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Functions of Slow Increase and Integer Sequences

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Abstract

We study some properties of functions that satisfy the condition $f'(x) = o\left(\frac{f(x)}{x}\right)$, for $x \to \infty$, i.e., $\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = 0$. We call these "functions of slow increase", since they satisfy the condition $\lim_{x\to\infty} \frac{f(x)}{x^\alpha} = 0$ for all $\alpha > 0$. A typical example of a function of slow increase is the function $f(x) = \log x$. As an application, we obtain some general results on sequence A_n of positive integers that satisfy the asymptotic formula $A_n \sim n^s f(n)$, where $f(x)$ is a function of slow increase.

1 Functions of Slow Increase

Definition 1. Let $f(x)$ be a function defined on the interval $[a, \infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The function $f(x)$ is *of slow increase* if the following condition holds.

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = 0. \tag{1}$$

Typical functions of slow increase are $f(x) = \log x$, $f(x) = \log^2 x$, $f(x) = \log\log x$, $f(x) = \frac{\log x}{\log\log x}$ and $\Psi : (0, \infty) \to (0, \infty)$, $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt$, which generalize the harmonic sum $H_n : N^* \to R$, $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ to $(0, \infty)$, namely $H_n = \Psi(n + 1) + \gamma$, where γ is Euler's constant.

We have the following theorems.

Theorem 2. *If $f(x)$ and $g(x)$ are functions of slow increase and C and α are positive constants then the following functions are of slow increase.*

$$f(x)+C, \qquad f(x)-C, \qquad Cf(x), \qquad f(x)g(x), \qquad f(x)^\alpha,$$

$$f(g(x)), \qquad \log f(x), \qquad f(x^\alpha), \qquad f(x^\alpha g(x)), \qquad f(x)+g(x).$$

If $f(x)$ and $g(x)$ are functions of slow increase, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) > 0$ then $\frac{f(x)}{g(x)}$ is a function of slow increase.

If $h(x)$ is a function such that $h(x) > 0$, $\lim_{x\to\infty} h(x) = \infty$ and with continuous derivative $h'(x) > 0$, then $h(\log x)$ is a function of slow increase if and only if $\lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0$.

If $h(x)$ is a function such that $h(x) > 0$, $\lim_{x\to\infty} h(x) = \infty$ and with continuous derivative $h'(x) > 0$, then $e^{h(x)}$ is a function of slow increase if and only if $\lim_{x\to\infty} xh'(x) = 0$.

If $f(x)$ is a function of slow increase the following limit holds.

$$\lim_{x\to\infty} \frac{\log f(x)}{\log x} = 0. \tag{2}$$

Proof. Use Definition 1. □

Theorem 3. *The function $f(x)$ is of slow increase if and only if $\frac{f(x)}{x^\alpha}$ has negative derivative (from a certain x_α) for all $\alpha > 0$.*

Proof. We have

$$\frac{d}{dx}\left(\frac{f(x)}{x^\alpha}\right) = \frac{f(x)}{x^{\alpha+1}}\left(\frac{xf'(x)}{f(x)} - \alpha\right). \tag{3}$$

Therefore if limit (1) holds we obtain that for all $\alpha > 0$,

$$\frac{d}{dx}\left(\frac{f(x)}{x^\alpha}\right) < 0 \tag{4}$$

for $x > x_\alpha$. On the other hand, if (4) holds (for $x > x_\alpha$), (3) gives

$$0 < \frac{xf'(x)}{f(x)} < \alpha.$$

Consequently we obtain (note that α is arbitrary)

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0.$$

That is, equation (1). □

The following theorem justifies the term "slow increase".

Theorem 4. *If the function $f(x)$ is of slow increase then*

$$\lim_{x\to\infty} \frac{f(x)}{x^\beta} = 0 \tag{5}$$

for all $\beta > 0$.

Proof. Let $\alpha > 0$ be such that $\alpha < \beta$. Then $\frac{f(x)}{x^\alpha}$ has a negative derivative (for $x > x_\alpha$), then it is decreasing, therefore it is bounded $0 < \frac{f(x)}{x^\alpha} < M$. So

$$\lim_{x\to\infty} \frac{f(x)}{x^\beta} = \lim_{x\to\infty} \frac{f(x)}{x^\alpha} . \frac{1}{x^{\beta-\alpha}} = 0.$$

□

Corollary 5. *If the function $f(x)$ is of slow increase then the following limits hold.*

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0, \tag{6}$$

$$\lim_{x\to\infty} f'(x) = 0. \tag{7}$$

Proof. Limit (6) is an immediate consequence of Theorem 4. Limit (7) is an immediate consequence of limit (6) and limit (1). □

Theorem 6. *If the function $f(x)$ is of slow increase then*

$$\sum_{i=1}^{\infty} i^\alpha f(i)^\beta = \infty \tag{8}$$

for all $\alpha > -1$ and for all β.

Proof. We have

$$\sum_{i=1}^{\infty} i^\alpha f(i)^\beta = \sum_{i=1}^{\infty} \left(i^{\alpha+1} f(i)^\beta\right) \frac{1}{i}. \tag{9}$$

Now, it is well-known that

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty. \tag{10}$$

On the other hand, we have (note that $\alpha + 1 > 0$)

$$\lim_{i\to\infty} i^{\alpha+1} f(i)^\beta = \infty. \tag{11}$$

Limit (11) is clearly true if $\beta \geq 0$. If $\beta < 0$ limit (11) is a direct consequence of Theorem 2 ($f(x)^{-\beta}$ is of slow increase) and Theorem 4.

Finally, equations (9), (10) and (11) give equation (8). □

Theorem 7. *If the function $f(x)$ is of slow increase then the following limit holds*

$$\lim_{x\to\infty} \frac{\int_a^x t^\alpha f(t)^\beta \, dt}{\frac{x^{\alpha+1}}{\alpha+1} f(x)^\beta} = 1 \tag{12}$$

for all $\alpha > -1$ and for all β.

Proof. We have (see (11))

$$\lim_{x\to\infty} \frac{x^{\alpha+1}}{\alpha+1} f(x)^\beta = \infty.$$

On the other hand, the function $t^\alpha f(t)^\beta$ is either increasing or decreasing.

Use Theorem 2 and Theorem 3 in the case $\alpha < 0$, $\beta > 0$ and $\alpha > 0$, $\beta < 0$. The others cases are trivial.

Consequently (8) implies,

$$\lim_{x\to\infty} \int_a^x t^\alpha f(t)^\beta \, dt = \infty.$$

Now, limit (12) is a direct consequence of the L'Hospital's rule and limit (1). □

Some particular cases of this theorem are the following:

If $\alpha = 0$ we have

$$\int_a^x f(t)^\beta \, dt \sim x f(x)^\beta. \tag{13}$$

If $\alpha = 0$ and $\beta = 1$ we have

$$\int_a^x f(t) \, dt \sim x f(x). \tag{14}$$

If $\alpha = 0$ and $\beta = -1$ we have

$$\int_a^x \frac{1}{f(t)} \, dt \sim \frac{x}{f(x)}. \tag{15}$$

Theorem 8. *If the function $f(x)$ is of slow increase and C is a constant then the following limit holds*

$$\lim_{x\to\infty} \frac{f(x+C)}{f(x)} = 1. \tag{16}$$

Proof. If $C > 0$, applying the Lagrange's Theorem we obtain

$$0 \le \frac{f(x+C) - f(x)}{f(x)} = \frac{C f'(\xi)}{f(x)}, \qquad (x < \xi < x + C). \tag{17}$$

Equations (17) and (7) give (16). In the same way can be proved the case $C < 0$. □

Theorem 9. *If the function $f(x)$ is of slow increase, $f'(x)$ is decreasing and $C > 0$ then the following limit holds*

$$\lim_{x\to\infty} \frac{f(Cx)}{f(x)} = 1. \tag{18}$$

Proof. Suppose that $C > 1$. Applying Lagrange's theorem we obtain

$$0 \le \frac{f(Cx) - f(x)}{f(x)} = \frac{(Cx - x) f'(\xi)}{f(x)} \le (C-1) \frac{x f'(x)}{f(x)}, \quad (x < \xi < Cx). \tag{19}$$

Equations (19) and (1) give (18).

Suppose that $C < 1$. Applying Lagrange's theorem we obtain

$$0 \leq \frac{f(x) - f(Cx)}{f(Cx)} = \frac{(x - Cx)f'(\xi)}{f(Cx)} \leq \frac{1 - C}{C} \frac{Cxf'(Cx)}{f(Cx)}, \quad (Cx < \xi < x). \tag{20}$$

Equations (20) and (1) give (18). □

Theorem 10. *If the function $f(x)$ is of slow increase, $f'(x)$ is decreasing and $0 < C_1 \leq g(x) \leq C_2$ then the following limit holds.*

$$\lim_{x \to \infty} \frac{f(g(x)x)}{f(x)} = 1. \tag{21}$$

Proof. We have

$$\frac{f(C_1 x)}{f(x)} \leq \frac{f(g(x)x)}{f(x)} \leq \frac{f(C_2 x)}{f(x)}. \tag{22}$$

Equation (22) and Theorem 9 give (21). □

2 Applications to Integer Sequences

In this section we consider only functions of slow increase that have decreasing derivative.

Let A_n be a strictly increasing sequence of positive integers such that

$$A_n \sim n^s f(n), \qquad (A_1 > 1) \tag{23}$$

and $f(x)$ is a function of slow increase.

Let $\psi(x)$ be the number of A_n that do not exceed x.

Example 11. If $A_n = p_n$ is the sequence of prime numbers we have (Prime Number Theorem) $s = 1$ and $f(x) = \log x$. If $A_n = c_{n,k}$ is the sequence of numbers with k prime factors we have $s = 1$ and $f(x) = \frac{(k-1)! \log x}{(\log \log x)^{k-1}}$ (see [2]). If $A_n = p_n^2$ we have $s = 2$ and $f(x) = \log^2 x$.

Remark 12. Note that: (i) Theorem 4 implies that $s \geq 1$ in equation (23).

(ii) There exists a strictly increasing sequence A_n that satisfies (23), for example $A_n = \lfloor n^s f(n) \rfloor$.

(iii) If the function $g(x)$ is of slow increase then $\frac{g(A_n)}{g(n)} \to l \Leftrightarrow \frac{g(n^s f(n))}{g(n)} \to l$ and $\frac{g(A_n)}{g(n)} \to \infty \Leftrightarrow \frac{g(n^s f(n))}{g(n)} \to \infty$, because (Theorem 10) $g(A_n) \sim g(n^s f(n))$.

Theorem 13. *If A_n satisfies (23) and $g(x)$ is a function of slow increase then the following equations hold*

$$A_{n+1} \sim A_n, \tag{24}$$

$$\lim_{n \to \infty} \frac{A_{n+1} - A_n}{A_n} = 0, \tag{25}$$

$$\log A_{n+1} \sim \log A_n, \tag{26}$$

$$g(A_{n+1}) \sim g(A_n), \tag{27}$$

$$\log A_n \sim s \log n, \tag{28}$$

$$\log\log A_n \sim \log\log n, \tag{29}$$

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 0.$$

Proof. Equation (24) is an immediate consequence of equation (23) and Theorem 8. Equation (25) is an immediate consequence of equation (24). Equations (26) and (27) are an immediate consequence of equation (24) and Theorem 10. Equation (28) is an direct consequence of equations (23) and (2). Equation (29) is an direct consequence of (28). The last limit is an immediate consequence of (23) ($(A_n/n) \to \infty$) and (24). □

Theorem 14. *If A_n satisfies (23) and $g(x)$ is a function of slow increase then the following equation holds (note that $l \geq 1$).*

$$g(A_n) \sim lg(n) \Leftrightarrow g(\psi(x)) \sim \frac{1}{l} g(x). \tag{30}$$

In particular (see (28) and (29))

$$\log A_n \sim s \log n \Leftrightarrow \log \psi(x) \sim \frac{1}{s} \log x, \tag{31}$$

$$\log\log A_n \sim \log\log n \Leftrightarrow \log\log \psi(x) \sim \log\log x. \tag{32}$$

Proof. We have

$$\begin{aligned} g(\psi(x)) \sim \frac{1}{l} g(x) \quad &\Rightarrow \quad g(\psi(A_n)) \sim \frac{1}{l} g(A_n) \Rightarrow g(n) \sim \frac{1}{l} g(A_n) \\ &\Rightarrow \quad g(A_n) \sim lg(n). \end{aligned}$$

On the other hand

$$g(A_n) \sim lg(n) \Rightarrow \ g(A_n) \sim lg(\psi(A_n)) \Rightarrow g(\psi(A_n)) \sim \frac{1}{l} g(A_n). \tag{33}$$

If $A_n \leq x < A_{n+1}$ we have

$$\frac{g(\psi(A_n))}{\frac{1}{l} g(A_{n+1})} \leq \frac{g(\psi(x))}{\frac{1}{l} g(x)} \leq \frac{g(\psi(A_n))}{\frac{1}{l} g(A_n)}.$$

Now, both sides have limit 1 (see (33) and (27)). □

We shall need the following well-known lemma (see [5, p. 332]).

Lemma 15. *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{i\to\infty} \frac{a_i}{b_i} = 1$. Then if $\sum_{i=1}^{\infty} b_i$ is divergent, the following limit holds.*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = 1.$$

In the following theorem we shall obtain information on $\psi(x)$ when $s = 1$ (see (23)) and $f(A_n) \sim f(n)$.

Theorem 16. *If $f(A_n) \sim f(n)$ then*

$$A_n \sim nf(n) \Leftrightarrow \psi(x) \sim \frac{x}{f(x)} \Leftrightarrow \psi(x) \sim \int_a^x \frac{1}{f(t)}\, dt \Leftrightarrow \sum_{A_i \le x} f(A_i) \sim x. \tag{34}$$

Besides if $g(x)$ is a function of slow increase and $g(A_n) \sim l' g(n)$ then

$$\psi(x) \sim \frac{\sum_{A_i \le x} g(A_i)^\beta}{g(x)^\beta} \tag{35}$$

for all β.

Proof. We have (note that $\frac{x}{f(x)} \to \infty$, see (6))

$$\begin{aligned} \psi(x) \sim \frac{x}{f(x)} \quad &\Rightarrow \quad \psi(A_n) \sim \frac{A_n}{f(A_n)} \Rightarrow n \sim \frac{A_n}{f(A_n)} \Rightarrow A_n \sim nf(A_n) \\ &\Rightarrow \quad A_n \sim nf(n). \end{aligned}$$

On the other hand

$$\begin{aligned} A_n \sim nf(n) \quad &\Rightarrow \quad A_n \sim \psi(A_n) f(n) \Rightarrow \psi(A_n) \sim \frac{A_n}{f(n)} \\ &\Rightarrow \quad \psi(A_n) \sim \frac{A_n}{f(A_n)}. \end{aligned} \tag{36}$$

If $A_n \le x < A_{n+1}$ we have (note that $\frac{x}{f(x)}$ is increasing, see Theorem 3)

$$\frac{\psi(A_n)}{\frac{A_{n+1}}{f(A_{n+1})}} \le \frac{\psi(x)}{\frac{x}{f(x)}} \le \frac{\psi(A_n)}{\frac{A_n}{f(A_n)}}. \tag{37}$$

Now, both sides have limit 1 (see (36), (24) and (27)). Consequently

$$A_n \sim nf(n) \Rightarrow \psi(x) \sim \frac{x}{f(x)}.$$

On the other hand (see (15))

$$\psi(x) \sim \frac{x}{f(x)} \Leftrightarrow \psi(x) \sim \int_a^x \frac{1}{f(t)}\, dt.$$

Note that (see (13))

$$\int_a^n g(x)^\beta \, dx \sim n g(n)^\beta.$$

Therefore as $g(x)^\beta$ is either increasing or decreasing,

$$\sum_{i=1}^{n} g(i)^\beta = \int_a^n g(x)^\beta \, dx + h(n) \sim n g(n)^\beta. \tag{38}$$

Equation (38), $g(A_n)^\beta \sim l'^\beta g(n)^\beta$ and Lemma 15 give

$$\sum_{i=1}^{n} g(A_i)^\beta \sim n l'^\beta g(n)^\beta.$$

That is

$$\sum_{A_i \leq A_n} g(A_i)^\beta \sim \psi(A_n) g(A_n)^\beta.$$

Consequently

$$\psi(A_n) \sim \frac{\sum_{A_i \leq A_n} g(A_i)^\beta}{g(A_n)^\beta}. \tag{39}$$

If $A_n \leq x < A_{n+1}$ we have ($\beta > 0$)

$$\frac{\psi(A_n)}{\frac{\sum_{A_i \leq A_n} g(A_i)^\beta}{g(A_n)^\beta}} \leq \frac{\psi(x)}{\frac{\sum_{A_i \leq x} g(A_i)^\beta}{g(x)^\beta}} \leq \frac{\psi(A_n)}{\frac{\sum_{A_i \leq A_n} g(A_i)^\beta}{g(A_{n+1})^\beta}}.$$

Now, both sides have limit 1 (see (39) and (27)). Therefore

$$\psi(x) \sim \frac{\sum_{A_i \leq x} g(A_i)^\beta}{g(x)^\beta}.$$

That is, equation (35). If $\beta < 0$, the proof of (35) is the same.

Consequently if $g(x) = f(x)$ and $\beta = 1$ we find that

$$\psi(x) \sim \frac{x}{f(x)} \Leftrightarrow \sum_{A_i \leq x} f(A_i) \sim x.$$

□

Example 17. Let us consider the sequence p_n of prime numbers, in this case we have (Prime Number Theorem) $p_n \sim n \log n$ and $\psi(x) = \pi(x) \sim x/(\log x)$. Let us consider the sequence $c_{n,k}$ of numbers with k prime factors, in this case we have $c_{n,k} \sim \frac{(k-1)! n \log n}{(\log \log n)^{k-1}}$ (see Example 11) and (Landau's Theorem) (see [1, 2]) $\psi(x) \sim \frac{x (\log \log x)^{k-1}}{(k-1)! \log x}$.

In the following general theorem we obtain information on $\psi(x)$ if $f(A_n) \sim l f(n)$. Theorem 16 is a particular case of this Theorem.

Theorem 18. *If $f(A_n) \sim l f(n)$ then*

$$A_n \sim n^s f(n) \;\Leftrightarrow\; \psi(x) \sim l^{\frac{1}{s}} \frac{x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}} \Leftrightarrow \psi(x) \sim \frac{l^{\frac{1}{s}}}{s} \int_a^x \frac{t^{-1+\frac{1}{s}}}{f(t)^{\frac{1}{s}}}\, dt$$
$$\Leftrightarrow \sum_{A_i \le x} f(A_i)^{\frac{1}{s}} \sim l^{\frac{1}{s}} x^{\frac{1}{s}}.$$

Besides if $g(x)$ is a function of slow increase and $g(A_n) \sim l' g(n)$ then

$$\psi(x) \sim \frac{\sum_{A_i \le x} g(A_i)^\beta}{g(x)^\beta} \tag{40}$$

for all β.

Proof. The proof that

$$A_n \sim n^s f(n) \Leftrightarrow \psi(x) \sim l^{\frac{1}{s}} \frac{x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}}$$

is the same as in Theorem 16. Now, see equation (12),

$$\int_a^x \frac{t^{-1+\frac{1}{s}}}{s f(t)^{\frac{1}{s}}}\, dt \sim \frac{x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}}.$$

Therefore

$$\psi(x) \sim l^{\frac{1}{s}} \frac{x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}} \Leftrightarrow \psi(x) \sim \frac{l^{\frac{1}{s}}}{s} \int_a^x \frac{t^{-1+\frac{1}{s}}}{f(t)^{\frac{1}{s}}}\, dt.$$

The proof of the equation (40) is the same as in Theorem 16. If $g(x) = f(x)$ and $\beta = 1/s$ then we find that

$$\psi(x) \sim l^{\frac{1}{s}} \frac{x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}} \Leftrightarrow \sum_{A_i \le x} f(A_i)^{\frac{1}{s}} \sim l^{\frac{1}{s}} x^{\frac{1}{s}}.$$

□

Example 19. Let us consider the following sequence of positive integers (see Theorem 22)

$$A_n = \sum_{i=1}^n p_i^k \sim \frac{n^{k+1}}{k+1} \log^k n$$

where k is a positive integer. In this case we have $s = k+1$, $f(x) = \frac{\log^k x}{k+1}$ and $l = (k+1)^k$. Consequently

$$\psi(x) \sim (k+1) \frac{x^{\frac{1}{k+1}}}{(\log x)^{\frac{k}{k+1}}}.$$

Let us consider the sequence P_n of the A_n powers. For example, if $A_n = p_n$ is the sequence of prime numbers, P_n is the sequence of prime powers . Let $\lambda(x)$ be the number of P_n that do not exceed x.

Theorem 20. *If A_n satisfies (23) then*

$$\lambda(x) \sim \psi(x). \tag{41}$$

Proof. The $A_i \le x$ are $A_1, A_2, \ldots, A_{\psi(x)}$. Let us write

$$A_i^{\alpha_i} = x, \qquad (i = 1, 2, \ldots, \psi(x)).$$

Therefore

$$\alpha_i = \frac{\log x}{\log A_i}, \qquad (i = 1, 2, \ldots, \psi(x)).$$

We have the following inequalities

$$\psi(x) \le \lambda(x) \le \sum_{i=1}^{\psi(x)} [\alpha_i] \le \sum_{i=1}^{\psi(x)} \alpha_i = \log x \sum_{i=1}^{\psi(x)} \frac{1}{\log A_i}. \tag{42}$$

Equation (28) gives

$$\frac{1}{\log A_n} \sim \frac{1}{s \log n}. \tag{43}$$

Note that (see (15))

$$\int_2^x \frac{1}{\log t}\, dt \sim \frac{x}{\log x}.$$

Now,

$$\frac{1}{\log A_1} + \sum_{i=2}^{\psi(x)} \frac{1}{s \log i} = \frac{1}{\log A_1} + \frac{1}{s} \sum_{i=2}^{\psi(x)} \frac{1}{\log i}$$
$$= \frac{1}{s} \int_2^{\psi(x)} \frac{1}{\log t}\, dt + O(1) \sim \frac{\psi(x)}{s \log \psi(x)}. \tag{44}$$

Equations (43), (44) and Lemma 15 give

$$\sum_{i=1}^{\psi(x)} \frac{1}{\log A_i} \sim \frac{\psi(x)}{s \log \psi(x)}. \tag{45}$$

Equations (42) and (45) give

$$\psi(x) \le \lambda(x) \le h(x) \frac{\psi(x) \log x}{s \log \psi(x)},$$

where $h(x) \to 1$. That is

$$1 \le \frac{\lambda(x)}{\psi(x)} \le h(x) \frac{\log x}{s \log \psi(x)}. \tag{46}$$

Finally, equations (31) and (46) give (41). □

Corollary 21. *The following limit holds.*

$$\lim_{x\to\infty} \frac{\sum_{i=1}^{\psi(x)} (\alpha_i - [\alpha_i])}{\psi(x)} = 0.$$

That is, the mean fractional part has limit zero.

Theorem 22. *If A_n satisfies (23) then the following asymptotic formulas hold*

$$\sum_{i=1}^{n} A_i^{\alpha} \sim \frac{n^{s\alpha+1} f(n)^{\alpha}}{s\alpha+1} \sim \frac{n}{s\alpha+1}\, A_n^{\alpha}, \qquad (\alpha > 0), \tag{47}$$

$$\sum_{A_i \le x} A_i^{\alpha} \sim \frac{\psi(x)}{s\alpha+1}\, x^{\alpha}, \qquad (\alpha > 0). \tag{48}$$

Proof. Let us consider the sum

$$1 + 2 + \cdots + (n' - 1) + \sum_{i=n'}^{n} \left(i^s f(i)\right)^{\alpha}, \tag{49}$$

where n' is a positive integer on interval $[a, \infty)$. Note that (see (23))

$$A_i^{\alpha} \sim \left(i^s f(i)\right)^{\alpha}. \tag{50}$$

Note that the function $x^s f(x)$ is increasing and therefore we have

$$\sum_{i=n'}^{n} \left(i^s f(i)\right)^{\alpha} = \int_{n'}^{n} x^{s\alpha} f(x)^{\alpha}\, dx + O\left(n^{s\alpha} f(n)^{\alpha}\right). \tag{51}$$

On the other hand (see (12))

$$\int_{n'}^{n} x^{s\alpha} f(x)^{\alpha}\, dx \sim \frac{n^{s\alpha+1} f(n)^{\alpha}}{s\alpha+1}. \tag{52}$$

Equations (49), (51) and (52) give

$$1 + 2 + \cdots + (n' - 1) + \sum_{i=n'}^{n} \left(i^s f(i)\right)^{\alpha} \sim \frac{n^{s\alpha+1} f(n)^{\alpha}}{s\alpha+1} \sim \frac{n}{s\alpha+1} A_n^{\alpha}. \tag{53}$$

Finally, (53), (50) and Lemma 15 give (47).

If we substitute $n = \psi(A_n)$ into equation (47) and proceed as in Theorem 14 and Theorem 16 then we obtain (48). □

Remark 23. Equations (47) and (48) when $A_n = p_n$ is the sequence of prime numbers were obtained by Sálat and Znám [6], more precise formulas when α is a positive integer were obtained by Jakimczuk [3]. Equations (47) and (48) when $A_n = c_{n,k}$ is the sequence of numbers with k prime factors were obtained by Jakimczuk [2].

Jakimczuk [4] proved the following theorem.

Theorem 24. *If A_n satisfies (23) then the following formulas hold*

$$\sum_{i=1}^{n} \log A_i = s\ n \log n - s\ n + n \log f(n) + o(n),$$

$$\lim_{n \to \infty} \frac{\sqrt[n]{A_1 A_2 \dots A_n}}{A_n} = \frac{1}{e^s}.$$

Proof. See [4]. In that proof we supposed that

$$\lim_{x \to \infty} \int_a^x \frac{t f'(t)}{f(t)}\ dt = \infty.$$

Consequently (L'Hospital's rule)

$$\lim_{x \to \infty} \frac{\int_a^x \frac{t f'(t)}{f(t)}\ dt}{x} = 0. \tag{54}$$

This supposition is unnecessary since if the integral converges then (54) also holds. □

Definition 25. The function of slow increase $f(x)$ is a *universal function* if and only if for all sequence A_n that satisfies (23) we have $f(A_n) \sim l f(n)$ where l depends of the sequence A_n.

Example 26. Equation (28) implies that $f(x) = \log x$ is an universal function, in this case $l = s$. Equation (29) implies that $f(x) = \log \log x$ is an universal function, in this case $l = 1$ does not depend of the sequence A_n.

Remark 27. Note that if $f(x)$ and $g(x)$ are universal functions then $f(x)^\alpha$ $(\alpha > 0)$, $C f(x)$ $(C > 0)$ and $f(x) g(x)$ are universal functions. If $f(x)/g(x)$ is a function of slow increase then is an universal function.

Theorem 28. *If $f(x)$ is an universal function and A_n satisfies (23) then we have*

$$\psi(x) \sim \frac{\sum_{A_i \leq x} f(A_i)^\beta}{f(x)^\beta}$$

for all β.

Proof. The proof is the same as in Theorem 16 and Theorem 18. □

Example 29. Since $f(x) = \log x$ is an universal function, we have for all sequences A_n satisfying (23) that

$$\psi(x) \sim \frac{\sum_{A_i \leq x} \log^\beta A_i}{\log^\beta x}.$$

In particular, if $\beta = 1$ we have

$$\psi(x) \sim \frac{\sum_{A_i \leq x} \log A_i}{\log x}.$$

Theorem 30. *There exist functions of slow increase that are not universal functions.*

Proof. We shall prove that the following function of slow increase

$$g(x) = e^{\frac{\log x}{\log \log x}},$$

is not an universal function. We shall prove that there exists a sequence A_n that satisfies (23) and

$$\lim_{n\to\infty} \frac{g(A_n)}{g(n)} = \infty.$$

Since A_n satisfies (23) we can write

$$A_n = h_1(n) n^s f(n),$$

where $h_1(n) \to 1$. Therefore

$$\frac{g(A_n)}{g(n)} = \exp\left(\frac{\log h_1(n) + s\log n + \log f(n)}{\log\log n + \log s + \log\left(1 + \frac{\log f(n)}{s \log n} + \frac{\log h_1(n)}{s\log n}\right)} - \frac{\log n}{\log\log n}\right). \tag{55}$$

If $s > 1$ (55) becomes (see (2))

$$\frac{g(A_n)}{g(n)} = \exp\left(h_2(n)\frac{s\log n}{\log\log n} - \frac{\log n}{\log\log n}\right),$$

where $h_2(n) \to 1$. That is

$$\frac{g(A_n)}{g(n)} = \exp\left(h_3(n)\frac{(s-1)\log n}{\log\log n}\right),$$

where $h_3(n) \to 1$. Consequently we have

$$\lim_{n\to\infty} \frac{g(A_n)}{g(n)} = \infty.$$

This proves the theorem. In particular this limit is true if $f(x) = g(x)$.

To complete, we shall examine the case $s = 1$. In this case (55) becomes (note that $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$)

$$\begin{aligned}
\frac{g(A_n)}{g(n)} &= \exp\left(\frac{\log h_1(n) + \log n + \log f(n)}{\log\log n + h_4(n)\frac{\log f(n)}{\log n} + h_4(n)\frac{\log h_1(n)}{\log n}} - \frac{\log n}{\log\log n}\right) \\
&= \exp\left(\frac{\log n + \log f(n)}{\log\log n + h_4(n)\frac{\log f(n)}{\log n} + h_4(n)\frac{\log h_1(n)}{\log n}} - \frac{\log n}{\log\log n} + o(1)\right) \\
&= \exp\left(\frac{\log\log n \log f(n) - h_4(n)\log f(n) - h_4(n)\log h_1(n)}{(\log\log n)^2 + h_4(n)\frac{\log\log n \log f(n)}{\log n} + h_4(n)\frac{\log\log n\log h_1(n)}{\log n}} + o(1)\right) \\
&= \exp\left(h_5(n)\frac{\log f(n)}{\log\log n} + o(1)\right),
\end{aligned}$$

where $h_4(n) \to 1$ and $h_5(n) \to 1$.

For example, if $f(x) = g(x)$ then $\lim_{n\to\infty} \frac{g(A_n)}{g(n)} = \infty$. If $f(x) = \log^\alpha x$ ($\alpha > 0$) then $\lim_{n\to\infty} \frac{g(A_n)}{g(n)} = e^\alpha$. If $f(x) = \log\log x$ then $\lim_{n\to\infty} \frac{g(A_n)}{g(n)} = 1$. □

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