m = 1$, 2, 3 and 4 appear in [7] as A000127, A001296, A001297, and A001298, respectively. Using the recurrence relation (2), it is possible to show, by induction, that for each $m \in \mathbb{N}$ there exists a polynomial $p_m(x)$ such that $S(n+m,n) = p_m(n)$, $n = 1,2,3,\ldots$. It is in fact reasonably straightforward to show that $p_m(x)$ has degree $2m$ and leading coefficient $\frac{1}{2^m m!}$. It follows from this, in conjunction with (5), that the mth diagonal of the table for $f(n,k)$ is a polynomial sequence of degree $2m$. On using (9), a similar result applies to the table for $g(n,k)$.

It is possible to generalize the results (3) and (7). We have

$$x^{n-1}I_0(n,x) = \left(x\cdots\left(x\left(x^{n-m}I_m(n,x)\right)'\right)'\cdots\right)',$$

where the nesting is to a depth of m. It follows from this that

$$x^{n-1}I_0(n,x) = \sum_{k=1}^{m} S(m,k)x^{k-1}\left(x^{n-m}I_m(n,x)\right)^{(k)}. \quad (10)$$

It is also straightforward to show that

$$(x^q h(x))^{(k)} = \sum_{j=0}^{k} x^{q-j} (q)_j \binom{k}{j} h^{(k-j)}(x), \tag{11}$$

where $(q)_0 = 1$ by definition. From (10) and (11) we may obtain

$$\begin{aligned} I_0(n,x) &= \frac{1}{x^{n-1}} \sum_{k=1}^{m} S(m,k) x^{k-1} \sum_{j=0}^{k} x^{n-m-j} (n-m)_j \binom{k}{j} I_m^{(k-j)}(n,x) \\ &= \sum_{k=1}^{m} S(m,k) \sum_{j=0}^{k} x^{k-j-m} (n-m)_j \binom{k}{j} I_m^{(k-j)}(n,x). \end{aligned}$$

This result, in conjunction with (1), allows us to find a linear differential equation satisfied by $y = I_m(n,x)$ for any $m \in \mathbb{N}$.

5 Tables

n	$f(n,1)$	$f(n,2)$	$f(n,3)$	$f(n,4)$	$f(n,5)$	$f(n,6)$	$f(n,7)$	$f(n,8)$
2	2	1						
3	1	5	1					
4	4	13	9	1				
5	1	35	45	14	1			
6	6	81	190	110	20	1		
7	1	189	721	686	224	27	1	
8	8	421	2583	3759	1932	406	35	1

Table 1: The coefficients $f(n,k)$ of the differential equation (4).

n	$g(n,1)$	$g(n,2)$	$g(n,3)$	$g(n,4)$	$g(n,5)$	$g(n,6)$	$g(n,7)$	$g(n,8)$
3	2	6	1					
4	−2	21	11	1				
5	46	50	69	17	1			
6	−150	201	318	162	24	1		
7	526	294	1421	1141	319	32	1	
8	−1498	1429	5481	7035	3120	562	41	1

Table 2: The coefficients $g(n,k)$ of the differential equation (8).

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