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Powerful Values of Quadratic Polynomials

Jean-Marie De Koninck and Nicolas Doyon
Départment de Mathématiques
Université Laval
Québec, PQ G1V 0A6
Canada
jmdk@mat.ulaval.ca
nicolas.doyon.1@ulaval.ca

Florian Luca
Instituto de Matemáticas
Universidad Nacional Autónoma de México
C. P. 58180
Morelia, Michoacán
México
fluca@matmor.unam.mx

Abstract

We study the set of those integers k such that n^2+k is powerful for infinitely many positive integers n. We prove that most integers k have this property.

1 Introduction

Given an arbitrary integer $k \neq 0$, Mollin and Walsh [1] have shown that there exist infinitely many ways of writing k as a difference of two nonsquare powerful numbers. A positive integer n is said to be *powerful* if $p^2 \mid n$ for each prime divisor p of n. For instance, 17 has two such representations below 10^9, namely $17 = 125 - 108$ and $17 = 173881809 - 173881792$. But what about representations $17 = P - n^2$, where P is a powerful number? It turns out that there are no such representation with $P < 10^9$. However, in view of Theorem 1 (below) we believe that infinitely many such representations should exist, even though the smallest is probably very large (see Table 1 in Section 3). In general, identifying all those integers $k \neq 0$

such that n^2+k is a powerful number for infinitely many positive integers n seems to be a very difficult problem. Indeed, already showing that one such n exists is not obvious.

Now, for a given $k \neq 0$, if one can find an integer n_0 such that n_0^2+k is a powerful number which is not a perfect square, that is if

$$n_0^2 + k = m_0^2 b^3, \tag{1}$$

for some integer m_0, with $b > 1$ squarefree, then (1) can be written as

$$n_0^2 - D m_0^2 = -k, \tag{2}$$

where $D > 1$ is a cube but not a square. Now since $x^2 - Dy^2 = -k$ is a generalized Pell equation with a given solution (see, for instance, Robertson [4]), it must have infinitely many solutions, thus providing infinitely many n's for which n^2+k is powerful. However, given an arbitrary integer k, finding a minimal solution (n_0, m_0) of (2) for an appropriate D is not easily achieved.

In this note, we use a different approach and show that for almost all integers k, there exist infinitely many positive integers n such that n^2+k is powerful. In fact, we prove the following result.

Theorem 1. *For any positive real number x, let $\mathcal{P}(x)$ be the set of integers k with $|k| \le x$ such that n^2+k is powerful for infinitely many positive integers n. Then $2x - \#\mathcal{P}(x) = o(x)$.*

Throughout this paper we use the Landau symbols O and o as well as the Vinogradov symbols $\gg$ and $\ll$ with their usual meaning. We let $\log_1 x = \max\{1, \log x\}$, where log stands for the natural logarithm, and for $i > 1$ we define $\log_i x = \log_1\big(\log_{i-1} x\big)$. When $i = 1$ we omit the subscript and thus understand that all the logarithms that will appear are ≥ 1. For a positive integer n we write $\phi(n)$ for the Euler function of n.

2 The Proof

For the proof, we first give a sufficient algebraic criterion on k which insures that n^2+k is powerful for infinitely many n. We then show that most integers k satisfy this condition.

We shall prove this only when $k > 0$, but the argument extends without any major modification to the case $k < 0$.

Proposition 2. *Assume that there exist positive integers y_1 and $d \mid ky_1^2+1$ such that*

$$u = \frac{1}{2y_1}\left(\frac{ky_1^2+1}{d} - d\right)$$

is a positive integer coprime to k. Let $D = u^2+k$ and assume further that y_1 is coprime to D. Then there exist infinitely many positive integers n such that n^2+k is powerful.

Proof. Since u is an integer, it follows that d and $(ky_1^2+1)/d$ are integers of the same parity. Put

$$x_1 = \frac{1}{2}\left(\frac{ky_1^2+1}{d}+d\right).$$

One checks immediately that $x_1^2-(uy_1)^2 = ky_1^2+1$, which can be rewritten as

$$x_1^2 - Dy_1^2 = 1. \tag{3}$$

Define the sequences $(x_m)_{m\geq 1}$ and $(y_m)_{m\geq 1}$ as

$$x_m + y_m\sqrt{D} = (x_1+y_1\sqrt{D})^m$$

for all $m \geq 1$. Then, for all $m \geq 1$,

$$\begin{aligned} x_m^2 - Dy_m^2 &= (x_m+y_m\sqrt{D})(x_m-y_m\sqrt{D}) \\ &= (x_1+y_1\sqrt{D})^m(x_1-y_1\sqrt{D})^m \\ &= (x_1^2-Dy_1^2)^m = 1 \end{aligned} \tag{4}$$

We now search for positive integers n such that $n^2+k = D\ell^2$ holds with some positive integer ℓ such that $D \mid \ell$. It is clear that such numbers n have the property that n^2+k is powerful. We rewrite this equation as

$$n^2 - D\ell^2 = -k. \tag{5}$$

Noting that $u^2 - D\cdot 1^2 = -k$ and using (4), if

$$n + \sqrt{D}\ell = (u+\sqrt{D})(x_m+\sqrt{D}y_m),$$

one checks by multiplying each side by its conjugate that the pair (n,ℓ) satisfies (5). Expanding we get

$$n = ux_m + Dy_m \qquad \text{and} \qquad \ell = uy_m + x_m.$$

It suffices to argue that there exist infinitely many m such that $D \mid \ell$. Since

$$x_m = \frac{1}{2}\left((x_1+y_1\sqrt{D})^m + (x_1-y_1\sqrt{D})^m\right) \equiv x_1^m \pmod{D},$$

and

$$y_m = \frac{1}{2\sqrt{D}}\left((x_1+y_1\sqrt{D})^m - (x_1-y_1\sqrt{D})^m\right) \equiv mx_1^{m-1}y_1 \pmod{D},$$

the relation $D \mid (uy_m+x_m)$ is equivalent to $D \mid x_1^{m-1}(umy_1+x_1)$. Since D and x_1 are coprime (in light of (3)), the above divisibility relation holds if and only if $muy_1 \equiv -x_1 \pmod{D}$. Since both u and y_1 are coprime to D, it follows that their product is invertible modulo D. Hence, if $m \equiv -x_1(uy_1)^{-1} \pmod{D}$, then

$$n = ux_m + Dy_m$$

has the property that n^2+k is powerful. This completes the proof of the proposition. ∎

It now remains to show that for most positive integers k one can choose integers y_1 and d such that the conditions from Proposition 2 are fulfilled. It is clear that for the purpose of making n^2+k powerful, we may assume that k is squarefree. Indeed, if $p^2 \mid k$, we may then take $n = pn'$ and note that

$$n^2 + k = p^2(n'^2 + (k/p^2)),$$

so we may replace k by k/p^2.

Theorem 3. *The set of squarefree positive integers k for which there exist positive integers y_1 and $d \mid ky_1^2+1$ such that the conditions of Proposition 2 are satisfied is of density 1.*

Proof. We let x be a large positive real number and we assume that $k \le x$ is a positive integer. We choose $y_1 = 12$.

The number d will always be a prime number in a certain arithmetical progression modulo 144, as follows. If $\gcd(k,6)=1$, we then take $d \equiv 1 \pmod{144}$. If $\gcd(k,6)=2$, then $d \equiv 91 \pmod{144}$. If $\gcd(k,6)=3$, we put $d \equiv 65 \pmod{144}$, and finally if $6 \mid k$, we then put $d \equiv 11 \pmod{144}$.

We first show that y_1 and D are coprime, from which it will follow that 6 and D are coprime.

If $\gcd(k,6)=1$, then since $d \equiv 1 \pmod{144}$, we get that $(144k+1)/d \equiv 1 \pmod{144}$. Hence, $(144k+1)/d - d \equiv 0 \pmod{144}$, which shows that $6 \mid u$. Since k is coprime to 6, we get that 6 is coprime to D.

If $\gcd(k,6)=2$, then $d \equiv 91 \pmod{144}$. In particular, $d \equiv 11 \pmod{16}$ and $d \equiv 1 \pmod 9$. Hence, $(144k+1)/d \equiv 3 \pmod{16}$ and $(144k+1)/d \equiv 1 \pmod 9$. Thus, $(144k+1)/d - d$ is congruent to 8 modulo 16 and to 0 modulo 9. Hence, u is an odd multiple of 3. Since 2 divides k but 3 doesn't, we get that 6 is coprime to D.

It is easily seen that the other two cases, namely $\gcd(k,6)=3$ and $\gcd(k,6)=6$, can be treated similarly.

Moreover, since $\gcd(u^2+k,12) = \gcd(D,y) = 1$ there is no prime $p \in \{2,3\}$ such that $p \mid \gcd(k,u)$.

Thus, it remains to show that for all positive integers $k \le x$ except $o(x)$ of them such a prime d can be chosen in such a way that there is no prime $p > 3$ dividing both u and k. Note that if $p > 3$ divides both u and k, then $(144k+1)/d \equiv d \pmod p$, so that $144k+1 \equiv d^2 \pmod p$, in which case $d^2 \equiv 1 \pmod p$. Thus, $d \equiv \pm 1 \pmod p$. We can reverse the argument to show that if $d \equiv \pm 1 \pmod p$, then $p \mid 2y_1u$ and $p \mid k$. Since $p > 3$ and the largest prime factor of y_1 is 3, the condition $d \equiv \pm 1 \pmod p$ guarantees that $p \mid \gcd(u,k)$.

For coprime positive integers $a,\ b$ we write

$$S(x;a,b) = \sum_{p \equiv a \pmod b} \frac{1}{p} - \frac{\log_2 x}{\phi(b)}.$$

A result of Pomerance (see Theorem 1 and Remark 1 in [3]) shows that, uniformly for all $a < b \le x$,

$$S(x;a,b) = \frac{1}{p(a,b)} + O\left(\frac{\log 2b}{\phi(b)}\right),$$

where $p(a,b)$ is the smallest prime number in the arithmetical progression $a \pmod b$.

Let $\omega(k;a,b)$ be the number of prime factors p of k which are congruent to $a \pmod b$. We let $b=144$, $a=1,91,65,11$ according to whether $\gcd(k,6)=1,2,3,6$, respectively. By a classical result of Turán [5], the estimate

$$\omega(144k+1;a,b)=\frac{\log_2 x}{\phi(b)}+O\left(\left(\frac{\log_2 x}{\phi(b)}\right)^{2/3}\right)$$

holds for all $k\le x$, with at most

$$O\left(\frac{x}{(\log_2 x)^{1/6}}\right)=o(x)$$

exceptions. From now on, we work only with such positive integers k. Note that $\omega(144k+1;a,b)$ gives the number of admissible values for d.

We now put $y=\log_2 x$, $z=x^{1/3}$ and show that the number of such $k\le x$ for which there exists a prime factor $p\in[y,z]$ dividing both u and k is $o(x)$. Indeed, let us fix p and d. Then $k\equiv 0 \pmod p$ and $144k+1\equiv 0 \pmod d$. This puts $k\le x$ into a certain arithmetical progression modulo pd.

Assume first that $pd\le x$. Clearly, the number of such positive integers $k\le x$ is $\le x/(pd)+1\ll x/(pd)$. Summing up this inequality for all $p\ge y$ and all $d\equiv \pm 1 \pmod p$, we get that the number of such numbers k is

$$\ll x\sum_{p\ge y}\frac{1}{p}\sum_{\substack{d\le x\\ d\equiv \pm 1 \pmod p}}\frac{1}{d}\ll x\log_2 x\sum_{p\ge y}\frac{1}{p^2}\ll\frac{x\log_2 x}{y\log y}=o(x).$$

We now look at those positive integers $k\le x$ such that $pd>x$. Write $144k+1=dm$, and note that $m\le 288x/d<288p\ll p$. Since $p\mid k$ and $d\equiv\pm1 \pmod p$, we get that $m\equiv \pm 1 \pmod p$. Fix m. Then k/p is in a certain residue class modulo m depending on p. Write $k/p=v+m\ell$. Then

$$dm=144p(k/p)+1=(144pv+1)+144pm\ell,$$

so that

$$d=w+144p\ell,$$

where $w=(144pv+1)/m$. Furthermore, $d\le 288x/m$. Hence, by a result of Montgomery and Vaughan [2], the number of such primes d does not exceed

$$\frac{4\cdot 144x}{m\phi(144p)\log(288x/(mp))}\ll\frac{x}{mp\log x}, \qquad (6)$$

where we used the fact that $m\le 288p\le 288x^{1/3}$, and therefore

$$\frac{288x}{mp}\gg x^{1/3}.$$

Summing up inequality (6) over all the possible values of $m \le 288p \le 288x^{1/3}$, and then afterwards over all $p \in [y, z]$, we get that the number of such k's is

$$\ll \frac{x}{\log x} \sum_{y \le p} \frac{1}{p} \sum_{\substack{m \le x \\ m \equiv \pm 1 \pmod{p}}} \frac{1}{m} \ll x \sum_{y \le p} \frac{1}{p^2} \ll \frac{x}{y \log y} = o(x).$$

From now on, we consider only those k such that if $d \equiv a \pmod{b}$ is a prime factor of $144k+1$, with the pair (a, b) being the appropriate one depending on the value of $\gcd(k, 6)$, then there exists a prime $p \in [5, y] \cup [x^{1/3}, x]$ such that $p \mid \gcd(k, u)$. First observe that k has at most 3 prime factors in $[x^{1/3}, x]$.

Moreover, for each prime $p > x^{1/3}$, there are at most 3 values of d such that $d \equiv \pm 1 \pmod{p}$. Indeed, if there were 4 or more, let d_1, d_2, d_3 and d_4 be 4 of them. We would then have

$$144x + 1 \ge 144k + 1 \ge d_1 d_2 d_3 d_4 \ge (x^{1/3} - 1)^4,$$

which is impossible for large x. Hence, there are at most 9 values of d which might be congruent to ± 1 modulo some prime factor $p > x^{1/3}$ of k. Since we have $(1 + o(1))\dfrac{\log_2 x}{\phi(b)}$ such primes d, it follows that we also have $(1 + o(1))\dfrac{\log_2 x}{\phi(b)}$ such prime factors d of $144k + 1$ with the property that each of them is congruent to ± 1 modulo a prime factor p of k in the interval $[5, y]$. We apply Turán's inequality from [5] again to conclude that all $k \le x$ have at most $1.5 \log_2 y < 2 \log_4 x$ prime factors $p < y$ with at most $o(x)$ exceptions.

We now write

$$M = \prod_{5 \le p < (\log_3 x)/2} p,$$

and look at those d such that $d \equiv 2 \pmod{M}$. Note that such d are in a certain arithmetical progression $A \pmod{B}$, where $B = bM = (\log_2 x)^{1/2 + o(1)}$. We apply again the results from [3] and [5] to infer that all positive integers $k \le x$ have $\omega(k; A, B)$ factors in the interval

$$\left[\frac{\log_2 x}{2\phi(B)}, \frac{2 \log_2 x}{\phi(B)} \right],$$

with $o(x)$ possible exceptions. Because $d \equiv 2 \pmod{M}$, we have that $d \not\equiv \pm 1 \pmod{p}$ for all $p < (\log_3 x)/2$. Hence, there exist at least

$$\mu := \lfloor (\log_2 x)/(4 \log_4 x \phi(B)) \rfloor > (\log_2 x)^{1/3}$$

such primes d which furthermore are congruent to either 1 or -1 modulo p for the **same** prime $p > (\log_3 x)/2$.

We now count how many such k's can there can be. Because of the above argument, we can write $144k + 1 = d_1 d_2 \cdots d_\mu Q < 288x$ (for some positive integer Q), where each $d_j \equiv \pm 1$

(mod p). Thus, the number of such k's is at most

$$\begin{aligned}\sum_{p>(\log_3 x)/2} \frac{288x}{\mu!}\left(\sum_{\substack{d\le x\\ d\equiv A \pmod{B}\\ d\equiv \pm 1 \pmod{p}}} \frac{1}{d}\right)^{\mu} &\ll x \sum_{p>(\log_3 x)/2}\left(\frac{2e\log_2 x+O(1)}{\mu\phi(B)(p-1)}\right)^{\mu}\\ &\ll x\sum_{p>(\log_3 x)/2}\left(\frac{O(\log_4 x)}{p}\right)^{(\log_2 x)^{1/3}}\\ &= o(x),\end{aligned}$$

which completes the proof of Theorem 3. □

3 Comments and Numerical Results

Although we proved that n^2+k is powerful for infinitely many n's only for most integers k, we do conjecture that this is actually true for all integers k. Indeed, fixing a squarefree integer k, the probability that n^2+k is powerful is of the order $\frac{1}{\sqrt{n^2+k}} \approx \frac{1}{n}$ for large n. This means that we should expect that

$$\#\{n \le x : n^2+k \text{ is powerful}\} \sim \sum_{n\le x}\frac{1}{n} \sim \log x \to \infty \quad \text{as } x\to\infty \tag{7}$$

for any squarefree k. From our proof it follows indeed that if $\#\{n : n^2+k \text{ is powerful}\} > 1$, then $\#\{n \le x : n^2+k \text{ is powerful}\} \gg \log x$.

Table 1 (resp. Table 2) provides, for each integer $1 \le k \le 50$, the smallest known value of n for which n^2+k (resp. n^2-k) is a powerful number without being a perfect square. These values of n were obtained by finding the minimal solution of $x^2 - Dy^2 = \pm k$ by considering various cubefull D's. Those $n > 10^9$ may not be the smallest n for which $n^2 \pm k$ is powerful.

Given three integers a, b, c, one could ask if the polynomial an^2+bn+c is powerful for infinitely many integers n. Assuming that an^2+bn+c is a powerful number which is not a square, we can then write $an^2+bn+c = Dm^2$ with $D > 1$ squarefree and $D \mid m$. We then have

$$n = \frac{-b \pm \sqrt{4aDm^2+b^2-4ac}}{2a}.$$

Since n is an integer, there exists an integer y such that $4aDm^2+b^2-4ac = y^2$, or, equivalently, $y^2 - aD(2m)^2 = b^2-4ac$ with $y \equiv \pm b \pmod{2a}$. But then the existence of one solution implies the existence of infinitely many. On the other hand, we also get that if there is an infinity of integers y for which $y^2-b^2+4ac = Dx^2$ where $D>1$ is squarefree and $2aD \mid x$, then there exist infinitely many n's for which an^2+bn+c is a powerful number.

The prediction (7) may at first seem at odd with the fact that some of the smallest n's obtained in Tables 1 and 2 are huge. However, the statement "n^2+k is powerful" is equivalent to "$n^2 + k = dm^2$ with d squarefree and $d \mid m$". Now for any fixed d, this last equation is a generalized Pell equation. While a solution may not exist for some values of d, when it does for a particular d, it is well known that the smallest solution can be surprisingly large. When solutions exist, it is still possible that none of them will be such that $d \mid m$. The size of the smallest solution such that $d \mid m$ can also be quite large. There are thus three possible reasons for the huge value of the smallest solution: a large value of d, a large value of the smallest solution to $n^2 + k = dm^2$ or a large value of the smallest solution such that $d \mid m$. We investigated this in Table 3, where $n = n_0$ is the smallest solution to $n^2 + k = dm^2$ not taking into account the restriction $d \mid m$. It turns out that the surprisingly large values of the smallest n in Table 1 are not due to a very large value of d but rather to a large value of n_0 (see for instance $k = 33$), or to the large size of the smallest solution n_0 for which $d \mid m$ (see for instance $k = 17$).

Table 1

k	n	$n^2 + k$
1	682	$5^3 \cdot 61^2$
2	5	3^3
3	37	$2^2 \cdot 7^3$
4	11	5^3
5	1879706	$3^5 \cdot 7^2 \cdot 23^3 \cdot 29^3$
6	463	$5^4 \cdot 7^3$
7	11	2^7
8	10	$2^2 \cdot 3^3$
9	2046	$3^2 \cdot 5^3 \cdot 61^2$
10	341881	$11^3 \cdot 9371^2$
11	31	$2^2 \cdot 3^5$
12	74	$2^4 \cdot 7^3$
13	70	17^3
14	5519	$3^3 \cdot 5^5 \cdot 19^2$
15	793	$2^7 \cdot 17^3$
16	22	$2^2 \cdot 5^3$
17	n_{17}	f_{17}
18	57	$3^3 \cdot 11^2$
19	559	$2^2 \cdot 5^7$
20	338	$2^3 \cdot 3^3 \cdot 23^2$
21	n_{21}	f_{21}
22	503259461	$47^3 \cdot 491^2 \cdot 3181^2$
23	45	2^{11}
24	926	$2^2 \cdot 5^4 \cdot 7^3$
25	190	$5^3 \cdot 17^2$

k	n	$n^2 + k$
26	109	$3^5 \cdot 7^2$
27	36	$3^3 \cdot 7^2$
28	62	$2^5 \cdot 11^2$
29	436	$3^2 \cdot 5^3 \cdot 13^2$
30	832836278711	$31^2 \cdot 59^2 \cdot 67^2 \cdot 79^3 \cdot 9679^2$
31	63	$2^5 \cdot 5^3$
32	88	$2^5 \cdot 3^5$
33	n_{33}	f_{33}
34	7037029	$5^5 \cdot 7^5 \cdot 971^2$
35	36	11^3
36	33	$3^2 \cdot 5^3$
37	n_{37}	f_{37}
38	5945	$3^2 \cdot 7^3 \cdot 107^2$
39	31	$2^3 \cdot 5^3$
40	52	$2^3 \cdot 7^3$
41	78	$5^3 \cdot 7^2$
42	720025	$13^3 \cdot 31^3 \cdot 89^2$
43	22364	$11^3 \cdot 613^2$
44	62	$2^4 \cdot 3^5$
45	96	$3^3 \cdot 7^3$
46	50927	$5^5 \cdot 11^2 \cdot 19^3$
47	39	$2^5 \cdot 7^2$
48	148	$2^6 \cdot 7^3$
49	524	$5^3 \cdot 13^3$
50	1325	$3^5 \cdot 5^2 \cdot 17^2$

Here, $n_{17} = 1952785824219551870$ with $f_{17} = 3^2 \cdot 13^3 \cdot 367^2 \cdot 7487^2 \cdot 5054107013^2$;
$n_{21} = 4580728614212333152148$ with $f_{21} = 5^2 \cdot 31^2 \cdot 37^3 \cdot 41611^2 \cdot 3155673955493^2$;
$n_{33} = 2451448196948930$ with $f_{33} = 7^2 \cdot 17^3 \cdot 29^3 \cdot 31992951041^2$;
$n_{37} = 1865111669472103216621387524 6076$ with $f_{37} = 317^3 \cdot 10219159057^2 \cdot 323370789682598407^2$.

Table 2

k	n	n^2-k	k	n	n^2-k
1	3	2^3	26	2537	23^5
2	11427	$7^3 \cdot 617^2$	27	51700	$13^3 \cdot 1103^2$
3	n_3	f_3	28	54	$2^3 \cdot 19^2$
4	6	2^5	29	426	$7^3 \cdot 23^2$
5	73	$2^2 \cdot 11^3$	30	83	19^3
6	62531004125	$19^3 \cdot 14831^2 \cdot 50909^2$	31	34	$3^2 \cdot 5^3$
7	n_7	f_7	32	40	$2^5 \cdot 7^2$
8	20	$2^3 \cdot 7^2$	33	3601	$2^8 \cdot 37^3$
9	15	$2^3 \cdot 3^3$	34	948281	$3^6 \cdot 47^3 \cdot 109^2$
10	n_{10}	f_{10}	35	531783519104	$29^3 \cdot 997^2 \cdot 3415409^2$
11	56	5^5	36	42	$2^6 \cdot 3^3$
12	47	13^3	37	73	$2^2 \cdot 3^3 \cdot 7^2$
13	16	3^5	38	16493	$11^2 \cdot 131^3$
14	33017	$5^2 \cdot 11^3 \cdot 181^2$	39	n_{39}	f_{39}
15	1138	109^3	40	632	$2^3 \cdot 3^3 \cdot 43^2$
16	68	$2^9 \cdot 3^2$	41	71	$2^3 \cdot 5^4$
17	23	2^9	42	691888331	$13^2 \cdot 79^3 \cdot 75797^2$
18	19	7^3	43	5016	$13^2 \cdot 53^3$
19	762488	$3^2 \cdot 5^3 \cdot 127^2 \cdot 179^2$	44	112	$2^2 \cdot 5^5$
20	146	$2^4 \cdot 11^3$	45	219	$2^2 \cdot 3^2 \cdot 11^3$
21	1552808	$43^3 \cdot 5507^2$	46	847	$3^3 \cdot 163^2$
22	47	3^7	47	180190	$53^3 \cdot 467^2$
23	6234	$7^2 \cdot 13^3 \cdot 19^2$	48	94	$2^2 \cdot 13^3$
24	32	$2^3 \cdot 5^3$	49	56	$3^2 \cdot 7^3$
25	45	$2^4 \cdot 5^3$	50	57135	$5^2 \cdot 7^3 \cdot 617^2$

Here, $n_3 = 15503069909027$ with $f_3 = 13^3 \cdot 239^2 \cdot 64866401293^2$;
$n_7 = 85227106679780$ with $f_7 = 3^3 \cdot 59^3 \cdot 36192438539^2$;
$n_{10} = 714571300448055826123252946343 31$ with $f_{10} = 3^3 \cdot 13^3 \cdot 43^2 \cdot 6823075915494777091540353511^2$;
$n_{39} = 82716851195974$ with $f_{39} = 7^2 \cdot 373^3 \cdot 29287^2 \cdot 56009^2$.

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Table 3

k	d	n_0	k	d	n_0
1	5	2	26	3	1
2	3	2	27	3	9
3	7	2	28	2	2
4	5	1	29	5	4
5	$3 \cdot 23 \cdot 29$	26258	30	79	7
6	7	1	31	10	3
7	2	1	32	6	8
8	3	2	33	$17 \cdot 29$	1310
9	5	6	34	35	1
10	11	1	35	11	3
11	3	1	36	5	3
12	7	4	37	317	61016
13	17	2	38	7	5
14	15	1	39	10	1
15	34	11	40	14	4
16	5	2	41	5	2
17	13	10	42	$31 \cdot 13$	19
18	3	3	43	11	1
19	5	1	44	3	2
20	6	2	45	21	12
21	37	4	46	95	7
22	47	5	47	2	5
23	2	3	48	7	8
24	7	2	49	65	4
25	5	10	50	3	5

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