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On Multiperiodic Infinite Recursions and Their Finite Core

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Abstract

We define *multiperiodic infinite recursions* and show that for such a recursion there is a finite linear recursion, the *finite core*, which gives almost the same type of recursion except for a different offset. Moreover, if we add the sequences produced by all multiperiodic infinite recursions with a given finite core, we almost obtain a multiple of the sequence associated with the finite core.

1 Introduction

Consider the problem to determine the number of all additive partitions of an integer n into terms from a (possibly infinite) set M of positive integers, where we consider two partitions as different even if they contain the same terms but in a different order. We denote this number as $S_M(n)$. Obviously, there is a recursive method to calculate $S_M(n)$: For $n < 0$, $S_M(n) = 0$, $S_M(0) = 1$, and

$$S_M(n) = \sum_{m \in M} S_M(n-m) \text{ for } n > 0, \tag{1}$$

which is well-defined even if M is infinite: all members of M are positive, so the sum in (1) has only finitely many non-zero terms.

One of the most famous examples of a sequence $(S_M(n))_{n \geq 0}$ is the sequence of Fibonacci numbers (A000045 in the Online Encyclopedia [3] with a shift of 1):

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Here, the underlying set is $M = \{1, 2\}$.

We can obtain almost the same sequence from an infinite underlying set, namely the set

$$M_2 = \{1, 3, 5, 7, 9, 11, \ldots\}$$

of odd positive integers. As sequence $(S_{M_2}(n))_{n\geq 0}$ we obtain

$$1, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

which looks (except for the beginning) quite like the Fibonacci numbers. Similarly, for the set

$$M_1 = \{2, 3, 4, 5, 6, 7, \ldots\}$$

of all positive integers ≥ 2 the sequence $(S_{M_1}(n))_{n\geq 0}$ is

$$1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

– again Fibonacci! However there is still another observation: If we add the last two sequences, we obtain the original Fibonacci sequence, except for the first item of the sequences. So we observe for all $n \in \mathbb{Z}$

$$S_{M_1}(n) + S_{M_2}(n) = S_M(n) + S_\emptyset(n). \tag{2}$$

Here the sequence $(S_\emptyset(n))_{n\geq 0} = (1, 0, 0, 0, \ldots)$ is just the characteristic function of the value 0. Of course this observation is not surprising because of the recursion for the Fibonacci numbers, nontheless we will see that similar observations hold for very different and more complicated recursive sequences, even if they are not self-similar sequences as the three examples above.

Let us analyze how the sets M, M_1 and M_2 are related. The numbers 1 and 2 of the set M are the key for this analysis. The set M_1 is *1-periodic*, starting from the value 2. The set M_2 is *2-periodic*, starting from the value 1. So M_i is built from M by deleting the entry i and introducing a period i instead, which applies for the remaining number and creates an *infinite periodic set*. A proof of generalizations of the relation (2) will be the main topic of this paper.

In order to be able to formulate these generalizations we have to introduce some basic notation about multisets. For us, a multiset $\mathcal{M}$ is a pair (M, w), consisting of a *support set* $M \subseteq \mathbb{N}_{\geq 1}$ and a *multiplicity function* $w : \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N}_0$ with

$$M = \{m \in \mathbb{N} \mid w(m) > 0\}.$$

We may also write

$$\mathcal{M} = [m_1, m_2, m_3, m_4, \ldots]$$

as a (possibly infinite) list of positive (possibly not distinct) integers such that every integer j with $j = m_i$ occurs only finitely often in the list, namely exactly $w(j)$ times. The multiset $\mathcal{M}$ is *finite* if M is finite, otherwise *infinite*. We will in particular denote finite multisets by lists of the form $[m_1, \ldots, m_N]$. We denote the empty multiset $(\emptyset, w)$ simply by $\emptyset$. Let $\mathcal{M}_1 = (M_1, w_1)$, $\mathcal{M}_2 = (M_2, w_2)$ be multisets. We say $\mathcal{M}_2 \subseteq \mathcal{M}_1$ if, for all $n \in \mathbb{N}_{\geq 1}$,

$w_1(n) - w_2(n) \geq 0$, and we write $\mathcal{M}_2 \subsetneqq \mathcal{M}_1$ if $\mathcal{M}_2 \subseteq \mathcal{M}_1$ and $\mathcal{M}_2 \neq \mathcal{M}_1$. If $\mathcal{M}_2 \subseteq \mathcal{M}_1$, the multiset $\mathcal{M}_1 - \mathcal{M}_2$ is defined as $(M_1', w_1 - w_2)$, where

$$M_1' = \{m \in M_1 \mid w_1(m) - w_2(m) > 0\}.$$

Let I be a set and, for all $i \in I$, $\mathcal{M}_i = (M_i, w_i)$ be a multiset. Then the *sum*

$$\bigoplus_{i \in I} \mathcal{M}_i = \left(\bigcup_{i \in I} M_i, \sum_{i \in I} w_i \right)$$

is defined in case $\sum_{i \in I} w_i(n) < \infty$ for all $n \in \mathbb{N}$. We further define for a multiset $\mathcal{M} = (M, w)$ and a function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$

$$\sum_{m \in \mathcal{M}} f(m) := \sum_{m \in M} w(m) f(m).$$

Linear recursions have been a classical topic in elementary and analytic combinatorics, for an overview see Flajolet and Sedgewick [1]. Thinking of recursions, the above model can be easily generalized from sets M to multisets $\mathcal{M}$. Multisets fit to the partition model in the following sense. Think of the elements of a multiset $\mathcal{M}$ being colored in distinct colors. Then we want to count the number $S_{\mathcal{M}}(n)$ of additive partitions of a number n into parts from $\mathcal{M}$, respecting the order of the associated colors. If $\mathcal{M}$ is a set, this coincides with our definition above. Again, for $n > 0$, we have $S_{\mathcal{M}}(-n) = 0$, $S_{\mathcal{M}}(0) = 1$ and the recursive formula

$$S_{\mathcal{M}}(n) = \sum_{m \in \mathcal{M}} S_{\mathcal{M}}(n - m). \tag{3}$$

From this linear recursion we obtain the ordinary generating function $F_{\mathcal{M}}(z)$ of the sequence $(S_{\mathcal{M}}(n))_{n \geq 0}$, namely

$$\begin{aligned} F_{\mathcal{M}}(z) &= F_{S_{\mathcal{M}}}(z) = \sum_{n \geq 0} S_{\mathcal{M}}(n) z^n = 1 + \sum_{m \in \mathcal{M}} z^m \sum_{n \geq 0} S_{\mathcal{M}}(n-m) z^{n-m} \\ &= \frac{1}{1 - \sum_{m \in \mathcal{M}} z^m}, \end{aligned} \tag{4}$$

which will have non-zero radius of convergence in the cases we will consider.

A multiset $\mathcal{M}'$ is *periodic* if there are finitely many positive numbers $m_2, \ldots, m_N$ and a positive number m_1 (the *period*), so that $\mathcal{M}'$ is the sum of all multisets of the form

$$[m_i, m_i + m_1, m_i + 2m_1, m_i + 3m_1, \ldots],$$

where $i = 2, \ldots, N$. We call the finite multiset $[m_1, m_2, \ldots, m_N]$ the *finite core* of $\mathcal{M}'$ and denote $\mathcal{M}'$ by $\mathcal{M}_{[m_1]}$. For periodic multisets we have the following results:

Theorem 1. *Let $N \geq 2$ and $\mathcal{M} = [m_1, m_2, \ldots, m_N]$ be a finite multiset. Then, for any $n \in \mathbb{Z}$,*

$$S_{\mathcal{M}_{[m_1]}}(n) = \sum_{i=1}^{N} S_{\mathcal{M}_{[m_1]}}(n - m_i) - S_{\emptyset}(n - m_1) + S_{\emptyset}(n).$$

Theorem 2. *Let $N \geq 2$ and $\mathcal{M} = [m_1, m_2, \ldots, m_N]$ be a finite multiset. Then, for any $n \in \mathbb{Z}$,*

$$\sum_{i=1}^{N} S_{\mathcal{M}_{[m_i]}}(n) = (N-1)S_{\mathcal{M}}(n) + S_{\emptyset}(n).$$

Theorem 1 states that infinite periodic recursions, i.e., recursions coming from periodic multisets, can be written as the same finite recursion as their finite core, except for different initial values. Therefore they can be solved by the well-known analysis of finite linear recursions (see, e.g., Matoušek and Nešetřil [2]). We, however, will not focus on the asymptotic analysis but on a different point, namely Theorem 2, which gives us an exact additive relation between all periodic multisets with the same finite core and this core. We have already observed this phenomenon for the Fibonacci numbers. We observe it as well for any finite multiset, e.g., for the set $M = \{2, 3, 6\}$, where $(S_M(n))_{n\geq 0}$ is the sequence A121833 in Sloane's *Encyclopedia* [3].

Instead of proving Theorems 1 and 2 directly, which could be done either elementary or analytic, we will formulate a more general setting and prove generalizations of the statements above. Let $\mathcal{M}$ be a finite multiset, and $\mathcal{P} = [p_1, \ldots, p_K] \subsetneqq \mathcal{M}$. Then the infinite *multiperiodic* multiset $\mathcal{M}_{\mathcal{P}}$ is defined as the union of all (one-element) multisets of the form

$$[m + k_1 p_1 + k_2 p_2 + \ldots + k_K p_K]$$

which are taken with multiplicity $\binom{\sum k_i}{k_1, \ldots, k_K}$, for all integers $k_1, \ldots, k_K \geq 0$ and all $m \in \mathcal{M} - \mathcal{P}$. This is well-defined, since, for any $n \in \mathbb{N}$, there are only finitely many $(K+1)$-tuples $(m, k_1, \ldots, k_K) \in \mathbb{N}^{K+1}$ with

$$m + k_1 p_1 + k_2 p_2 + \ldots + k_K p_K = n.$$

(Recall that the multiplicities in our multisets are always finite, and $m, p_i > 0$.) Again, we call $\mathcal{M}$ the *finite core* of $\mathcal{M}_{\mathcal{P}}$. The recursion for $\mathcal{M}_{\mathcal{P}}$ is

$$S_{\mathcal{M}_{\mathcal{P}}}(n) = \sum_{m \in \mathcal{M} - \mathcal{P}} \sum_{k_1=0}^{\infty} \cdots \sum_{k_K=0}^{\infty} \binom{\sum k_i}{k_1, k_2, \ldots, k_K} S_{\mathcal{M}_{\mathcal{P}}}(n - m - \sum_{i=1}^{K} k_i p_i) + S_{\emptyset}(n)$$

for all $n \in \mathbb{Z}$.

In Section 2 we will formulate and prove the generalizations of Theorems 1 and 2 concerning multiperiodicity. In the Section 3 we discuss a further generalization.

2 Main results

The following theorem generalizes Theorem 1.

Theorem 3. *Let $N > K \geq 1$ and $\mathcal{M} = [m_1, m_2, \ldots, m_N]$ be a finite multiset. Let $\mathcal{P} = [m_1, m_2, \ldots, m_K]$ and $\mathcal{M}^* = \mathcal{M} - \mathcal{P}$. Then, for any $n \in \mathbb{Z}$,*

$$S_{\mathcal{M}_{\mathcal{P}}}(n) = \sum_{i=1}^{N} S_{\mathcal{M}_{\mathcal{P}}}(n - m_i) - \sum_{s=1}^{K} S_{\emptyset}(n - m_s) + S_{\emptyset}(n).$$

Proof.

$$
\begin{aligned}
F_{\mathcal{M}_{\mathcal{P}}}(z) &= \frac{1}{1-\sum\limits_{m\in\mathcal{M}^*}\sum\limits_{k_1=0}^{\infty}\sum\limits_{k_2=0}^{\infty}\cdots\sum\limits_{k_K=0}^{\infty}\binom{\sum_{j=1}^K k_j}{k_1,\ldots,k_K} z^{m+k_1m_1+k_2m_2+\ldots+k_Km_K}} \\
&= \frac{1}{1-\sum\limits_{m\in\mathcal{M}^*} z^m \sum\limits_{\nu=0}^{\infty}\sum\limits_{\substack{k_1,\ldots,k_K\geq 0\\ \sum_{j=1}^K k_j=\nu}}\binom{\nu}{k_1,\ldots,k_K}(z^{m_1})^{k_1}\cdots(z^{m_K})^{k_K}} \\
&= \frac{1}{1-\sum\limits_{m\in\mathcal{M}^*} z^m \sum\limits_{\nu=0}^{\infty}(z^{m_1}+z^{m_2}+\cdots+z^{m_K})^{\nu}} \\
&= \frac{1}{1-\sum\limits_{m\in\mathcal{M}^*} z^m \frac{1}{1-z^{m_1}-z^{m_2}-\cdots-z^{m_K}}} \\
&= \frac{1-\sum\limits_{s=1}^{K} z^{m_s}}{1-\sum\limits_{i=1}^{N} z^{m_i}}.
\end{aligned}
$$

In the first equation we used (4), in the third equation the multinomial theorem, in the fourth the geometric series. The last expression is the generating function $F_{\mathcal{M}}$ as in (4) except for the different offset as in the statement of the theorem. □

We conclude that the sequence function of a multiperiodic multiset is dominated by the sequence function of its finite core:

Corollary 4. *Let $N > K \geq 1$, $\mathcal{M} = [m_1, m_2, \ldots, m_N]$ be a finite multiset and $\mathcal{P} = [m_1, \ldots, m_K]$. Then, for any $n \in \mathbb{Z}$,*

$$S_{\mathcal{M}_{\mathcal{P}}}(n) \leq S_{\mathcal{M}}(n).$$

Proof. This is because of $-\sum_{s=1}^K S_{\emptyset}(n-m_s) \leq 0$ in the statement of Theorem 3. □

The next corollary could be used for an elementary proof of the main Theorem 6, whereas our analytic proof uses Theorem 3 directly.

Corollary 5. *Let $N > K \geq 1$ and $\mathcal{M} = [m_1, m_2, \ldots, m_N]$ be a finite multiset. Then, for any $n \in \mathbb{Z}$,*

$$S_{\mathcal{M}}(n) + (K-1)S_{\mathcal{M}_{[m_1,\ldots,m_K]}}(n) - \sum_{j=1}^{K} S_{\mathcal{M}_{[m_1,\ldots,m_{K+1}]-[m_j]}}(n) = KS_{\mathcal{M}}(n-m_{K+1}).$$

Proof. We prove the corollary by induction on n. For $n \le 0$ both sides are zero. Let $n > 0$ and assume it has been proved for $n-1$. Then

$$
\begin{aligned}
& S_{\mathcal{M}}(n) - K S_{\mathcal{M}}(n - m_{K+1}) \\
\overset{\text{by (3)}}{=} \quad & \sum_{i=1}^{N} \left[S_{\mathcal{M}}(n - m_i) - K S_{\mathcal{M}}(n - m_{K+1} - m_i) \right] - K S_{\emptyset}(n - m_{K+1}) \\
= \quad & \sum_{i=1}^{N} \left[\sum_{j=1}^{K} S_{\mathcal{M}_{[m_1,\ldots,m_{K+1}]-[m_j]}}(n - m_i) - (K-1) S_{\mathcal{M}_{[m_1,\ldots,m_K]}}(n - m_i) \right] \\
& - K S_{\emptyset}(n - m_{K+1}) \\
= \quad & \sum_{j=1}^{K} S_{\mathcal{M}_{[m_1,\ldots,m_{K+1}]-[m_j]}}(n) - (K-1) S_{\mathcal{M}_{[m_1,\ldots,m_K]}}(n) \\
& + \sum_{j=1}^{K} \left[\sum_{s=1}^{K+1} S_{\emptyset}(n - m_s) - S_{\emptyset}(n - m_j) \right] \\
& - (K-1) \sum_{s=1}^{K} S_{\emptyset}(n - m_s) - K S_{\emptyset}(n - m_{K+1}) \\
= \quad & \sum_{j=1}^{K} S_{\mathcal{M}_{[m_1,\ldots,m_{K+1}]-[m_j]}}(n) - (K-1) S_{\mathcal{M}_{[m_1,\ldots,m_K]}}(n).
\end{aligned}
$$

The second step is by induction hypothesis, and the third step by Theorem 3. $\square$

Now we formulate our main result, which generalizes Theorem 2:

Theorem 6. *Let $N > K \ge 1$ and $\mathcal{M} = [m_1, m_2, \ldots, m_N]$ be a finite multiset. Then, for any $n \in \mathbb{Z}$,*

$$
\binom{N-1}{K} S_{\mathcal{M}}(n) + \binom{N-1}{K-1} S_{\emptyset}(n) = \sum_{\{i_1,\ldots,i_K\} \subseteq \{1,\ldots,N\}} S_{\mathcal{M}_{[m_{i_1},\ldots,m_{i_K}]}}(n).
$$

Proof. By Theorem 3 we have

$$\begin{aligned}
\sum_{\{i_1,\ldots,i_K\}\subseteq\{1,\ldots,N\}} F_{\mathcal{M}_{[m_{i_1},\ldots,m_{i_K}]}}(z) &= \sum_{\{i_1,\ldots,i_K\}\subseteq\{1,\ldots,N\}} \frac{1-\sum_{s=1}^{K} z^{m_{i_s}}}{1-\sum_{j=1}^{N} z^{m_j}} \\
&= \frac{\binom{N}{K}\cdot 1-\binom{N}{K}\frac{K}{N}\sum_{j=1}^{N} z^{m_j}}{1-\sum_{j=1}^{N} z^{m_j}} \\
&= \frac{\binom{N-1}{K}}{1-\sum_{j=1}^{N} z^{m_j}} + \binom{N-1}{K-1}.
\end{aligned}$$

In the last step we use

$$\binom{N}{K}\frac{K}{N} = \binom{N-1}{K-1}, \quad \text{resp.} \quad \binom{N}{K} = \binom{N-1}{K} + \binom{N-1}{K-1}.$$

We obtain the required linear combination of the generating functions of the sequences $(S_{\mathcal{M}}(n))_{n\geq 0}$ and $(S_{\emptyset}(n))_{n\geq 0}$. □

Corollary 7. *Let $N \geq 2$ and $\mathcal{M} = [m_1,\ldots,m_N]$ be a finite multiset. Then, for any $n \in \mathbb{Z}$,*

$$\sum_{\emptyset\neq\mathcal{P}\subsetneq\mathcal{M}} S_{\mathcal{M}_{\mathcal{P}}}(n) = \left(2^{N-1}-1\right)\left(S_{\mathcal{M}}(n)+S_{\emptyset}(n)\right).$$

Proof. This follows from adding up the formula from Theorem 6 for all $K = 1,\ldots,N-1$. □

3 Final remark

Our results can be generalized to finite cores which are arbitrary (finite) linear recursions. Instead of the initial value $S_{\mathcal{M}}(0) = 1$, which was motivated by the application of unordered partitions, we may also assume arbitrary (finite) initial values, i.e., for a multiset $\mathcal{M}$ of positive integers and a $(D+1)$-vector $(c_0, c_1, \ldots, c_D)$ of complex numbers c_d we define the *c-sequence function* $S_{\mathcal{M}}^{(c)}$ by the recursion

$$S_{\mathcal{M}}^{(c)}(n) = \sum_{m\in\mathcal{M}} S_{\mathcal{M}}^{(c)}(n-m) + \Phi^{(c)}(n)$$

for all $n \in \mathbb{Z}$, where

$$\Phi^{(c)}(n) = \sum_{d=0}^{D} c_d S_\emptyset(n-d).$$

Let further for a finite multiset $\mathcal{P}$ of positive integers

$$\Phi^{(c)}_{\mathcal{P}}(n) = \Phi^{(c)}(n) - \sum_{d=0}^{D} c_d \sum_{m\in\mathcal{P}} S_\emptyset(n-m-d).$$

Then we can restate Theorems 3 and 6 in the following way:

Theorem 8. *Let $N > K \geq 1$ and $\mathcal{M} = [m_1, m_2, \dots, m_N]$ be a finite multiset. Let $\mathcal{P} = [m_1, m_2, \dots, m_K]$. Then, for any $n \in \mathbb{Z}$,*

$$S^{(c)}_{\mathcal{M}_{\mathcal{P}}}(n) = \sum_{i=1}^{N} S^{(c)}_{\mathcal{M}_{\mathcal{P}}}(n-m_i) + \Phi^{(c)}_{\mathcal{P}}(n).$$

Theorem 9. *Let $N > K \geq 1$ and $\mathcal{M} = [m_1, m_2, \dots, m_N]$ be a finite multiset. Then, for any $n \in \mathbb{Z}$,*

$$\binom{N-1}{K} S^{(c)}_{\mathcal{M}}(n) + \binom{N-1}{K-1} \Phi^{(c)}(n) = \sum_{\{i_1,\dots,i_K\}\subseteq\{1,\dots,N\}} S^{(c)}_{\mathcal{M}_{[m_{i_1},\dots,m_{i_K}]}}(n).$$

For the proof of these generalizations we remark that we simply have to multiply the equations in the analytic proofs of Theorems 3 and 6 by

$$\sum_{d=0}^{D} c_d z^d,$$

which stands for the changed offset $\Phi^{(c)}(n)$. The term $\Phi^{(c)}_{\mathcal{P}}(n)$ comes from the product

$$\left(1 - \sum_{m\in\mathcal{P}} z^m\right)\left(\sum_{d=0}^{D} c_d z^d\right).$$

This completes the discussion of recursions coming from multiperiodic multisets.

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(Concerned with sequence A000045 and A121833.)

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