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Series of Error Terms for Rational Approximations of Irrational Numbers

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Abstract

Let p_n/q_n be the n-th convergent of a real irrational number α, and let $\varepsilon_n = \alpha q_n - p_n$. In this paper we investigate various sums of the type $\sum_m \varepsilon_m$, $\sum_m |\varepsilon_m|$, and $\sum_m \varepsilon_m x^m$. The main subject of the paper is bounds for these sums. In particular, we investigate the behaviour of such sums when α is a quadratic surd. The most significant properties of the error sums depend essentially on Fibonacci numbers or on related numbers.

1 Statement of results for arbitrary irrationals

Given a real irrational number α and its regular continued fraction expansion

$$\alpha = \langle a_0; a_1, a_2, \dots \rangle \qquad (a_0 \in \mathbb{Z},\, a_\nu \in \mathbb{N} \text{ for } \nu \geq 1)\,,$$

the convergents p_n/q_n of α form a sequence of best approximating rationals in the following sense: for any rational p/q satisfying $1 \leq q < q_n$ we have

$$\left| \alpha - \frac{p_n}{q_n} \right| < \left| \alpha - \frac{p}{q} \right| .$$

The convergents p_n/q_n of α are defined by finite continued fractions

$$\frac{p_n}{q_n} = \langle a_0; a_1, \dots, a_n \rangle .$$

The integers p_n and q_n can be computed recursively using the initial values $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, $q_0 = 1$, and the recurrence formulae

$$p_n \;=\; a_n p_{n-1} + p_{n-2}\,, \qquad q_n \;=\; a_n q_{n-1} + q_{n-2} \tag{1}$$

with $n \geq 1$. Then p_n/q_n is a rational number in lowest terms satisfying the inequalities

$$\frac{1}{q_n + q_{n+1}} \;<\; |q_n\alpha - p_n| \;<\; \frac{1}{q_{n+1}} \qquad (n \geq 0)\,. \tag{2}$$

The error terms $q_n\alpha - p_n$ alternate, i.e., $\operatorname{sgn}(q_n\alpha - p_n) = (-1)^n$. For basic facts on continued fractions and convergents see [4, 5, 8].
Throughout this paper let

$$\rho \;=\; \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\rho} \;=\; -\frac{1}{\rho} \;=\; \frac{1-\sqrt{5}}{2}\,.$$

The Fibonacci numbers F_n are defined recursively by $F_{-1} = 1$, $F_0 = 0$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 1$. In this paper we shall often apply *Binet's formula*,

$$F_n \;=\; \frac{1}{\sqrt{5}}\Big(\rho^n - \Big(-\frac{1}{\rho}\Big)^n\Big) \qquad (n \geq 0)\,. \tag{3}$$

While preparing a talk on the subject of so-called *leaping convergents* relying on the papers [2, 6, 7], the author applied results for convergents to the number $\alpha = e = \exp(1)$. He found two identities which are based on formulas given by Cohn [1]:

$$\sum_{n=0}^{\infty}(q_n e - p_n) \;=\; 2\int_0^1 \exp(t^2)\,dt - 2e + 3 \;=\; 0.4887398\ldots\,,$$

$$\sum_{n=0}^{\infty}|q_n e - p_n| \;=\; 2e\int_0^1 \exp(-t^2)\,dt - e \;=\; 1.3418751\ldots\,.$$

These identities are the starting points of more generalized questions concerning error series of real numbers α.

1.) What is the maximum size M of the series $\sum_{m=0}^{\infty}|q_m\alpha - p_m|$? One easily concludes that $M \geq (1+\sqrt{5})/2$, because $\sum_{m=0}^{\infty}\big|q_m(1+\sqrt{5})/2 - p_m\big| = (1+\sqrt{5})/2$.

2.) Is there a method to compute $\sum_{m=0}^{\infty}|q_m\alpha - p_m|$ explicitly for arbitrary real quadratic irrationals ?

The series $\sum_{m=0}^{\infty}|q_m\alpha - p_m| \in [0, M]$ measures the approximation properties of α on average. The smaller this series is, the better rational approximations α has. Nevertheless, α can be a Liouville number and $\sum_{m=0}^{\infty}|q_m\alpha - p_m|$ takes a value close to M. For example, let us consider the numbers

$$\alpha_n \;=\; \langle\, 1; \underbrace{1, \ldots, 1}_{n}, a_{n+1}, a_{n+2}, \ldots \rangle$$

for even positive integers n, where the elements $a_{n+1}, a_{n+2}, \ldots$ are defined recursively in the following way. Let $p_k/q_k = \langle 1; \underbrace{1, \ldots, 1}_{k} \rangle$ for $k = 0, 1, \ldots, n$ and set

$$\begin{array}{lllllll} a_{n+1} & := & q_n^n\,, & q_{n+1} & = & a_{n+1}q_n + q_{n-1} & = & q_n^n\big(q_n + q_{n-1}\big)\,, \\ a_{n+2} & := & q_{n+1}^{n+1}\,, & q_{n+2} & = & a_{n+2}q_{n+1} + q_n & = & q_{n+1}^{n+1}\big(q_n^{n+1} + q_{n-1}\big)\,, \\ a_{n+3} & := & q_{n+2}^{n+2}\,, & q_{n+3} & = & a_{n+3}q_{n+2} + q_{n+1} & = & \ldots \end{array}$$

and so on. In the general case we define a_{k+1} by $a_{k+1} = q_k^k$ for $k = n, n+1, \ldots$. Then we have with (1) and (2) that

$$0 < \left| \alpha_n - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} < \frac{1}{a_{k+1}q_k^2} = \frac{1}{q_k^{k+2}} \quad (k \geq n)\,.$$

Hence α_n is a Liouville number. Now it follows from (9) in Theorem 2 below with $2k = n$ and $n_0 = (n/2) - 1$ that

$$\sum_{m=0}^{\infty} |q_m\alpha_n - p_m| > \sum_{m=0}^{n-1} |q_m\alpha_n - p_m| = (F_{n-1} - 1)(\rho - \alpha_n) + \rho - \rho^{1-n} \geq \rho - \frac{1}{\rho^{n-1}}\,.$$

We shall show by Theorem 2 that $M = \rho$, such that the error sums of the Liouville numbers α_n tend to this maximum value ρ for increasing n.
We first treat infinite sums of the form $\sum_n |q_n\alpha - p_n|$ for arbitrary real irrational numbers $\alpha = \langle 1; a_1, a_2, \ldots \rangle$, when we may assume without loss of generality that $1 < \alpha < 2$.

Proposition 1. *Let $\alpha = \langle 1; a_1, a_2, \ldots \rangle$ be a real irrational number. Then for every integer $m \geq 0$, the following two inequalities hold: Firstly,*

$$|q_{2m}\alpha - p_{2m}| + |q_{2m+1}\alpha - p_{2m+1}| < \frac{1}{\rho^{2m}}\,, \tag{4}$$

provided that either

$$a_{2m}a_{2m+1} > 1 \qquad \text{or} \qquad \big(a_{2m} = a_{2m+1} = 1 \quad \text{and} \quad a_1a_2\cdots a_{2m-1} > 1\big)\,. \tag{5}$$

Secondly,

$$|q_{2m}\alpha - p_{2m}| + |q_{2m+1}\alpha - p_{2m+1}| = \frac{1}{\rho^{2m}} + F_{2m}(\rho - \alpha) \qquad (0 \leq m \leq k)\,, \tag{6}$$

provided that

$$a_1 = a_2 = \ldots = a_{2k+1} = 1\,. \tag{7}$$

In the second term on the right-hand side of (6), $\rho - \alpha$ takes positive or negative values according to the parity of the smallest subscript $r \geq 1$ with $a_r > 1$: For odd r we have $\rho > \alpha$, otherwise, $\rho < \alpha$.
Next, we introduce a set $\mathcal{M}$ of irrational numbers, namely

$$\mathcal{M} := \big\{ \alpha \in \mathbb{R} \setminus \mathbb{Q} \;\big|\; \exists\, k \in \mathbb{N} : \alpha = \langle 1; 1, \ldots, 1, a_{2k+1}, a_{2k+2}, \ldots \rangle \wedge a_{2k+1} > 1 \big\}\,.$$

Note that $\rho > \alpha$ for $\alpha \in \mathcal{M}$. Our main result for real irrational numbers is given by the subsequent theorem.

Theorem 2. *Let $1 < \alpha < 2$ be a real irrational number and let $g, n \geq 0$ be integers with $n \geq 2g$. Set $n_0 := \lfloor n/2 \rfloor$. Then the following inequalities hold.*
1.) *For $\alpha \notin \mathcal{M}$ we have*

$$\sum_{\nu=2g}^{n} |q_\nu \alpha - p_\nu| \;\leq\; \rho^{1-2g} - \rho^{-2n_0-1}\,, \tag{8}$$

with equality for $\alpha = \rho$ and every odd $n \geq 0$.
2.) *For $\alpha \in \mathcal{M}$, say $\alpha = \langle 1; 1, \ldots, 1, a_{2k+1}, a_{2k+2}, \ldots \rangle$ with $a_{2k+1} > 1$, we have*

$$\sum_{\nu=2g}^{n} |q_\nu \alpha - p_\nu| \;\leq\; (F_{2k-1} - F_{2g-1})(\rho - \alpha) + \rho^{1-2g} - \rho^{-2n_0-1}\,, \tag{9}$$

with equality for $n = 2k - 1$.
3.) *We have*

$$\sum_{\nu=2g}^{\infty} |q_\nu \alpha - p_\nu| \;\leq\; \rho^{1-2g}\,, \tag{10}$$

with equality for $\alpha = \rho$.

In particular, for any positive ε and any even integer n satisfying

$$n \;\geq\; \frac{\log(\rho/\varepsilon)}{\log \rho}\,,$$

it follows that

$$\sum_{\nu=n}^{\infty} |q_\nu \alpha - p_\nu| \;\leq\; \varepsilon\,.$$

For $\nu \geq 1$ we know by $q_2 \geq 2$ and by (2) that $|q_\nu \alpha - p_\nu| < 1/q_{\nu+1} \leq 1/q_2 \leq 1/2$, which implies $|q_\nu \alpha - p_\nu| = \|q_\nu \alpha\|$, where $\|\beta\|$ denotes the distance of a real number β to the nearest integer. For $\alpha = \langle a_0; a_1, a_2, \ldots \rangle$, $|q_0 \alpha - p_0| = \alpha - a_0 = \{\alpha\}$ is the fractional part of α. Therefore, we conclude from Theorem 2 that

$$\sum_{\nu=1}^{\infty} \|q_\nu \alpha\| \;\leq\; \rho - \{\alpha\}\,.$$

In particular, we have for $\alpha = \rho$ that

$$\sum_{\nu=1}^{\infty} \|q_\nu \rho\| \;=\; 1\,.$$

The following theorem gives a simple bound for $\sum_m (q_m \alpha - p_m)$.

Theorem 3. *Let α be a real irrational number. Then the series $\sum_{m=0}^{\infty} (q_m \alpha - p_m) x^m$ converges absolutely at least for $|x| < \rho$, and*

$$0 \;<\; \sum_{m=0}^{\infty} (q_m \alpha - p_m) \;<\; 1\,.$$

Both the upper bound 1 and the lower bound 0 are best possible.

The proof of this theorem is given in Section 3. We shall prove Proposition 1 and Theorem 2 in Section 4, using essentially the properties of Fibonacci numbers.

2 Statement of Results for Quadratic Irrationals

In this section we state some results for error sums involving real quadratic irrational numbers α. Any quadratic irrational α has a periodic continued fraction expansion,

$$\alpha = \langle a_0; a_1, \dots, a_\omega, T_1, \dots, T_r, T_1, \dots, T_r, \dots \rangle = \langle a_0; a_1, \dots, a_\omega, \overline{T_1, \dots, T_r} \rangle ,$$

say. Then there is a linear three-term recurrence formula for $z_n = p_{rn+s}$ and $z_n = q_{rn+s}$ $(s = 0, 1, \dots, r-1)$, [3, Corollary 1]. This recurrence formula has the form

$$z_{n+2} = Gz_{n+1} \pm z_n \qquad (rn > \omega) .$$

Here, G denotes a positive integer, which depends on α and r, but not on n and s. The number G can be computed explicitly from the numbers $T_1, \dots, T_r$ of the continued fraction expansion of α. This is the basic idea on which the following theorem relies.

Theorem 4. *Let α be a real quadratic irrational number. Then*

$$\sum_{m=0}^{\infty} (q_m\alpha - p_m)x^m \in \mathbb{Q}[\alpha](x) .$$

It is not necessary to explain further technical details of the proof. Thus, the generating function of the sequence $(q_m\alpha - p_m)_{m\geq 0}$ is a rational function with coefficients from $\mathbb{Q}[\alpha]$.

Example 5. Let $\alpha = \sqrt{7} = \langle 2; \overline{1, 1, 1, 4} \rangle$. Then

$$\sum_{m=0}^{\infty} (q_m\sqrt{7} - p_m)x^m = \frac{x^3 - (2+\sqrt{7})x^2 + (3+\sqrt{7})x - (5+2\sqrt{7})}{x^4 - (8+3\sqrt{7})} . \tag{11}$$

In particular, for $x = 1$ and $x = -1$ we obtain

$$\sum_{m=0}^{\infty} (q_m\sqrt{7} - p_m) = \frac{21 - 5\sqrt{7}}{14} = 0.555088817\dots ,$$

$$\sum_{m=0}^{\infty} |q_m\sqrt{7} - p_m| = \frac{7 + 5\sqrt{7}}{14} = 1.444911182\dots .$$

Next, we consider the particular quadratic surds

$$\alpha = \frac{n + \sqrt{4+n^2}}{2} = \langle n; n, n, n, \dots \rangle$$

and compute the generating function of the error terms $q_m\alpha - p_m$.

Corollary 6. *Let $n \geq 1$ and $\alpha = (n + \sqrt{4+n^2})/2$. Then*

$$\sum_{m=0}^{\infty} (q_m\alpha - p_m)x^m = \frac{1}{x+\alpha},$$

particularly

$$\sum_{m=0}^{\infty} (q_m\alpha - p_m) = \frac{1}{\alpha+1}, \quad \sum_{m=0}^{\infty} |q_m\alpha - p_m| = \frac{1}{\alpha-1}, \quad \sum_{m=0}^{\infty} \frac{q_m\alpha - p_m}{m+1} = \log\Big(1 + \frac{1}{\alpha}\Big).$$

For the number $\rho = (1+\sqrt{5})/2$ we have $p_m = F_{m+2}$ and $q_m = F_{m+1}$. Hence, using $1/(\rho+1) = (3-\sqrt{5})/2 = 1 + \overline{\rho}$, $1/(\rho - 1) = \rho$, and $1 + 1/\rho = \rho$, we get from Corollary 6

$$\sum_{m=0}^{\infty} (F_{m+1}\rho - F_{m+2}) = 1 + \overline{\rho}, \quad \sum_{m=0}^{\infty} |F_{m+1}\rho - F_{m+2}| = \rho, \quad \sum_{m=0}^{\infty} \frac{F_{m+1}\rho - F_{m+2}}{m+1} = \log \rho. \tag{12}$$

Similarly, we obtain for the number $\alpha = \sqrt{7}$ from (11):

$$\sum_{m=0}^{\infty} \frac{q_m\sqrt{7} - p_m}{m+1} = \int_0^1 \frac{x^3 - (2+\sqrt{7})x^2 + (3+\sqrt{7})x - (5+2\sqrt{7})}{x^4 - (8+3\sqrt{7})}\, dx = 0.5568649708\ldots$$

3 Proof of Theorem 3

Throughout this paper we shall use the abbreviations $\varepsilon_m(\alpha) = \varepsilon_m := q_m\alpha - p_m$ and $\varepsilon(\alpha) = \sum_{m=0}^{\infty} |\varepsilon_m(\alpha)|$. The sequence $(|\varepsilon_m|)_{m\geq 0}$ converges strictly decreasing to zero. Since $\varepsilon_0 > 0$ and $\varepsilon_m\varepsilon_{m+1} < 0$, we have

$$\varepsilon_0 + \varepsilon_1 < \sum_{m=0}^{\infty} \varepsilon_m < \varepsilon_0.$$

Put $a_0 = \lfloor \alpha \rfloor$, $\theta := \varepsilon_0 = \alpha - a_0$, so that $0 < \theta < 1$. Moreover,

$$\varepsilon_0 + \varepsilon_1 = \theta + a_1\alpha - (a_0a_1 + 1) = \theta + a_1\theta - 1 = \theta + \Big\lfloor \frac{1}{\theta} \Big\rfloor \theta - 1.$$

Choosing an integer $k \geq 1$ satisfying

$$\frac{1}{k+1} < \theta < \frac{1}{k},$$

we get

$$\theta + \Big\lfloor \frac{1}{\theta} \Big\rfloor \theta - 1 > \frac{1}{k+1} + \frac{k}{k+1} - 1 = 0,$$

which proves the lower bound for $\sum \varepsilon_m$.
In order to estimate the radius of convergence for the series $\sum \varepsilon_m x^m$ we first prove the inequality

$$q_m \geq F_{m+1} \qquad (m \geq 0), \tag{13}$$

which follows inductively. We have $q_0 = 1 = F_1$, $q_1 = a_1 \geq 1 = F_2$, and

$$q_m \;=\; a_m q_{m-1} + q_{m-2} \;\geq\; q_{m-1} + q_{m-2} \;\geq\; F_m + F_{m-1} \;=\; F_{m+1} \qquad (m \geq 2)\,,$$

provided that (13) is already proven for q_{m-1} and q_{m-2}. With Binet's formula (3) and (13) we conclude that

$$q_{m+1} \;\geq\; \frac{1}{\sqrt{5}}\Big(\rho^{m+2} - \Big(-\frac{1}{\rho}\Big)^{m+2}\Big) \;\geq\; \frac{1}{\sqrt{5}}\rho^m \qquad (m \geq 0)\,. \tag{14}$$

Hence, we have

$$|\varepsilon_m|x^m \;=\; |q_m\alpha - p_m|x^m \;<\; \frac{x^m}{q_{m+1}} \;\leq\; \sqrt{5}\Big(\frac{x}{\rho}\Big)^m \qquad (m \geq 0)\,.$$

It follows that the series $\sum \varepsilon_m x^m$ converges absolutely at least for $|x| < \rho$. In order to prove that the upper bound 1 is best possible, we choose $0 < \varepsilon < 1$ and a positive integer n satisfying

$$\frac{1}{n}\Big(1 + \frac{\rho\sqrt{5}}{\rho - 1}\Big) \;<\; \varepsilon\,.$$

Put

$$\alpha_n \;:=\; \langle\, 0; 1, \overline{n}\,\rangle \;=\; \frac{1}{2} - \frac{1}{n} + \frac{1}{2}\sqrt{1 + \frac{4}{n^2}} \;>\; 1 - \frac{1}{n}\,.$$

With $p_0 = 0$ and $q_0 = 1$ we have by (1), (2), and (14),

$$\begin{aligned}
\sum_{m=0}^{\infty}(q_m\alpha_n - p_m) \;&\geq\; \alpha_n - \sum_{m=1}^{\infty}|q_m\alpha_n - p_m| \\
&>\; 1 - \frac{1}{n} - \sum_{m=1}^{\infty}\frac{1}{q_{m+1}} \;\geq\; 1 - \frac{1}{n} - \sum_{m=1}^{\infty}\frac{1}{nq_m} \\
&\geq\; 1 - \frac{1}{n} - \frac{\sqrt{5}}{n}\sum_{m=1}^{\infty}\frac{1}{\rho^{m-1}} \\
&=\; 1 - \frac{1}{n}\Big(1 + \frac{\rho\sqrt{5}}{\rho - 1}\Big) \;>\; 1 - \varepsilon\,.
\end{aligned}$$

For the lower bound 0 we construct quadratic irrational numbers $\beta_n \;:=\; \langle\, 0; \overline{n}\,\rangle$ and complete the proof of the theorem by similar arguments. □

4 Proofs of Proposition 1 and Theorem 2

Lemma 7. *Let $\alpha = \langle a_0; a_1, a_2, \ldots\rangle$ be a real irrational number with convergents p_m/q_m. Let $n \geq 1$ be a subscript satisfying $a_n > 1$. Then*

$$q_{n+k} \;\geq\; F_{n+k+1} + F_{k+1}F_n \qquad (k \geq 0)\,. \tag{15}$$

In the case $n \equiv k+1 \equiv 0 \pmod 2$ we additionally assume that $n \geq 4$, $k \geq 3$. Then

$$F_{n+k+1} + F_{k+1}F_n \; > \; \rho^{n+k}\,. \tag{16}$$

When $\alpha - \rho \notin \mathbb{Z}$, the inequality (15) with $m = n+k$ is stronger than (13).

Proof. We prove (15) by induction on k. Using (1) and (13), we obtain for $k = 0$ and $k = 1$, respectively,

$$q_n \;=\; a_n q_{n-1} + q_{n-2} \;\geq\; 2F_n + F_{n-1} \;=\; (F_n + F_{n-1}) + F_n \;=\; F_{n+1} + F_1 F_n\,,$$

$$q_{n+1} \;=\; a_{n+1} q_n + q_{n-1} \;\geq\; q_n + q_{n-1} \;\geq\; (F_{n+1} + F_n) + F_n \;=\; F_{n+2} + F_2 F_n\,.$$

Now, let $k \geq 0$ and assume that (15) is already proven for q_{n+k} and q_{n+k+1}. Then

$$\begin{aligned} q_{n+k+2} &\geq q_{n+k+1} + q_{n+k} \\ &\geq (F_{n+k+2} + F_{k+2}F_n) + (F_{n+k+1} + F_{k+1}F_n) \\ &= F_{n+k+3} + F_{k+3}F_n\,. \end{aligned}$$

This corresponds to (15) with k replaced by $k+2$. In order to prove (16) we express the Fibonacci numbers F_m by Binet's formula (3). Hence, we have

$$\begin{aligned} & F_{n+k+1} + F_{k+1}F_n \\ =\;& \rho^{n+k}\left(\rho\left(\frac{1}{5} + \frac{1}{\sqrt{5}}\right) + (-1)^{n+k}\left(\frac{1}{\sqrt{5}} - \frac{1}{5}\right)\frac{1}{\rho^{2n+2k+1}} + \frac{1}{5}\left(\frac{(-1)^{n+1}}{\rho^{2n-1}} + \frac{(-1)^{k}}{\rho^{2k+1}}\right)\right). \end{aligned}$$

Case 1: Let $n \equiv k \equiv 1 \pmod 2$.
In particular, we have $k \geq 1$. Then

$$\begin{aligned} & F_{n+k+1} + F_{k+1}F_n \\ =\;& \rho^{n+k}\left(\rho\left(\frac{1}{5} + \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{5}\right)\frac{1}{\rho^{2n+2k+1}} + \frac{1}{5}\left(\frac{1}{\rho^{2n-1}} - \frac{1}{\rho^{2k+1}}\right)\right) \\ >\;& \rho^{n+k}\left(\rho\left(\frac{1}{5} + \frac{1}{\sqrt{5}}\right) - \frac{1}{5\rho^3}\right) \;=\; \rho^{n+k}\,. \end{aligned}$$

Case 2: Let $n \equiv 1 \pmod 2$, $k \equiv 0 \pmod 2$.
In particular, we have $n \geq 1$ and $k \geq 0$. First, we assume that $k \geq 2$. Then, by similar computations as in Case 1, we obtain

$$\begin{aligned} & F_{n+k+1} + F_{k+1}F_n \\ =\;& \rho^{n+k}\left(\rho\left(\frac{1}{5} + \frac{1}{\sqrt{5}}\right) - \left(\frac{1}{\sqrt{5}} - \frac{1}{5}\right)\frac{1}{\rho^{2n+2k+1}} + \frac{1}{5}\left(\frac{1}{\rho^{2n-1}} + \frac{1}{\rho^{2k+1}}\right)\right) \;>\; \rho^{n+k}\,. \end{aligned}$$

For $k = 0$ and some odd $n \geq 1$ we get

$$F_{n+k+1} + F_{k+1}F_n \; > \; \rho^n \left(\rho \left(\frac{1}{5} + \frac{1}{\sqrt{5}} \right) - \left(\frac{1}{\sqrt{5}} - \frac{1}{5} \right) \frac{1}{\rho^3} + \frac{1}{5\rho} \right) \; > \; \rho^n \,.$$

Case 3: Let $n \equiv 0 \pmod 2$, $k \equiv 1 \pmod 2$.
By the assumption of the lemma, we have $n \geq 4$ and $k \geq 3$. Then

$$\begin{aligned} & F_{n+k+1} + F_{k+1}F_n \\ = \; & \rho^{n+k} \left(\rho \left(\frac{1}{5} + \frac{1}{\sqrt{5}} \right) - \left(\frac{1}{\sqrt{5}} - \frac{1}{5} \right) \frac{1}{\rho^{2n+2k+1}} + \frac{1}{5} \left(-\frac{1}{\rho^{2n-1}} - \frac{1}{\rho^{2k+1}} \right) \right) \; > \; \rho^{n+k} \,. \end{aligned}$$

Case 4: Let $n \equiv k \equiv 0 \pmod 2$.
In particular, we have $n \geq 2$. Then

$$\begin{aligned} & F_{n+k+1} + F_{k+1}F_n \\ = \; & \rho^{n+k} \left(\rho \left(\frac{1}{5} + \frac{1}{\sqrt{5}} \right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{5} \right) \frac{1}{\rho^{2n+2k+1}} + \frac{1}{5} \left(-\frac{1}{\rho^{2n-1}} + \frac{1}{\rho^{2k+1}} \right) \right) \; > \; \rho^{n+k} \,. \end{aligned}$$

This completes the proof of Lemma 7. □

Lemma 8. *Let m be an integer. Then*

$$\frac{\rho^{2m}}{F_{2m+2}} \; < \; 1 \qquad (m \geq 1) \,, \tag{17}$$

$$\rho^{2m} \Big(\frac{1}{F_{2m+3}} + \frac{1}{F_{2m+3} + F_{2m+1}} \Big) \; < \; 1 \qquad (m \geq 0) \,. \tag{18}$$

Proof. For $m \geq 1$ we estimate Binet's formula (3) for F_{2m+2} using $4m + 2 \geq 6$:

$$F_{2m+2} \; = \; \frac{\rho^{2m}}{\sqrt{5}} \left(\rho^2 - \frac{1}{\rho^{4m+2}} \right) \; \geq \; \frac{\rho^{2m}}{\sqrt{5}} \Big(\rho^2 - \frac{1}{\rho^6} \Big) \; > \; \rho^{2m} \,.$$

Similarly, we prove (18) by

$$F_{2n+1} \; = \; \frac{1}{\sqrt{5}} \left(\rho^{2n+1} + \frac{1}{\rho^{2n+1}} \right) \; > \; \frac{\rho^{2n+1}}{\sqrt{5}} \qquad (n \geq 0) \,.$$

Hence,

$$\rho^{2m} \Big(\frac{1}{F_{2m+3}} + \frac{1}{F_{2m+3} + F_{2m+1}} \Big) \; < \; \rho^{2m} \left(\frac{\sqrt{5}}{\rho^{2m+3}} + \frac{\sqrt{5}}{\rho^{2m+3} + \rho^{2m+1}} \right) \; < \; 1 \,.$$

The lemma is proven. □

Proof of Proposition 1: Firstly, we assume the hypotheses in (5) and prove (4). As in the proof of Theorem 3, put $a_0 = \lfloor \alpha \rfloor$, $\theta := \alpha - a_0$, $a_1 = \lfloor 1/\theta \rfloor$ with $0 < \theta < 1$ and $\varepsilon_0 = \theta < 1$. Then

$$|\varepsilon_0| + |\varepsilon_1| \;=\; \theta + (a_0 a_1 + 1) - a_1 \alpha \;=\; \theta + 1 - a_1 \theta \;=\; \theta + 1 - \left\lfloor \frac{1}{\theta} \right\rfloor \theta\,.$$

We have $0 < \theta < 1/2$, since otherwise for $\theta > 1/2$, we obtain $a_1 = \lfloor 1/\theta \rfloor = 1$. With $a_0 = a_1 = 1$ the conditions in (5) are unrealizable both. Hence, there is an integer $k \geq 2$ with

$$\frac{1}{k+1} \;<\; \theta \;<\; \frac{1}{k}\,.$$

Obviously, it follows that $[1/\theta] = k$, and therefore

$$\theta + 1 - \left\lfloor \frac{1}{\theta} \right\rfloor \theta \;<\; \frac{1}{k} + 1 - \frac{k}{k+1} \;=\; \frac{2k+1}{k(k+1)} \;\leq\; \frac{5}{6} \qquad (k \geq 2)\,.$$

Altogether, we have proven that

$$|\varepsilon_0| + |\varepsilon_1| \;\leq\; \frac{5}{6} \;<\; 1\,. \tag{19}$$

Therefore we already know that the inequality (4) holds for $m = 0$. Thus, we assume $m \geq 1$ in the sequel. Noting that $\varepsilon_{2m} > 0$ and $\varepsilon_{2m+1} < 0$ hold for every integer $m \geq 0$, we may rewrite (4) as follows:

$$\big(0 \;<\; \big) \quad (p_{2m+1} - p_{2m}) - \alpha (q_{2m+1} - q_{2m}) \;<\; \frac{1}{\rho^{2m}} \qquad (m \geq 0)\,. \tag{20}$$

We distinguish three cases according to the conditions in (5).

Case 1: Let $a_{2m+1} \geq 2$.
Additionally, we apply the trivial inequality $a_{2m+2} \geq 1$. Then, using (2), (13), and (18),

$$\begin{aligned}
|\varepsilon_{2m}| + |\varepsilon_{2m+1}| \quad &< \quad \frac{1}{q_{2m+1}} + \frac{1}{q_{2m+2}} \\
&\leq \quad \frac{1}{2q_{2m} + q_{2m-1}} + \frac{1}{q_{2m+1} + q_{2m}} \\
&\leq \quad \frac{1}{2q_{2m} + q_{2m-1}} + \frac{1}{3q_{2m} + q_{2m-1}} \\
&\leq \quad \frac{1}{2F_{2m+1} + F_{2m}} + \frac{1}{3F_{2m+1} + F_{2m}} \\
&< \quad \frac{1}{\rho^{2m}} \qquad (m \geq 0)\,.
\end{aligned}$$

Case 2: Let $a_{2m+1} = 1$ and $a_{2m} \geq 2$.
Here, we have $p_{2m+1} - p_{2m} = p_{2m} + p_{2m-1} - p_{2m} = p_{2m-1}$, and similarly $q_{2m+1} - q_{2m} = q_{2m-1}$.

Therefore, by (20), it suffices to show that $0 < p_{2m-1} - \alpha q_{2m-1} < \rho^{-2m}$ for $m \geq 1$. This follows with (2), (13), and (17) from

$$\begin{aligned} 0 &< p_{2m-1} - \alpha q_{2m-1} < \frac{1}{q_{2m}} \\ &\leq \frac{1}{2q_{2m-1} + q_{2m-2}} \leq \frac{1}{2F_{2m} + F_{2m-1}} \\ &= \frac{1}{F_{2m+2}} < \frac{1}{\rho^{2m}} \qquad (m \geq 1)\,. \end{aligned}$$

Case 3: Let $a_{2m} = a_{2m+1} = 1 \wedge a_1 a_2 \cdots a_{2m-1} > 1$.
Since $a_{2m+1} = 1$, we again have (as in Case 2):

$$0 < |\varepsilon_{2m}| + |\varepsilon_{2m+1}| = p_{2m-1} - \alpha q_{2m-1} < \frac{1}{q_{2m}}\,. \tag{21}$$

By the hypothesis of Case 3, there is an integer n satisfying $1 \leq n \leq 2m-1$ and $a_n \geq 2$. We define an integer $k \geq 1$ by setting $2m = n + k$. Then we obtain using (15) and (16),

$$q_{2m} = q_{n+k} \geq F_{n+k+1} + F_{k+1} F_n > \rho^{n+k} = \rho^{2m}\,.$$

From the identity $n+k = 2m$ it follows that the particular condition $n \equiv k+1 \equiv 0 \pmod 2$ in Lemma 7 does not occur. Thus, by (21), we conclude that the desired inequality (4).
In order to prove (6), we now assume the hypothesis (7), i.e., $a_1 a_2 \cdots a_{2k+1} = 1$ and $0 \leq m \leq k$. From $2m-1 \leq 2k-1$ and $a_0 = a_1 = \ldots = a_{2k-1} = 1$ it is clear that $p_{2m-1} = F_{2m+1}$ and $q_{2m-1} = F_{2m}$. Since $a_{2k+1} = 1$ and $0 \leq m \leq k$, we have

$$\begin{aligned} & |q_{2m}\alpha - p_{2m}| + |q_{2m+1}\alpha - p_{2m+1}| \\ = \; & p_{2m-1} - \alpha q_{2m-1} = F_{2m+1} - \alpha F_{2m} = F_{2m+1} - \rho F_{2m} + (\rho - \alpha) F_{2m}\,. \end{aligned}$$

From Binet's formula (3) we conclude that

$$F_{2m+1} - \rho F_{2m} = \frac{1}{\sqrt{5}} \left(\frac{1}{\rho^{2m+1}} + \frac{1}{\rho^{2m-1}} \right) = \rho^{-2m}\,,$$

which finally proves the desired identity (6) in Proposition 1. □

Lemma 9. *Let $k \geq 1$ be an integer, and let $\alpha := \langle 1; 1, \ldots, 1, a_{2k+1}, a_{2k+2} \ldots \rangle$ be a real irrational number with partial quotients $a_{2k+1} > 1$ and $a_\mu \geq 1$ for $\mu \geq 2k+2$. Then we have the inequalities*

$$(F_{2k-1} - 1)(\rho - \alpha) < \frac{1}{\rho^{2k}} - |\varepsilon_{2k}| - |\varepsilon_{2k+1}| \tag{22}$$

for $a_{2k+1} \geq 3$, and

$$(F_{2k-1} - 1)(\rho - \alpha) < \frac{1}{\rho^{2k}} + \frac{1}{\rho^{2k+2}} - |\varepsilon_{2k}| - |\varepsilon_{2k+1}| - |\varepsilon_{2k+2}| - |\varepsilon_{2k+3}| \tag{23}$$

for $a_{2k+1} = 2$.

One may conjecture that (22) also holds for $a_{2k+1} = 2$.

Example 10. Let $\alpha = \langle 1; 1, 1, 1, 1, 2, \overline{1} \rangle = (21\rho + 8)/(13\rho + 5) = (257 - \sqrt{5})/158$. With $k = 2$ and $a_5 = 2$, we have on the one side

$$\rho - \alpha = F_2(\rho - \alpha) = \frac{40\sqrt{5} - 89}{79} = 0.005604\ldots,$$

on the other side,

$$\frac{1}{\rho^4} - |\varepsilon_4| - |\varepsilon_5| = \frac{1}{\rho^4} - \frac{4\sqrt{5} - 1}{79} = 0.045337\ldots$$

Proof of Lemma 9:

Case 1: Let $n := a_{2k+1} \geq 3$.
Then there is a real number η satisfying $0 < \eta < 1$ and

$$r_{2k+1} := \langle a_{2k+1}; a_{2k+2}, \ldots \rangle = n + \eta =: 1 + \beta.$$

It is clear that $n - 1 < \beta < n$. From the theory of regular continued fractions (see [5, formula (16)]) it follows that

$$\begin{aligned} \alpha &= \langle 1; 1, \ldots, 1, a_{2k+1}, a_{2k+2} \ldots \rangle = \frac{F_{2k+2} r_{2k+1} + F_{2k+1}}{F_{2k+1} r_{2k+1} + F_{2k}} \\ &= \frac{F_{2k+2}(1 + \beta) + F_{2k+1}}{F_{2k+1}(1 + \beta) + F_{2k}} = \frac{\beta F_{2k+2} + F_{2k+3}}{\beta F_{2k+1} + F_{2k+2}}. \end{aligned}$$

Similarly, we have

$$\rho = \frac{F_{2k+2}\rho + F_{2k+1}}{F_{2k+1}\rho + F_{2k}},$$

hence, by some straightforward computations,

$$\rho - \alpha = \frac{1 + \beta - \rho}{(\rho F_{2k+1} + F_{2k})(\beta F_{2k+1} + F_{2k+2})} < \frac{n}{(\rho F_{2k+1} + F_{2k})(\beta F_{2k+1} + F_{2k+2})}. \qquad (24)$$

Here, we have applied the identities

$$F_{2k+2}^2 - F_{2k+1} F_{2k+3} = -1, \qquad F_{2k+1}^2 - F_{2k} F_{2k+2} = 1,$$

and the inequality $1 + \beta - \rho < 1 + n - \rho < n$. Since $\beta > n - 1$ and, by (2),

$$\begin{aligned} |\varepsilon_{2k}| &< \frac{1}{q_{2k+1}} = \frac{1}{n F_{2k+1} + F_{2k}}, \\ |\varepsilon_{2k+1}| &< \frac{1}{q_{2k+2}} = \frac{1}{a_{2k+2} q_{2k+1} + F_{2k+1}} \leq \frac{1}{(n+1) F_{2k+1} + F_{2k}}. \end{aligned}$$

(22) follows from

$$\frac{n(F_{2k-1}-1)}{\big(\rho F_{2k+1}+F_{2k}\big)\big((n-1)F_{2k+1}+F_{2k+2}\big)} < \frac{1}{\rho^{2k}} - \frac{1}{nF_{2k+1}+F_{2k}} - \frac{1}{(n+1)F_{2k+1}+F_{2k}}\,. \quad (25)$$

In order to prove (25), we need three inequalities for Fibonacci numbers, which rely on Binet's formula. Let $\delta := 1/\rho^4$. Then, for all integers $s \geq 1$, we have

$$\frac{\rho^{2s+1}}{\sqrt{5}} < F_{2s+1} < \frac{(1+\delta)\rho^{2s+1}}{\sqrt{5}} \qquad \text{and} \qquad \frac{(1-\delta)\rho^{2s}}{\sqrt{5}} \leq F_{2s}\,. \quad (26)$$

We start to prove (25) by observing that

$$\sqrt{5}\left(\frac{1+\delta}{\rho^2(\rho^2+1-\delta)} + \frac{1}{3\rho+1-\delta} + \frac{1}{4\rho+1-\delta}\right) < 1\,.$$

Here, the left-hand side can be diminished by noting that

$$\frac{1}{\rho} > \frac{n}{(n-1)\rho+(1-\delta)\rho^2}\,.$$

By $n \geq 3$ we get

$$\sqrt{5}\left(\frac{(1+\delta)n}{\rho\big(\rho^2+1-\delta\big)\big((n-1)\rho+(1-\delta)\rho^2\big)} + \frac{1}{n\rho+1-\delta} + \frac{1}{(n+1)\rho+1-\delta}\right) < 1\,,$$

or, equivalently,

$$\begin{aligned} &\frac{(1+\delta)n\rho^{2k-1}/\sqrt{5}}{\big(\rho\cdot\rho^{2k+1}/\sqrt{5}+(1-\delta)\rho^{2k}/\sqrt{5}\big)\big((n-1)\rho^{2k+1}/\sqrt{5}+(1-\delta)\rho^{2k+2}/\sqrt{5}\big)} \\ < \;& \frac{1}{\rho^{2k}} - \frac{1}{n\rho^{2k+1}/\sqrt{5}+(1-\delta)\rho^{2k}/\sqrt{5}} - \frac{1}{(n+1)\rho^{2k+1}/\sqrt{5}+(1-\delta)\rho^{2k}/\sqrt{5}}\,. \end{aligned}$$

From this inequality, (25) follows easily by applications of (26) with $s \in \{2k-1, 2k, 2k+1, 2k+2\}$.

Case 2: Let $a_{2k+1} = 2$.

Case 2.1: Let $k \geq 2$.
We first consider the function

$$f(\beta) := \frac{1-\rho+\beta}{\beta F_{2k+1}+F_{2k+2}} \qquad (1 \leq \beta \leq 2)\,.$$

The function f increases monotonically with β, therefore we have

$$f(\beta) \leq f(2) = \frac{3-\rho}{2F_{2k+1}+F_{2k+2}}\,,$$

and consequently we conclude from the identity stated in (24) that

$$\rho - \alpha \le \frac{3-\rho}{(\rho F_{2k+1} + F_{2k})(2F_{2k+1} + F_{2k+2})}\,.$$

Hence, (23) follows from the inequality

$$\frac{(3-\rho)F_{2k-1}}{(\rho F_{2k+1} + F_{2k})(2F_{2k+1} + F_{2k+2})} + \frac{1}{q_{2k+1}} + \frac{1}{q_{2k+2}} + \frac{1}{q_{2k+3}} + \frac{1}{q_{2k+4}} < \frac{1}{\rho^{2k}} + \frac{1}{\rho^{2k+2}}\,. \tag{27}$$

On the left-hand side we now replace the q's by certain smaller terms in Fibonacci numbers. For q_{2k+2}, q_{2k+3}, and q_{2k+4}, we find lower bounds by (15) in Lemma 7:

$$\begin{aligned} q_{2k+1} &= a_{2k+1}q_{2k} + q_{2k-1} = 2F_{2k+1} + F_{2k}\,, \\ q_{2k+2} &\ge F_{2k+3} + F_2F_{2k+1} = F_{2k+3} + F_{2k+1}\,, \\ q_{2k+3} &\ge F_{2k+4} + F_3F_{2k+1} = F_{2k+4} + 2F_{2k+1}\,, \\ q_{2k+4} &\ge F_{2k+5} + F_4F_{2k+1} = F_{2k+5} + 3F_{2k+1}\,. \end{aligned}$$

Substituting these expressions into (27), we then conclude that (23) from

$$\frac{(3-\rho)F_{2k-1}}{(\rho F_{2k+1} + F_{2k})(2F_{2k+1} + F_{2k+2})} + \frac{1}{2F_{2k+1} + F_{2k}} + \frac{1}{F_{2k+3} + F_{2k+1}}$$
$$+ \frac{1}{F_{2k+4} + 2F_{2k+1}} + \frac{1}{F_{2k+5} + 3F_{2k+1}} < \frac{1}{\rho^{2k}}\Big(1 + \frac{1}{\rho^2}\Big)\,. \tag{28}$$

We apply the inequalities in (26) for all $s \ge 2$ when δ is replaced by $\delta := 1/\rho^8$. Using this redefined number δ, we have

$$\sqrt{5}\Big(\frac{(3-\rho)(1+\delta)}{\rho\big(\rho^2+1-\delta\big)\big(2\rho + (1-\delta)\rho^2\big)} + \frac{1}{2\rho+1-\delta} + \frac{1}{\rho^3+\rho} + \frac{1}{(1-\delta)\rho^4 + 2\rho} + \frac{1}{\rho^5 + 3\rho}\Big) - \frac{1}{\rho^2}$$
$$< 1\,,$$

or, equivalently,

$$\begin{aligned} & \frac{(3-\rho)(1+\delta)\rho^{2k-1}/\sqrt{5}}{\big(\rho\cdot\rho^{2k+1}/\sqrt{5} + (1-\delta)\rho^{2k}/\sqrt{5}\big)\big(2\rho^{2k+1}/\sqrt{5} + (1-\delta)\rho^{2k+2}/\sqrt{5}\big)} \\ + & \frac{1}{2\rho^{2k+1}/\sqrt{5} + (1-\delta)\rho^{2k}/\sqrt{5}} + \frac{1}{\rho^{2k+3}/\sqrt{5} + \rho^{2k+1}/\sqrt{5}} \\ + & \frac{1}{(1-\delta)\rho^{2k+4}/\sqrt{5} + 2\rho^{2k+1}/\sqrt{5}} + \frac{1}{\rho^{2k+5}/\sqrt{5} + 3\rho^{2k+1}/\sqrt{5}} \\ < & \frac{1}{\rho^{2k}}\Big(1 + \frac{1}{\rho^2}\Big)\,. \end{aligned}$$

From this inequality, (28) follows by applications of (26) with $s \in \{2k-1, 2k, 2k+1, 2k+2, 2k+3, 2k+4, 2k+5\}$ for $k \ge 2$ (which implies $s \ge 3$).

Case 2.2: Let $k = 1$.
From the hypotheses we have $a_{2k+1} = a_3 = 2$. To prove (23) it suffices to check the inequality in (28) for $k = 1$. We have

$$\begin{array}{lllll} F_{2k-1} = F_1 = 1\,, & F_{2k} = F_2 = 1\,, & F_{2k+1} = F_3 = 2\,, \\ F_{2k+2} = F_4 = 3\,, & F_{2k+3} = F_5 = 5\,, \\ F_{2k+4} = F_6 = 8\,, & F_{2k+5} = F_7 = 13\,. \end{array}$$

Then (28) is satisfied because

$$\rho^2\left(\frac{3-\rho}{7(1+2\rho)} + \frac{1}{5} + \frac{1}{7} + \frac{1}{12} + \frac{1}{19}\right) - \frac{1}{\rho^2} < 1\,.$$

This completes the proof of Lemma 9. □

Proof of Theorem 2: In the sequel we shall use the identity

$$F_{2g} + F_{2g+2} + F_{2g+4} + \ldots + F_{2n} = F_{2n+1} - F_{2g-1} \qquad (n \geq g \geq 0)\,, \tag{29}$$

which can be proven by induction by applying the recurrence formula of Fibonacci numbers. Note that $F_{-1} = 1$. Next, we prove (8).

Case 1: Let $\alpha \notin \mathcal{M}$, $\alpha = \langle 1; a_1, a_2, \ldots \rangle = \langle 1; 1, \ldots, 1, a_{2k}, a_{2k+1}, \ldots \rangle$ with $a_{2k} > 1$ for some subscript $k \geq 1$. This implies $\alpha > \rho$.

Case 1.1: Let $0 \leq n < 2k$.
Then $n_0 = \lfloor n/2 \rfloor \leq k - 1$. In order to treat $|\varepsilon_{2m}| + |\varepsilon_{2m+1}|$, we apply (6) with k replaced by $k - 1$ in Proposition 1. For α the condition (7) with k replaced by $k - 1$ is fulfilled. Note that the term $F_{2m}(\rho - \alpha)$ in (6) is negative. Therefore, we have

$$\begin{aligned} S(n) \;&:=\; \sum_{\nu=2g}^{n} |\varepsilon_\nu| \;\leq\; \sum_{m=g}^{\lfloor n/2 \rfloor} \big(|\varepsilon_{2m}| + |\varepsilon_{2m+1}|\big) \\ &<\; \sum_{m=g}^{n_0} \frac{1}{\rho^{2m}} \;=\; \frac{\rho^{2-2g} - \rho^{-2n_0}}{\rho^2 - 1} \;=\; \rho^{1-2g} - \rho^{-2n_0-1}\,. \end{aligned}$$

Case 1.2: Let $n \geq 2k$.

Case 1.2.1: Let $k \geq g$.
Here, we get

$$S(n) \leq \sum_{m=g}^{k-1} \big(|\varepsilon_{2m}| + |\varepsilon_{2m+1}|\big) + \big(|\varepsilon_{2k}| + |\varepsilon_{2k+1}|\big) + \sum_{m=k+1}^{n_0} \big(|\varepsilon_{2m}| + |\varepsilon_{2m+1}|\big)\,. \tag{30}$$

When $n_0 \le k$, the right-hand sum is empty and becomes zero. The same holds for the left-hand sum for $k = g$.
a) We estimate the left-hand sum as in the preceding case applying (6), $\rho - \alpha < 0$, and the hypothesis $a_1 a_2 \cdots a_{2k-1} = 1$:

$$\sum_{m=g}^{k-1} \big(|\varepsilon_{2m}| + |\varepsilon_{2m+1}|\big) \;<\; \sum_{m=g}^{k-1} \frac{1}{\rho^{2m}}\,.$$

b) Since $a_{2k} > 1$, the left-hand condition in (5) allows us to apply (4) for $m = k$:

$$|\varepsilon_{2k}| + |\varepsilon_{2k+1}| \;<\; \frac{1}{\rho^{2k}}\,.$$

c) We estimate the right-hand sum in (30) again by (4). To check the conditions in (5), we use $a_1 a_2 \cdots a_{2m-1} > 1$, which holds by $m \ge k+1$ and $a_{2k} > 1$. Hence,

$$\sum_{m=k+1}^{n_0} \big(|\varepsilon_{2m}| + |\varepsilon_{2m+1}|\big) \;<\; \sum_{m=k+1}^{n_0} \frac{1}{\rho^{2m}}\,.$$

Altogether, we find with (30) that

$$S(n) \;<\; \sum_{m=g}^{n_0} \frac{1}{\rho^{2m}} \;=\; \rho^{1-2g} - \rho^{-2n_0-1}\,. \tag{31}$$

Case 1.2.2: Let $k < g$.
In order to estimate $|\varepsilon_{2m}| + |\varepsilon_{2m+1}|$ for $g \le m \le n_0$, we use $k + 1 \le g$ and the arguments from c) in Case 1.2.1. Again, we obtain the inequality (31). The results from Case 1.1 and Case 1.2 prove (8) for $a_{2k} > 1$ with $k \ge 1$. It remains to investigate the following case.

Case 2: Let $\alpha \notin \mathcal{M}$, $\alpha = \langle 1; a_1, a_2, \ldots\rangle$ with $a_1 > 1$.
For $m = 0$ (provided that $g = 0$) the first condition in (5) is fulfilled by $a_{2m}a_{2m+1} = a_0 a_1 = a_1 > 1$. For $m \ge 1$ we know that $a_1 a_2 \cdots a_{2m-1} > 1$ always satisfies one part of the second condition. Therefore, we apply the inequality from (4):

$$S(n) \;<\; \sum_{m=g}^{n_0} \frac{1}{\rho^{2m}} \;=\; \rho^{1-2g} - \rho^{-2n_0-1}\,.$$

Next, we prove (9). Let $\alpha \in \mathcal{M}$, $\alpha = \langle 1; a_1, a_2, \ldots\rangle = \langle 1; 1, \ldots, 1, a_{2k+1}, a_{2k+2}, \ldots\rangle$ with $a_{2k+1} > 1$ for some subscript $k \ge 1$. This implies $\rho > \alpha$.

Case 3.1: Let $0 \le n < 2k$.
Then $n_0 = \lfloor n/2 \rfloor \le k - 1$. In order to treat $|\varepsilon_{2m}| + |\varepsilon_{2m+1}|$, we apply (6) with k replaced by $k - 1$ in Proposition 1. For α the condition (7) with k replaced by $k - 1$ is fulfilled. Note

that the term $F_{2m}(\rho-\alpha)$ in (6) is positive. Therefore we have, using (29),

$$\begin{aligned} S(n) &\le \sum_{m=g}^{n_0}\Big(\frac{1}{\rho^{2m}}+(\rho-\alpha)F_{2m}\Big)\\ &= \rho^{1-2g}-\rho^{-2n_0-1}+(\rho-\alpha)\sum_{m=g}^{n_0}F_{2m}\\ &= \rho^{1-2g}-\rho^{-2n_0-1}+(\rho-\alpha)(F_{2n_0+1}-F_{2g-1})\\ &\le \rho^{1-2g}-\rho^{-2n_0-1}+(F_{2k-1}-F_{2g-1})(\rho-\alpha)\,. \end{aligned}$$

Here we have used that $2n_0+1\le 2k-1$.

Case 3.2: Let $n\ge 2k$.
Our arguments are similar to the proof given in Case 1.2, using $a_1a_2\cdots a_{2k-1}=1$ and $a_{2k+1}>1$.

Case 3.2.1: Let $k\ge g$.
Applying (29) again, we obtain

$$\begin{aligned} S(n) &\le \sum_{m=g}^{k-1}\big(|\varepsilon_{2m}|+|\varepsilon_{2m+1}|\big)+\big(|\varepsilon_{2k}|+|\varepsilon_{2k+1}|\big)+\sum_{m=k+1}^{n_0}\big(|\varepsilon_{2m}|+|\varepsilon_{2m+1}|\big)\\ &< \sum_{m=g}^{k-1}\Big(\frac{1}{\rho^{2m}}+(\rho-\alpha)F_{2m}\Big)+\frac{1}{\rho^{2k}}+\sum_{m=k+1}^{n_0}\frac{1}{\rho^{2m}}\\ &= \sum_{m=g}^{n_0}\frac{1}{\rho^{2m}}+(\rho-\alpha)\sum_{m=g}^{k-1}F_{2m}\\ &= \rho^{1-2g}-\rho^{-2n_0-1}+(F_{2k-1}-F_{2g-1})(\rho-\alpha)\,. \end{aligned}$$

Case 3.2.2: Let $k<g$.
From $g\ge k+1$ we get

$$S(n)\le\sum_{m=g}^{n_0}\frac{1}{\rho^{2m}}=\rho^{1-2g}-\rho^{-2n_0-1}\,.$$

The results of Case 3.1 and Case 3.2 complete the proof of (9).
For the inequality (10) we distinguish whether α belongs to $\mathcal{M}$ or not.

Case 4.1: Let $\alpha\notin\mathcal{M}$.
Then (10) is a consequence of the inequality in (8):

$$\sum_{\nu=2g}^{\infty}|\varepsilon_\nu|\le\lim_{n_0\to\infty}\big(\rho^{1-2g}-\rho^{-2n_0-1}\big)=\rho^{1-2g}\,.$$

Case 4.2: Let $\alpha\in\mathcal{M}$.
There is a subscript $k\ge 1$ satisfying $\alpha=\langle 1;1,\dots,1,a_{2k+1},a_{2k+2},\dots\rangle$ and $a_{2k+1}>1$. To

simplify arguments, we introduce the function $\chi(k,g)$ defined by $\chi(k,g)=1$ (if $k>g$), and $\chi(k,g)=0$ (if $k\le g$). We have

$$\begin{aligned} S &:= \sum_{\nu=2g}^{\infty}|\varepsilon_\nu| = \sum_{\nu=2g}^{2k-1}|\varepsilon_\nu| + \sum_{\nu=\max\{2k,2g\}}^{\infty}|\varepsilon_\nu| \\ &= \chi(k,g)\left((F_{2k-1}-F_{2g-1})(\rho-\alpha)+\rho^{1-2g}-\rho^{-2k+1}\right) + \sum_{m=\max\{k,g\}}^{\infty}\left(|\varepsilon_{2m}|+|\varepsilon_{2m+1}|\right) \\ &\le (F_{2k-1}-F_{2g-1})(\rho-\alpha)+\rho^{1-2g}-\rho^{-2k+1} + \sum_{m=k}^{\infty}\left(|\varepsilon_{2m}|+|\varepsilon_{2m+1}|\right) \\ &\le (F_{2k-1}-1)(\rho-\alpha)+\rho^{1-2g}-\rho^{-2k+1} + \sum_{m=k}^{\infty}\left(|\varepsilon_{2m}|+|\varepsilon_{2m+1}|\right), \qquad (32) \end{aligned}$$

where we have used (9) with $n=2k-1$ and $n_0=\lfloor n/2\rfloor = k-1$.

Case 4.2.1: Let $a_{2k+1}\ge 3$.
The conditions in Lemma 9 for (22) are satisfied. Moreover, the terms $|\varepsilon_{2m}|+|\varepsilon_{2m+1}|$ of the series in (32) for $m\ge k+1$ can be estimated using (4), since $a_1a_2\cdots a_{2k+1}>1$. Therefore, we obtain

$$\begin{aligned} S &< \frac{1}{\rho^{2k}} - \frac{1}{\rho^{2k-1}} + \rho^{1-2g} + \sum_{m=k+1}^{\infty}\left(|\varepsilon_{2m}|+|\varepsilon_{2m+1}|\right) \\ &< \frac{1}{\rho^{2k}} - \frac{1}{\rho^{2k-1}} + \rho^{1-2g} + \frac{1}{\rho^{2k+1}} = \rho^{1-2g}. \end{aligned}$$

Case 4.2.2: Let $a_{2k+1}=2$.
Now the conditions in Lemma 9 for (23) are satisfied. Thus, from (32) and (4) we have

$$\begin{aligned} S &< \frac{1}{\rho^{2k}} + \frac{1}{\rho^{2k+2}} - \frac{1}{\rho^{2k-1}} + \rho^{1-2g} + \sum_{m=k+2}^{\infty}\left(|\varepsilon_{2m}|+|\varepsilon_{2m+1}|\right) \\ &< \frac{1}{\rho^{2k}} + \frac{1}{\rho^{2k+2}} - \frac{1}{\rho^{2k-1}} + \rho^{1-2g} + \sum_{m=k+2}^{\infty}\frac{1}{\rho^{2m}} \\ &= \frac{1}{\rho^{2k}} + \frac{1}{\rho^{2k+2}} - \frac{1}{\rho^{2k-1}} + \rho^{1-2g} + \frac{1}{\rho^{2k+3}} = \rho^{1-2g}. \end{aligned}$$

This completes the proof of Theorem 2. □

5 Concluding remarks

In this section we state some additional identities for error sums $\varepsilon(\alpha)$. For this purpose let $\alpha=\langle a_0;a_1,a_2,\dots\rangle$ be the continued fraction expansion of a real irrational number. Then the numbers α_n are defined by

$$\alpha = \langle a_0;a_1,a_2,\dots,a_{n-1},\alpha_n\rangle \qquad (n=0,1,2,\dots).$$

Proposition 11. *For every real irrational number α we have*

$$\varepsilon(\alpha) = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1}{\alpha_k}$$

and

$$\begin{pmatrix} \varepsilon(\alpha) \\ \cdot \end{pmatrix} = \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \alpha \end{pmatrix} .$$

Next, let $\alpha = \langle a_0; a_1, a_2, \dots \rangle$ with $a_0 \geq 1$ be a real number with convergents p_m/q_m $(m \geq 0)$, where $p_{-1} = 1$, $q_{-1} = 0$. Then the convergents $\overline{p}_m/\overline{q}_m$ of the number $1/\alpha = \langle 0; a_0, a_1, a_2, \dots \rangle$ satisfy the equations $\overline{q}_m = p_{m-1}$ and $\overline{p}_m = q_{m-1}$ for $m \geq 0$, since we know that $\overline{p}_{-1} = 1$, $\overline{p}_0 = 0$ and $\overline{q}_{-1} = 0$, $\overline{q}_0 = 1$. Therefore we obtain a relation between $\varepsilon(\alpha)$ and $\varepsilon(1/\alpha)$:

$$\begin{aligned} \varepsilon(1/\alpha) &= \sum_{m=0}^{\infty} \left| \frac{\overline{q}_m}{\alpha} - \overline{p}_m \right| = \sum_{m=0}^{\infty} \left| \frac{p_{m-1}}{\alpha} - q_{m-1} \right| = \frac{1}{\alpha} \sum_{m=0}^{\infty} |q_{m-1}\alpha - p_{m-1}| \\ &= \frac{1}{\alpha} \Big(|q_{-1}\alpha - p_{-1}| + \sum_{m=0}^{\infty} |q_m\alpha - p_m| \Big) = \frac{1}{\alpha} \big(1 + \varepsilon(\alpha) \big) . \end{aligned}$$

This proves

Proposition 12. *For every real number $\alpha > 1$ we have*

$$\varepsilon(1/\alpha) = \frac{1 + \varepsilon(\alpha)}{\alpha} .$$

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