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The Composition of the gcd and Certain Arithmetic Functions

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Abstract

We provide mean value results for sums of the composition of the gcd and arithmetic functions belonging to certain classes. Some applications are also given.

1 Introduction

In what follows, $f : \mathbb{N} \longrightarrow \mathbb{C}$ is an arithmetic function with Dirichlet series $F(s)$ and $\gcd(a,b)$ is the gcd of a and b. The Dirichlet convolution product $f \star g$ of f and g is defined by

$$(f \star g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right).$$

The classical arithmetic functions τ, σ, μ, φ, ω are respectively the number and sum of divisors, the Möbius function, the Euler totient function and the number of distinct prime factors. Finally, γ is the Euler–Mascheroni constant, $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$ and we set $\psi(t) := t - \lfloor t \rfloor - 1/2$.

In 1885, E. Cesáro [3] proved the following identity.

Lemma 1. *For every positive integer n, we have*

$$\sum_{i=1}^{n} f(\gcd(i,n)) = (f \star \varphi)(n).$$

This follows from

$$\sum_{i=1}^{n} f(\gcd(i,n)) = \sum_{d|n} f(d) \sum_{\substack{k\leq n/d \\ \gcd(k,n/d)=1}} 1 = \sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right) = (f\star\varphi)(n).$$

It should be mentioned that such an identity also occurs with some other convolution products where the summation is over some subset of the set of the divisors of n. For instance, L. Tóth [13] showed that

$$\sum_{i\in\mathrm{Reg}(n)} f(\gcd(i,n)) = \sum_{\substack{d|n \\ \gcd(d,n/d)=1}} f(d)\varphi\left(\frac{n}{d}\right)$$

where the notation $i \in \mathrm{Reg}(n)$ means that $1 \leq i \leq n$ and there exists an integer x such that $i^2x \equiv i \pmod{n}$.

Lemma 1 has a lot of interesting applications.

(a) With $f = \mathrm{Id}$ we get

$$\sum_{i=1}^{n} \gcd(i,n) = (\mathrm{Id}\star\varphi)(n)$$

which is Pillai's function [11].

(b) With $f = \mu$ we get

$$\sum_{i=1}^{n} \mu(\gcd(i,n)) = (\mu\star\varphi)(n) \tag{1}$$

and thus the number of primitive Dirichlet characters modulo n is equal to $\sum_{i=1}^{n}\mu(\gcd(i,n))$. In particular, if m is an odd positive integer then

$$\sum_{i=1}^{2m} \mu(\gcd(i,2m)) = 0.$$

(c) With $f = \tau$ we have, using $\tau\star\varphi = \sigma$

$$\sum_{i=1}^{n} \tau(\gcd(i,n)) = \sigma(n) \tag{2}$$

so that

$$\sum_{i=1}^{n} \tau(\gcd(i,n)) \ll n\log\log n$$

which should be compared to the classical estimate $\sum_{i=1}^{n}\tau(i) \ll n\log n$.

(d) With $f = 2^\omega$ we easily get

$$\sum_{i=1}^{n} 2^{\omega(\gcd(i,n))} = \Psi(n) \tag{3}$$

where $\Psi(n) := (\mu^2 \star \mathrm{Id})(n) = n \prod_{p|n}(1+p^{-1})$ is the Dedekind arithmetic function.

(e) Applying Lemma 1 twice with $f = \tau$ and $f = \sigma$ respectively, and using $\tau \star \varphi = \sigma$ and $\sigma \star \varphi = \mathrm{Id} \times \tau$, we obtain

$$\tau(n) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{\gcd(i,n)} \tau(\gcd(i,j,n)).$$

The aim of this paper is to estimate the sums

$$\sum_{n \le x} \left(\sum_{i=1}^{n} f(\gcd(i,n)) \right) \tag{4}$$

for $x \ge 1$ sufficiently large and an arithmetic function f verifying certain hypotheses. In section 2, we provide a result for four classes of multiplicative functions and then give some applications in section 3. The aim of section 4 is to provide a refinement of an estimate given in Theorem 8.

2 Main result

This section is devoted to the proof of a unified theorem which gives estimates for sums of the type (4). To this end, we first need some specific notation. More precisely, we consider the four following classes of real-valued multiplicative functions.

1. $f \in \mathcal{M}_1(\alpha)$ if there exists a real number $\alpha \ge 0$ such that

$$\sum_{n \le x} |(f \star \mu)(n)| \ll x (\log x)^\alpha. \tag{5}$$

2. $f \in \mathcal{M}_2(\alpha)$ if there exists a real number $\alpha \in [0, 3/2]$ such that, for every positive integer m we have

$$\sum_{n \le x} |(f \star \mu)(n)| = x \sum_{i=0}^{\lfloor \alpha \rfloor + m} A_i (\log x)^{\alpha - i} + O\left(\frac{x}{(\log x)^m} \right) \qquad (A_i \in \mathbb{R}). \tag{6}$$

$$\sum_{n \le x} ((f \star \mu)(n))^2 \ll x (\log x)^\beta \qquad (\beta \ge 0). \tag{7}$$

$f(p^l) - f(p^{l-1})$ is bounded for all $l \ge 1$ and primes p. (8)

The sequence $p \longmapsto f(p) - 1$ is ultimately monotone. (9)

3. $f \in \mathcal{M}_3(A)$ if there exist $A > 0,\ B,\ C,\ D \in \mathbb{R}$ and an integrable function R defined on $[1, +\infty)$ such that

$$\sum_{n \le x} \frac{f(n)}{n} = Ax + B\log x + C\psi(x) + D + R(x)$$

and $R(x) \ll x^{-a}(\log x)^E$ with $a, E \ge 0$.

4. $f \in \mathcal{M}_4(A, \alpha, \beta)$ if there exist $A > 0$, $\alpha \ge 1$ and $\alpha > \beta \ge 0$ such that

$$\sum_{n \le x} \frac{f(n)}{n} = Ax^{\alpha} + O\left(x^{\beta}\right). \tag{10}$$

Finally, we define $0 \le \theta_f \le \frac{1}{2}$ and $\Delta_f \ge 0$ such that

$$\sum_{n \le \sqrt{x}} \frac{f(n)}{n} \psi\left(\frac{x}{n}\right) \ll x^{\theta_f} (\log x)^{\Delta_f} \tag{11}$$

and we set $\theta := \max(\theta_f, \theta_{\mathrm{Id}})$ and $\Delta := \max(\Delta_f, \Delta_{\mathrm{Id}})$.

Remark 2. It is known [5] that one can take

$$\theta_{\mathrm{Id}} = \frac{131}{416} \doteq 0.3149\ldots \quad \text{and} \quad \Delta_{\mathrm{Id}} = \frac{26947}{8320} \doteq 3.2388\ldots$$

The following result gives further information when $f \in \mathcal{M}_3(A)$ and $f \in \mathcal{M}_4(A, \alpha, \beta)$.

Lemma 3. *Let $f \in \mathcal{M}_3(A)$. Then we have for x sufficiently large*

$$\sum_{n \le x} \frac{f(n)}{n^2} = A\log x + A + G - \frac{B}{x} + \frac{C\psi(x)}{x} + O\left(\frac{(\log x)^E}{x^{a+1}} + \frac{1}{x^2}\right)$$

where

$$G := B + C\left(\frac{1}{2} - \gamma\right) + D + \int_1^{\infty} \frac{R(t)\mathrm{d}t}{t^2}. \tag{12}$$

Let $f \in \mathcal{M}_4(A, \alpha, \beta)$. Then we have for x sufficiently large

$$\sum_{n \le x} \frac{f(n)}{n^2} = A\alpha E_{\alpha}(x) + O\left(\mathcal{R}_{\beta}(x)\right)$$

where

$$E_{\alpha}(x) := \begin{cases} \dfrac{x^{\alpha-1}}{\alpha - 1}, & \text{if } \alpha > 1; \\ \\ \log x, & \text{if } \alpha = 1 \end{cases} \tag{13}$$

and

$$\mathcal{R}_\beta(x) := \begin{cases} x^{\beta-1}, & \text{if } \alpha > \beta > 1; \\ \log x, & \text{if } \alpha > \beta = 1; \\ 1, & \text{if } \alpha \geq 1 > \beta \geq 0. \end{cases} \tag{14}$$

Proof. Let $f \in \mathcal{M}_3(A)$. Using Abel summation, we get

$$\begin{aligned}\sum_{n\leq x} \frac{f(n)}{n^2} &= A + \frac{B\log x}{x} + \frac{C\psi(x)}{x} + \frac{D}{x} + \frac{R(x)}{x} \\ &+ \int_1^x \frac{1}{t^2}\left(At + B\log t + C\psi(t) + D + R(t)\right)\mathrm{d}t \\ &= A\log x + A + B + D + \int_1^\infty \frac{R(t)\mathrm{d}t}{t^2} - \frac{B}{x} \\ &+ C\left(\frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)\mathrm{d}t}{t^2}\right) + \frac{R(x)}{x} - \int_x^\infty \frac{R(t)\mathrm{d}t}{t^2}.\end{aligned}$$

The estimate

$$\int_1^x \frac{\psi(t)\mathrm{d}t}{t^2} = \frac{1}{2} - \gamma + O\left(\frac{1}{x^2}\right)$$

which can be proven by using Euler–MacLaurin's summation formula, gives

$$\sum_{n\leq x} \frac{f(n)}{n^2} = A\log x + A + G - \frac{B}{x} + \frac{C\psi(x)}{x} + O\left(\frac{(\log x)^E}{x^{a+1}} + \frac{1}{x^2}\right).$$

The proof for $f \in \mathcal{M}_4(A, \alpha, \beta)$ is similar and somewhat simpler, so we omit the details. □

Now we can state our main result.

Theorem 4. *Let f be a real-valued multiplicative function with Dirichlet series $F(s)$.*

1. *If $f \in \mathcal{M}_1(\alpha)$, then we have for x sufficiently large*

$$\sum_{n\leq x}\left(\sum_{i=1}^n f(\gcd(i,n))\right) = \frac{x^2F(2)}{2\zeta(2)} + O\left\{x\prod_{p\leq x}\left(1 + \sum_{l=1}^\infty \frac{\left|f\left(p^l\right) - f\left(p^{l-1}\right)\right|}{p^l}\right) + x(\log x)^\alpha\right\}.$$

2. *If $f \in \mathcal{M}_2(\alpha)$, then we have for x sufficiently large*

$$\sum_{n\leq x}\left(\sum_{i=1}^n f(\gcd(i,n))\right) = \frac{x^2F(2)}{2\zeta(2)} + O\left\{x(\log x)^{\frac{2}{3}(\alpha+1)}(\log\log x)^{\frac{4}{3}(\alpha+1)}\right\}.$$

3. If $f \in \mathcal{M}_3(A)$, then we have for x sufficiently large

$$\sum_{n\le x}\left(\sum_{i=1}^{n} f(\gcd(i,n))\right) = \frac{Ax^2\log x}{2\zeta(2)}+\frac{x^2}{2\zeta(2)}\left\{A\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right)+G\right\}+O\left\{\left(x^{1+\theta}+x^r\right)(\log x)^{\Gamma}\right\}$$

where θ is given in (11), G is given in (12) and

$$\Gamma := \max(2,\Delta,E+1) \quad \textit{and} \quad r := \begin{cases} \frac{1}{2}(3-a), & \text{if } 0\le a<1;\\ 0, & \text{if } a\ge 1.\end{cases}$$

4. If $f \in \mathcal{M}_4(A,\alpha,\beta)$, then we have for x sufficiently large

$$\sum_{n\le x}\left(\sum_{i=1}^{n} f(\gcd(i,n))\right) = \frac{A\alpha x^2 E_\alpha(x)}{2\zeta(\alpha+1)}+O\left\{x\prod_{p\le x}\left(1+\sum_{l=1}^{\infty}\frac{\left|f\left(p^l\right)-f\left(p^{l-1}\right)\right|}{p^l}\right)+x^2\mathcal{R}_\beta(x)\right\}$$

where $E_\alpha(x)$ is given in (13) and $\mathcal{R}_\beta(x)$ is given in (14).

Proof. Let f be a real-valued multiplicative function with Dirichlet series $F(s)$.

1. Set $g := f\star\mu$. Since $\varphi=\mu\star\mathrm{Id}$, we have, using Lemma 1

$$\begin{aligned}\sum_{n\le x}\left(\sum_{i=1}^{n} f(\gcd(i,n))\right) &= \sum_{n\le x}(g\star \mathrm{Id})(n) = \sum_{d\le x} g(d)\sum_{k\le x/d} k\\ &= \frac{1}{2}\sum_{d\le x} g(d)\left\lfloor\frac{x}{d}\right\rfloor\left(\left\lfloor\frac{x}{d}\right\rfloor+1\right)\\ &= \frac{1}{2}\sum_{d\le x} g(d)\left\{\frac{x^2}{d^2}-\frac{2x}{d}\psi\left(\frac{x}{d}\right)-\left(\frac{1}{4}-\psi\left(\frac{x}{d}\right)^2\right)\right\}\\ &= \frac{x^2}{2}\sum_{d\le x}\frac{g(d)}{d^2}-x\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)+O\left(\sum_{d\le x}|g(d)|\right).\end{aligned}$$

Using (5) it is easily seen that the series $\sum_{d\ge1} g(d)d^{-2}$ is absolutely convergent, and hence we have

$$\begin{aligned}\sum_{n\le x}\left(\sum_{i=1}^{n} f(\gcd(i,n))\right) &= \frac{x^2}{2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2}-x\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)+O\left(\sum_{d\le x}|g(d)|+x^2\sum_{d>x}\frac{|g(d)|}{d^2}\right)\\ &= \frac{x^2F(2)}{2\zeta(2)}-x\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)+O\left(x(\log x)^\alpha+x^2\sum_{d>x}\frac{|g(d)|}{d^2}\right).\end{aligned}$$

Now by Abel summation and (5), we get

$$x^2\sum_{d>x}\frac{|g(d)|}{d^2} = -\sum_{d\le x}|g(d)|+2x^2\int_x^\infty\frac{1}{t^3}\left(\sum_{d\le t}|g(d)|\right)\mathrm{d}t \ll x(\log x)^\alpha$$

and the inequality $|\psi(x/d)| \leq 1/2$ gives

$$\sum_{d\leq x} \frac{g(d)}{d}\psi\left(\frac{x}{d}\right) \ll \sum_{d\leq x} \frac{|g(d)|}{d} \ll \prod_{p\leq x}\left(1+\sum_{l=1}^{\infty}\frac{\left|g\left(p^l\right)\right|}{p^l}\right).$$

2. The proof is the same as before except that we are able to treat the sum $\sum_{d\leq x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)$ more efficiently. Using (6), (7), (8) and (9) we see that the function $d \longmapsto g(d)d^{-1}$ satisfies the conditions of Theorem 1 of [9] which gives

$$\sum_{d\leq xe^{-(\log x)^{1/6}}} \frac{g(d)}{d}\psi\left(\frac{x}{d}\right) \ll (\log x)^{\frac{2}{3}(\alpha+1)}(\log\log x)^{\frac{4}{3}(\alpha+1)}$$

and, using (6) and partial summation, we get

$$\sum_{xe^{-(\log x)^{1/6}}<d\leq x} \frac{g(d)}{d}\psi\left(\frac{x}{d}\right) \ll \sum_{xe^{-(\log x)^{1/6}}<d\leq x} \frac{|g(d)|}{d} \ll (\log x)^{\alpha+1/6}.$$

Note that $\alpha \in [0,3/2]$ implies $(\log x)^{2(\alpha+1)/3} \geq (\log x)^{\alpha+1/6}$.

3. Set $h := f \star \mathrm{Id}$. Using Dirichlet's hyperbola principle, we have

$$
\begin{aligned}
\sum_{n\le x}\frac{h(n)}{n} &= \sum_{n\le x}\sum_{d|n}\frac{f(d)}{d} = \sum_{n\le x}\left(\frac{f}{\mathrm{Id}}\star\mathbf{1}\right)(n)\\
&= \sum_{n\le\sqrt{x}}\frac{f(n)}{n}\sum_{m\le x/n}1 + \sum_{n\le\sqrt{x}}\sum_{m\le x/n}\frac{f(m)}{m} - \lfloor\sqrt{x}\rfloor\sum_{n\le\sqrt{x}}\frac{f(n)}{n}\\
&= \sum_{n\le\sqrt{x}}\frac{f(n)}{n}\left(\frac{x}{n}-\frac{1}{2}-\psi\left(\frac{x}{n}\right)\right)\\
&+ \sum_{n\le\sqrt{x}}\left\{\frac{Ax}{n}+B\log\frac{x}{n}+C\psi\left(\frac{x}{n}\right)+D+O\left(\left(\frac{n}{x}\right)^a(\log x)^E\right)\right\}\\
&- \left(\sqrt{x}-\frac{1}{2}-\psi(\sqrt{x})\right)\sum_{n\le\sqrt{x}}\frac{f(n)}{n}\\
&= x\sum_{n\le\sqrt{x}}\frac{f(n)}{n^2} - \sum_{n\le\sqrt{x}}\frac{f(n)}{n}\psi\left(\frac{x}{n}\right) + Ax\sum_{n\le\sqrt{x}}\frac{1}{n} + B\sum_{n\le\sqrt{x}}\log\frac{x}{n}\\
&+ C\sum_{n\le\sqrt{x}}\psi\left(\frac{x}{n}\right) + D\left(\sqrt{x}-\frac{1}{2}-\psi(\sqrt{x})\right) + O\left(x^{(1-a)/2}(\log x)^E\right)\\
&- \left(\sqrt{x}-\psi(\sqrt{x})\right)\left(A\sqrt{x}+\frac{B}{2}\log x + C\psi(\sqrt{x}) + D + O\left(x^{-a/2}(\log x)^E\right)\right)\\
&= x\sum_{n\le\sqrt{x}}\frac{f(n)}{n^2} + Ax\left(\frac{\log x}{2}+\gamma-\frac{\psi(\sqrt{x})}{\sqrt{x}}+O\left(x^{-1}\right)\right)\\
&+ B\left(\frac{\sqrt{x}\log x}{2}+\sqrt{x}+O(\log x)\right) - \frac{D}{2} - Ax - \frac{B\sqrt{x}\log x}{2}\\
&+ (A-C)\sqrt{x}\psi(\sqrt{x}) + \frac{B}{2}\psi(\sqrt{x})\log x + C\psi(\sqrt{x})^2\\
&+ O\left(x^\theta(\log x)^\Delta + x^{(1-a)/2}(\log x)^E\right)\\
&= x\sum_{n\le\sqrt{x}}\frac{f(n)}{n^2} + \frac{Ax\log x}{2} + Ax(\gamma-1) + B\sqrt{x} - C\sqrt{x}\psi(\sqrt{x})\\
&+ O\left(x^\theta(\log x)^\Delta + x^{(1-a)/2}(\log x)^E + \log x\right).
\end{aligned}
$$

Now by Lemma 3 we get

$$
\begin{aligned}
\sum_{n\le x}\frac{h(n)}{n} &= x\left\{\frac{A\log x}{2} + A + G - \frac{B}{\sqrt{x}} + \frac{C\psi(\sqrt{x})}{\sqrt{x}} + O\left(\frac{(\log x)^E}{x^{(a+1)/2}}+\frac{1}{x}\right)\right\}\\
&+ \frac{Ax\log x}{2} + Ax(\gamma-1) + B\sqrt{x} - C\sqrt{x}\psi(\sqrt{x})\\
&+ O\left(x^\theta(\log x)^\Delta + x^{(1-a)/2}(\log x)^E + \log x\right)\\
&= Ax\log x + (A\gamma+G)x + O\left(x^\theta(\log x)^\Delta + x^{(1-a)/2}(\log x)^E + \log x\right).
\end{aligned}
$$

An Abel summation then gives

$$\sum_{n\le x} h(n) = \frac{Ax^2\log x}{2} + \frac{x^2}{4}\left(A(2\gamma+1)+2G\right) + O\left(x^{1+\theta}(\log x)^{\Delta} + x^{(3-a)/2}(\log x)^{E} + x\log x\right)$$

and hence

$$\begin{aligned}
\sum_{n\le x}\left(\sum_{i=1}^{n} f(\gcd(i,n))\right) &= \sum_{d\le x}\mu(d)\sum_{k\le x/d} h(k)\\
&= \frac{Ax^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}\log\frac{x}{d} + \frac{x^2}{4}\left(A(2\gamma+1)+2G\right)\sum_{d\le x}\frac{\mu(d)}{d^2}\\
&+ O\left(x^{1+\theta}(\log x)^{\Delta} + x^{(3-a)/2}(\log x)^{E}\sum_{d\le x}\frac{1}{d^{(3-a)/2}} + x(\log x)^2\right)\\
&= \frac{Ax^2\log x}{2\zeta(2)} - \frac{A\zeta'(2)}{2\zeta(2)^2}x^2 + \frac{x^2}{4\zeta(2)}\left(A(2\gamma+1)+2G\right)\\
&+ O\left(\left(x^{1+\theta}+x^{r}\right)(\log x)^{\Gamma}\right)
\end{aligned}$$

which is the asserted result.

4. The proof is similar to the points 1 and 2. We use $g := f \star \mu$ and we have as above

$$\begin{aligned}
\sum_{n\le x}\left(\sum_{i=1}^{n} f(\gcd(i,n))\right) &= \sum_{d\le x} g(d)\sum_{k\le x/d} k\\
&= \frac{1}{2}\sum_{d\le x} g(d)\left\lfloor\frac{x}{d}\right\rfloor\left(\left\lfloor\frac{x}{d}\right\rfloor+1\right)\\
&= \frac{x^2}{2}\sum_{d\le x}\frac{g(d)}{d^2} + O\left(x\sum_{d\le x}\frac{|g(d)|}{d}\right)\\
&= \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}\sum_{k\le x/d}\frac{f(k)}{k^2} + O\left(x\sum_{d\le x}\frac{|g(d)|}{d}\right)
\end{aligned}$$

and using Lemma 3 gives the desired result.

The proof of Theorem 4 is complete. □

3 Applications

We first introduce some additional notation. The functions $\mu, \tau, \sigma, \varphi$, Id and $\mathbf{1}$ have their usual meanings and we add the following multiplicative functions.

- $\beta(n)$ is the number of square-full divisors of n.
- $a(n)$ is the number of non-isomorphic abelian groups of order n.
- $\tau^{(e)}(n)$ and $\sigma^{(e)}(n)$ are respectively the number and the sum of exponential divisors of n.
- If $k \geq 2$ is any fixed integer, μ_k is the characteristic function of the set of k-free integers, τ_k is the k-th Piltz divisor function defined by $\tau_k = \underbrace{\mathbf{1} \star \cdots \star \mathbf{1}}_{k \text{ times}}$ with $\tau_2 = \tau$, $\tau_{(k)}(n)$ is the number of k-free divisors of n with $\tau_{(2)} = 2^\omega$ and $\gamma_k(n)$ is the greatest k-free divisor of n.
- If $\mathbb{K}/\mathbb{Q}$ is any fixed number field of degree $d \geq 2$, $\nu_{\mathbb{K}}(n)$ is the number of nonzero integral ideals of norm n. The Dedekind zeta-function of $\mathbb{K}$ is denoted by $\zeta_{\mathbb{K}}$.

The following lemma gives the distribution of these functions into the classes $\mathcal{M}_i$.

Lemma 5. *Let $k \geq 2$ be a fixed integer. We have the following distribution.*

$\mathcal{M}_1(\alpha)$	$\mathcal{M}_2(\alpha)$	$\mathcal{M}_3(A)$	$\mathcal{M}_4(A,\alpha,\beta)$
$\beta \in \mathcal{M}_1(0)$	$\tau \in \mathcal{M}_2(0)$	$\varphi \in \mathcal{M}_3\left(\zeta(2)^{-1}\right)$	$\gamma_k \in \mathcal{M}_4\left(A_k, 1, \frac{1}{k}\right)$
$\tau^{(e)} \in \mathcal{M}_1(0)$	$\tau_{(k)} \in \mathcal{M}_2(0)$	$\sigma \in \mathcal{M}_3(\zeta(2))$	
$\mu_k \in \mathcal{M}_1(0)$	$\mu \in \mathcal{M}_2(1)$	$\sigma^{(e)} \in \mathcal{M}_3(2\kappa)$	
$a \in \mathcal{M}_1(0)$			

where $\kappa \doteq 0.568$ *and*

$$A_k := \prod_p \left(1 - \frac{1}{p^{k-1}(p+1)}\right). \tag{15}$$

Proof. In the sequel, $P(n)$ is the number of unrestricted partitions of n.

1. For the class $\mathcal{M}_1(\alpha)$, use

$$(\beta \star \mu)(n) = \begin{cases} 1, & \text{if } n \text{ is square-full;} \\ 0, & \text{otherwise} \end{cases}$$

$$(\mu_k \star \mu)(n) = \begin{cases} (-1)^{\omega(m)}, & \text{if } n = m^k \text{ and } \mu_2(m) = 1; \\ 0, & \text{otherwise} \end{cases}$$

and for the function a we have $|(a\star\mu)(p)| = P(1)-1 = 0$ and for all integers $l \geq 2$ we have $\left|(a\star\mu)\left(p^l\right)\right| = |P(l)-P(l-1)| < 2\times 5^{l/4}$ (see [7] for instance) so that the function $|a\star\mu|$ satisfies Wirsing's conditions (i.e. $0 \leq f(p^l) \leq \lambda_1\lambda_2^l$ for some real numbers $\lambda_1 > 0$ and $0 \leq \lambda_2 < 2$) and hence

$$\sum_{n\leq x} |(a\star\mu)(n)| \ll \frac{x}{\log x}\exp\left(\sum_{p\leq x}\frac{|(a\star\mu)(p)|}{p}\right) \ll \frac{x}{\log x}.$$

For the function $\tau^{(e)}$ we use $(\tau^{(e)}\star\mu)(p) = \tau(1)-1 = 0$ and for all integer $l \geq 2$ we have $(\tau^{(e)}\star\mu)\left(p^l\right) = \tau(l)-\tau(l-1)$ (see [16]) so that

$$\sum_{n\leq x}\left|(\tau^{(e)}\star\mu)(n)\right| \ll \frac{x}{\log x}.$$

2. For the class $\mathcal{M}_2(\alpha)$, we use the fact that $\tau\star\mu = \mathbf{1}$ and $\tau_{(k)}\star\mu = \mu_k$ which proves the result for τ and $\tau_{(k)}$. The function μ needs more work. First we have

$$(\mu\star\mu)(n) = \begin{cases}(-2)^{\omega(a)}, & \text{if } n = ab^2 \text{ with } (a,b)=1 \text{ and } \mu_2(a)=\mu_2(b)=1;\\ 0, & \text{otherwise}\end{cases}$$

so that the conditions (7), (8) and (9) are easily checked. We now prove the following identity

$$\sum_{n\leq x}|(\mu\star\mu)(n)| = A_0 x\log x + A_1 x + O\left(x^{1/2}(\log x)^3\right) \tag{16}$$

where $A_0 = \prod_p\left(1-2p^{-2}+p^{-4}\right) \approx 0.3695\ldots$ and $A_1 = A_0\left(2\gamma - 1 + 4\sum_p \frac{\log p}{p^2-1}\right)$ which implies condition (6).

To do this we first set $f(n) := |(\mu\star\mu)(n)|$ which is multiplicative with Dirichlet series $F(s) = \zeta(s)^2 H(s)$, where $H(s) := \prod_p\left(1-2p^{-2s}+p^{-4s}\right)$ is absolutely convergent in the half-plane $\sigma > 1/2$. Moreover, if we set $H(s) := \sum_{n=1}^{\infty} h(n)n^{-s}$, then we have from the Euler product

$$h(n) = \begin{cases}(-2)^{\omega(a)}, & \text{if } n = a^2b^4 \text{ with } (a,b)=1 \text{ and } \mu_2(a)=\mu_2(b)=1;\\ 0, & \text{otherwise}\end{cases}$$

so that

$$\sum_{n\leq x}\frac{|h(n)|}{n^{1/2}} \leq \sum_{a\leq x^{1/2}}\frac{2^{\omega(a)}}{a}\sum_{b\leq (x/a^2)^{1/4}}\frac{1}{b^2} \ll (\log x)^2.$$

Now we are able to show (16). From the factorization $F(s) = \zeta(s)^2H(s)$, we infer that

$$
\begin{aligned}
\sum_{n\le x} f(n) &= \sum_{n\le x} (\tau\star h)(n) = \sum_{d\le x} h(d) \sum_{k\le x/d} \tau(k) \\
&= \sum_{d\le x} h(d) \left\{ \frac{x}{d}\log\frac{x}{d} + \frac{x}{d}(2\gamma-1) + O\left(\left(\frac{x}{d}\right)^{1/2}\right)\right\} \\
&= x(\log x + 2\gamma - 1)\sum_{d\le x}\frac{h(d)}{d} - x\sum_{d\le x}\frac{h(d)\log d}{d} + O\left\{x^{1/2}\sum_{d\le x}\frac{|h(d)|}{d^{1/2}}\right\} \\
&= H(1)x\log x + x\left\{H(1)(2\gamma-1)+H'(1)\right\} + O\left(x^{1/2}(\log x)^2\right) \\
&+ O\left(x\log x\sum_{d>x}\frac{|h(d)|}{d} + x\sum_{d>x}\frac{|h(d)|\log d}{d}\right)
\end{aligned}
$$

and we conclude the proof by using Abel summation to get

$$
\sum_{d>x}\frac{|h(d)|}{d} \ll x^{-1/2}(\log x)^2 \quad \text{and} \quad \sum_{d>x}\frac{|h(d)|\log d}{d} \ll x^{-1/2}(\log x)^3.
$$

3. For the class $\mathcal{M}_3(A)$, we first have the well known estimates

$$
\begin{aligned}
\sum_{n\le x}\frac{\varphi(n)}{n} &= \frac{x}{\zeta(2)} + O\left((\log x)^{2/3}(\log\log x)^{4/3}\right). \\
\sum_{n\le x}\frac{\sigma(n)}{n} &= x\zeta(2) - \frac{\log x}{2} + O\left((\log x)^{2/3}\right).
\end{aligned}
$$

For the function $\sigma^{(e)}$ one can deduce from the results proven in [9, 10] that

$$
\sum_{n\le x}\frac{\sigma^{(e)}(n)}{n} = 2\kappa x + O\left((\log x)^{5/3}\right).
$$

4. For the class $\mathcal{M}_4(A,\alpha,\beta)$, use (see [12])

$$
\sum_{n\le x}\frac{\gamma_k(n)}{n} = A_k x + O\left(x^{1/k}\right)
$$

where A_k is given in (15).

The proof is complete. □

For the function $\nu_{\mathbb{K}}$, we have the following result in the case of Galois extensions.

Lemma 6. *Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of degree $d \geq 2$. Then, for every positive integer n, we have*

$$|(\nu_{\mathbb{K}} \star \mu)(n)| \leq \tau_{d-1}(n)$$

so that $\nu_{\mathbb{K}} \in \mathcal{M}_1(d-2)$.

Proof. The result is obvious for $n = 1$ and, by multiplicativity, it suffices to prove the inequality for prime powers. Let p be any prime number and $l \geq 1$ be any integer. Since $\mathbb{K}/\mathbb{Q}$ is Galois, all prime ideals above p have the same residual degree denoted by f_p and we set g_p to be the number of those prime ideals. If $f_p = 1$ then $\nu_{\mathbb{K}}(p^m) = \tau_{g_p}(p^m)$ for all integer $m \geq 0$ so that

$$(\nu_{\mathbb{K}} \star \mu)\left(p^l\right) = \nu_{\mathbb{K}}\left(p^l\right) - \nu_{\mathbb{K}}\left(p^{l-1}\right) = \tau_{g_p}\left(p^l\right) - \tau_{g_p}\left(p^{l-1}\right) = \left(\tau_{g_p} \star \mu\right)\left(p^l\right) = \tau_{g_p-1}\left(p^l\right).$$

If $f_p \geq 2$ and since $\mathbb{K}/\mathbb{Q}$ is Galois, we have

$$\nu_{\mathbb{K}}\left(p^l\right) = \begin{cases} \dbinom{g_p + l/f_p - 1}{l/f_p}, & \text{if } f_p \mid l; \\ \\ 0, & \text{otherwise} \end{cases}$$

so that

$$(\nu_{\mathbb{K}} \star \mu)\left(p^l\right) = \begin{cases} -1, & \text{if } l = 1; \\ \\ \dbinom{g_p + l/f_p - 1}{l/f_p}, & \text{if } l \geq 2 \text{ and } f_p \mid l \text{ and } f_p \nmid (l-1); \\ \\ -\dbinom{g_p + (l-1)/f_p - 1}{(l-1)/f_p}, & \text{if } l \geq 2 \text{ and } f_p \nmid l \text{ and } f_p \mid (l-1); \\ \\ 0, & \text{otherwise.} \end{cases}$$

We have the trivial inequality $g_p \leq d$ and if $l, f_p \geq 2$ then $l/f_p \leq l/2 \leq l-1$ so that, using the fact that if $g \geq 1$ and $0 \leq x \leq y$ then $\binom{g+x-1}{x} \leq \binom{g+y-1}{y}$, we have in every case

$$\left|(\nu_{\mathbb{K}} \star \mu)\left(p^l\right)\right| \leq \binom{g_p + l - 2}{l} \leq \binom{d + l - 2}{l} = \tau_{d-1}\left(p^l\right)$$

which concludes the proof. □

Remark 7. One can also have the same inequality in some nonnormal cases. For instance, let $\mathbb{K}_3$ be a cubic field with negative discriminant, so that $\mathbb{K}_3$ is not Galois. The factorization of prime numbers into prime ideals is nevertheless well known and one can prove [1] that we have in fact the following situation.

Factorization of (p)	l	$(\nu_{\mathbb{K}_3} \star \mu)\left(p^l\right)$
Completely split	any	$l+1$
inert	$l \equiv 0 \pmod 3$	1
inert	$l \equiv 1 \pmod 3$	-1
inert	$l \equiv -1 \pmod 3$	0
split	$l \equiv 0 \pmod 2$	1
split	$l \equiv 1 \pmod 2$	0
ramified	any	1
Completely ramified	any	0

so that we also have in this case $\left|(\nu_{\mathbb{K}_3} \star \mu)\left(p^l\right)\right| \leq \tau\left(p^l\right)$ and hence $\nu_{\mathbb{K}_3} \in \mathcal{M}_1(1)$.

Now collecting all those results with Theorem 4 we obtain the following estimates.

Theorem 8. *For x sufficiently large, we have*

(i)

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \beta(\gcd(i,n))\right) = \frac{x^2\zeta(4)\zeta(6)}{2\zeta(12)} + O(x).$$

(ii)

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} a(\gcd(i,n))\right) = \frac{x^2}{2}\prod_{k=2}^{\infty}\zeta(2k) + O(x).$$

(iii) Let $k \geq 2$ be a fixed integer. Then we have

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \mu_k(\gcd(i,n))\right) = \frac{x^2}{2\zeta(2k)} + O(x).$$

(iv) If $\widetilde{\tau}(l) := \tau(l) - \tau(l-1) - \tau(l-2) + \tau(l-3)$ for $l \geq 5$ then we have

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \tau^{(e)}(\gcd(i,n))\right) = \frac{x^2\zeta(4)}{2}\prod_{p}\left(1 + \sum_{l=5}^{\infty}\frac{\widetilde{\tau}(l)}{p^{2l}}\right) + O(x).$$

(v) *Let Ψ be the Dedekind arithmetical function. Then we have*

$$\sum_{n\le x}\Psi(n)=\frac{x^2\zeta(2)}{2\zeta(4)}+O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

More generally, let $k\ge 2$ be a fixed integer. Then we have

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\tau_{(k)}(\gcd(i,n))\right)=\frac{x^2\zeta(2)}{2\zeta(2k)}+O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

(vi)

$$\sum_{n\le x}\sigma(n)=\frac{x^2\zeta(2)}{2}+O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

(vii) *Let $\mathcal{N}(n)$ be the number of primitive Dirichlet characters modulo n. Then we have*

$$\sum_{n\le x}\mathcal{N}(n)=\frac{x^2}{2\zeta(2)^2}+O\left(x(\log x)^{4/3}(\log\log x)^{8/3}\right).$$

(viii) *Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of degree $d\ge 2$. Then we have*

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}(\gcd(i,n))\right)=\frac{x^2\zeta_{\mathbb{K}}(2)}{2\zeta(2)}+O\left(x(\log x)^{d-1}\right).$$

(ix) *Let $\mathbb{K}_3/\mathbb{Q}$ be a cubic field with negative discriminant. Let $\mathbb{K}_6/\mathbb{Q}$ be a normal closure of $\mathbb{K}_3$ and $L(s,\psi,\mathbb{K}_6/\mathbb{Q})$ be the Artin L-function associated to the character ψ of a two-dimensional finite representation of $\mathrm{Gal}\,(\mathbb{K}_6/\mathbb{Q})\simeq\mathcal{S}_3$. Then we have*

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}_3}(\gcd(i,n))\right)=\frac{x^2L(2,\psi,\mathbb{K}_6/\mathbb{Q})}{2}+O\left(x(\log x)^2\right).$$

(x)

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\varphi(\gcd(i,n))\right)=\frac{x^2\log x}{2\zeta(2)^2}+\frac{x^2}{2\zeta(2)^2}\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}+C_\varphi\right)+O\left(x^{3/2}(\log x)^{26947/8320}\right)$$

where $C_\varphi\in\mathbb{R}$.

(xi)

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\gcd(i,n)\right)=\frac{x^2\log x}{2\zeta(2)}+\frac{x^2}{2\zeta(2)}\left(2\gamma-\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(x^{547/416}(\log x)^{26947/8320}\right).$$

(xii)

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\sigma^{(e)}(\gcd(i,n))\right)=\frac{\kappa x^2\log x}{\zeta(2)}+\frac{x^2}{\zeta(2)}\left\{\kappa\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right)+C_{\sigma^{(e)}}\right\}$$
$$+O\left(x^{3/2}(\log x)^{26947/8320}\right)$$

where $\kappa \doteq 0.568$ *and* $C_{\sigma^{(e)}}\in\mathbb{R}$.

(xiii)

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\sigma(\gcd(i,n))\right)=\frac{x^2\log x}{2}+\frac{x^2}{2}\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}+C_\sigma\right)+O\left(x^{3/2}(\log x)^{26947/8320}\right)$$

where $C_\sigma\in\mathbb{R}$.

(xiv) Let $k\ge 2$ *be a fixed integer. Then we have*

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\gamma_k(\gcd(i,n))\right)=\frac{A_k x^2\log x}{2\zeta(2)}+O\left(x^2\right)$$

where A_k *is given in* (15).

Remark 9. The first estimate in (v) and estimate (vi) are slightly weaker than the result obtained by Walfisz [15] who proved that one can remove the factor $(\log\log x)^{4/3}$. This is due to the use of the Walfisz–Pétermann's result in its whole generality which does not take account of these particular cases. For instance with (vi), we have here $\tau\star\mu=\mathbf{1}$ and Walfisz showed precisely that (see also section 4)

$$\sum_{n\le x}\frac{\mathbf{1}(n)}{n}\psi\left(\frac{x}{n}\right)=\sum_{n\le x}\frac{1}{n}\psi\left(\frac{x}{n}\right)\ll(\log x)^{2/3}.$$

Using this estimate gives Walfisz's result. More generally, it can be shown by induction that, for every integer $k\ge 1$, we have (see [8] for instance)

$$\sum_{n\le x}\frac{\tau_k(n)}{n}\psi\left(\frac{x}{n}\right)\ll(\log x)^{k-1/3}.$$

Using the method of Theorem 4 we get for $k\ge 2$

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\tau_k(\gcd(i,n))\right)=\frac{x^2\zeta(2)^{k-1}}{2}+O\left(x(\log x)^{k-4/3}\right). \qquad (17)$$

Remark 10. Estimate (xi) has first been obtained in [4] and later rediscovered in [2].

4 Quadratic fields

Let $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ be a quadratic field of discriminant D and set $d = |D|$. χ is the primitive Dirichlet character associated to $\mathbb{K}$ so that $\chi(\cdot) = (d/\cdot)$ where (a/b) is a Kronecker symbol. Finally let $L(s,\chi)$ be the L-function associated to χ. It is known that for $\sigma > 1$, we have the factorization $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi)$, and hence using estimate (viii) of Theorem 8 we obtain

$$\sum_{n \le x} \left(\sum_{i=1}^{n} \nu_{\mathbb{K}}(\gcd(i,n)) \right) = \frac{x^2 L(2,\chi)}{2} + O\left(x \log x\right).$$

The purpose of this section is to show that the error term can be improved to

$$\sum_{n \le x} \left(\sum_{i=1}^{n} \nu_{\mathbb{K}}(\gcd(i,n)) \right) = \frac{x^2 L(2,\chi)}{2} + O\left(d^{1/2} x (\log x)^{2/3}\right). \tag{18}$$

The identity $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi)$ implies that $\nu_{\mathbb{K}} \star \mu = \chi$ and thus it is easy to verify that $\nu_{\mathbb{K}}$ satisfies hypotheses (6) and (7) of the class $\mathcal{M}_2(\alpha)$ with $\alpha = 0$. We also have $\left|(\nu_{\mathbb{K}} \star \mu)\left(p^l\right)\right| = \left|\chi\left(p^l\right)\right| \le 1$ of hypothesis (8). In fact, the only condition which fails is that $p \longmapsto (\nu_{\mathbb{K}} \star \mu)(p) = \chi(p)$ is not ultimately monotone since $\chi(p) = 1,\ 0,\ -1$ depending on whether p completely splits, is ramified or is inert in $\mathbb{K}$. The following result is a first step into the direction of (18).

Lemma 11. *For every real number $x \ge 1$ sufficiently large and every real number T such that $1 \le T \le x$, we have*

$$\sum_{n \le x} \left(\sum_{i=1}^{n} \nu_{\mathbb{K}}(\gcd(i,n)) \right) = \frac{x^2 L(2,\chi)}{2} - x \sum_{n \le T} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) + O\left(x^2 T^{-2} d^{1/2} \log d + T\right).$$

In particular, we have

$$\sum_{n \le x} \left(\sum_{i=1}^{n} \nu_{\mathbb{K}}(\gcd(i,n)) \right) = \frac{x^2 L(2,\chi)}{2} - x \sum_{n \le x^{1/2}} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) + O\left(x d^{1/2} \log d\right).$$

Proof. Using Dirichlet's hyperbola principle, we get

$$\begin{aligned}
\sum_{n \le x} \left(\sum_{i=1}^{n} \nu_{\mathbb{K}}(\gcd(i,n)) \right) &= \sum_{n \le x} (\chi \star \mathrm{Id})(n) \\
&= \sum_{n \le T} \chi(n) \sum_{k \le x/n} k + \sum_{n \le x/T} n \sum_{k \le x/n} \chi(k) - \sum_{n \le T} \chi(n) \sum_{n \le x/T} n \\
&= \frac{1}{2} \sum_{n \le T} \chi(n) \left\lfloor \frac{x}{n} \right\rfloor \left(\left\lfloor \frac{x}{n} \right\rfloor + 1 \right) + O\left\{ \sum_{n \le x/T} n \left| \sum_{k \le x/n} \chi(k) \right| \right\} \\
&\quad + O\left\{ \left(\sum_{n \le x/T} n \right) \left| \sum_{n \le T} \chi(n) \right| \right\}
\end{aligned}$$

and the use of the Pólya–Vinogradov inequality and the estimate

$$\left\lfloor \frac{x}{n} \right\rfloor \left(\left\lfloor \frac{x}{n} \right\rfloor + 1 \right) = \frac{x^2}{n^2} - \frac{2x}{n} \psi\left(\frac{x}{n}\right) + O(1)$$

give

$$\sum_{n \le x} \left(\sum_{i=1}^{n} \nu_{\mathbb{K}}(\gcd(i,n)) \right) = \frac{x^2}{2} \sum_{n \le T} \frac{\chi(n)}{n^2} - x \sum_{n \le T} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) + O\left(x^2 T^{-2} d^{1/2} \log d + T\right).$$

We conclude the proof by noticing that

$$\sum_{n \le T} \frac{\chi(n)}{n^2} = L(2,\chi) - \sum_{n > T} \frac{\chi(n)}{n^2}$$

and we get by Abel summation and the Pólya–Vinogradov inequality the estimate

$$\left| \sum_{n > T} \frac{\chi(n)}{n^2} \right| \le \frac{4 d^{1/2} \log d}{T^2}$$

giving the asserted result. □

Remark 12. The choice of $T = x^{1/2}$ is obviously not the best possible since $T = x^{2/3}$ provides an error-term of the form $O\left(x^{2/3} d^{1/2} \log d\right)$, but it will be sufficient for our purpose.

Using Lemma 11, we can see that (18) follows at once from the estimate

$$\sum_{n \le x^{1/2}} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) \ll d^{1/2} (\log x)^{2/3} \tag{19}$$

where χ is any primitive real Dirichlet character of modulus $d \ge 2$. For x large we set $w(x) := \exp\left(c(\log x)^{2/3}\right)$ where $c > 0$ is an absolute constant. Since

$$\sum_{n \le w(x)} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) \ll \sum_{n \le w(x)} \frac{1}{n} \ll (\log x)^{2/3}$$

it is sufficient to prove

$$\sum_{w(x) < n \le x^{1/2}} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) \ll d^{1/2} (\log x)^{2/3} \tag{20}$$

so that we set $N \ge 1$ to be a large integer satisfying $w(x) < N \le x^{1/2}$ and consider sums of the type

$$S_N(\chi) := \sum_{N < n \le N_1} \chi(n) \psi\left(\frac{x}{n}\right) \tag{21}$$

with $N_1 \ge 1$ integer such that $N < N_1 \le 2N$.

The proof of (20) uses ideas developed by Walfisz in [15] and exploited by Pétermann [9] and Pétermann & Wu [10]. The first step to estimate (21) is to use an approximation of the function ψ by trigonometric polynomials as it was shown by Vaaler [14].

Lemma 13. *For all real number $x \geq 1$ and all integer $H \geq 1$ we have*

$$\psi(x) = -\sum_{0<|h|\leq H} \Phi\left(\frac{h}{H+1}\right)\frac{e(hx)}{2\pi i h} + \mathcal{R}_H(x)$$

where $\Phi(t) := \pi t\,(1-|t|)\cot(\pi t) + |t|$ for $0<|t|<1$ and

$$|\mathcal{R}_H(x)| \leq \frac{1}{2H+2}\sum_{|h|\leq H}\left(1-\frac{|h|}{H+1}\right)e(hx).$$

Moreover, we have $0 < \Phi(t) < 1$ for $0<|t|<1$.

Note that

$$\sum_{|h|\leq H}\left(1-\frac{|h|}{H+1}\right)e(hx) = \frac{1}{H+1}\left|\sum_{h=0}^{H} e(hx)\right|^2$$

so that the sum in the error-term is a nonnegative real number. Using this useful tool by multiplying by $\chi(n)$ and summing over $(N, N_1]$ we get

$$S_N(\chi) = -\frac{1}{2\pi i}\sum_{0<|h|\leq H}\Phi\left(\frac{h}{H+1}\right)\frac{1}{h}\sum_{N<n\leq N_1}\chi(n)e\left(\frac{hx}{n}\right) + \sum_{N<n\leq N_1}\chi(n)\mathcal{R}_H\left(\frac{x}{n}\right)$$

with

$$\begin{aligned}\left|\sum_{N<n\leq N_1}\chi(n)\mathcal{R}_H\left(\frac{x}{n}\right)\right| &\leq \sum_{N<n\leq N_1}\left|\mathcal{R}_H\left(\frac{x}{n}\right)\right| \\ &\leq \frac{1}{2H+2}\sum_{|h|\leq H}\left(1-\frac{|h|}{H+1}\right)\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right) \\ &= \frac{N}{2H+2} + \frac{1}{2H+2}\sum_{0<|h|\leq H}\left(1-\frac{|h|}{H+1}\right)\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right) \\ &\leq \frac{N}{2H+2} + \frac{1}{H+1}\sum_{1\leq h\leq H}\left|\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)\right|\end{aligned}$$

so that

$$\begin{aligned}S_N(\chi) = &-\frac{1}{2\pi i}\sum_{0<|h|\leq H}\Phi\left(\frac{h}{H+1}\right)\frac{1}{h}\sum_{N<n\leq N_1}\chi(n)e\left(\frac{hx}{n}\right) \\ &+ O\left\{NH^{-1} + H^{-1}\sum_{1\leq h\leq H}\left|\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)\right|\right\}.\end{aligned}$$

Since χ is primitive, we can expand it as a linear combination of additive characters using Gauss sums, which gives

$$S_N(\chi) = -\frac{1}{2\pi i \tau(\overline{\chi})} \sum_{a \bmod d} \overline{\chi}(a) \sum_{0<|h|\le H} \Phi\left(\frac{h}{H+1}\right) \frac{1}{h} \sum_{N<n\le N_1} e\left(\frac{hx}{n} + \frac{an}{d}\right)$$
$$+ O\left\{ NH^{-1} + H^{-1} \sum_{1\le h\le H} \left| \sum_{N<n\le N_1} e\left(\frac{hx}{n}\right) \right| \right\}$$

where $\tau(\overline{\chi})$ is the Gauss sum associated to $\overline{\chi}$. Since χ is primitive, we have $|\tau(\overline{\chi})| = d^{1/2}$ so that

$$S_N(\chi) \ll NH^{-1} + d^{-1/2} \sum_{a \bmod d} \sum_{1\le h\le H} \frac{1}{h} \left| \sum_{N<n\le N_1} e\left(\frac{hx}{n} + \frac{an}{d}\right) \right| + H^{-1} \sum_{1\le h\le H} \left| \sum_{N<n\le N_1} e\left(\frac{hx}{n}\right) \right|.$$

The second step is given by the following lemma which lies at the heart of Walfisz's method.

Lemma 14. *Suppose that $e^{200} \le N < N_1 \le 2N$ and $T \ge N^2$ and let $\alpha \in \mathbb{R}$. Then there exists $c_0 > 0$ such that uniformly in α we have*

$$\sum_{N<n\le N_1} e\left(\frac{T}{n} + \alpha n\right) \ll N \exp\left\{-c_0 \frac{(\log N)^3}{(\log TN^{-1})^2}\right\}.$$

Proof. Set $G_\alpha(y) := T/y + \alpha y$ for $N \le y \le 2N$. The case $\alpha = 0$ is Lemma 2.5 of [10]. The proof of this result uses a general theorem obtained by Karatsuba (see [6] Theorem 1, [9] Lemma C or [10] Lemma 2.4) which requires conditions on derivatives of orders ≥ 2 of G_α. Since $G_\alpha^{(j)}(y) = G_0^{(j)}(y)$ for all integer $j \ge 2$, we can see that the linear phase $e(\alpha n)$ does not affect Karatsuba's result and thus we can closely follow the proof of Lemma 2.5 of [10] giving the asserted estimate. It should be mentioned that the condition $T \ge N^2$ ensures that Karatsuba's theorem is used with derivatives of G_α having orders ≥ 2. □

Applying Lemma 14 we obtain with $e^{200} \le N < N_1 \le 2N$ and $N \le x^{1/2}$

$$S_N(\chi) \ll NH^{-1} + Nd^{1/2} \exp\left\{-c_0 \frac{(\log N)^3}{(\log HxN^{-1})^2}\right\} \log H$$

and choosing $H = [\exp\{(\log N)^3/(\log x)^2\}]$ gives for $e^{c(\log x)^{2/3}} \le N \le x^{1/2}$

$$S_N(\chi) \ll \frac{Nd^{1/2}(\log N)^3}{(\log x)^2} \exp\left\{-c_1(\log N)^3/(\log x)^2\right\}$$

with some absolute constants $c, c_1 > 0$ depending only on c_0 and where we have used the bounds $N \ge e^{c(\log x)^{2/3}}$ and $H \le x^{1/8}$. An application of Abel summation yields

$$\sum_{N<n\le N_1} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) \ll \frac{d^{1/2}(\log N)^3}{(\log x)^2} \exp\left\{-c_1(\log N)^3/(\log x)^2\right\}$$

for $e^{c(\log x)^{2/3}} \le N \le x^{1/2}$ and a similar argument to that used in the proof of Lemma 2.3 of [10] finally gives

$$\sum_{w(x)<n\le x^{1/2}} \frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) \ll d^{1/2}(\log x)^{2/3}$$

which completes the proof of (20) and (19). The following result has thus been proved.

Theorem 15. *Let $\mathbb{K}/\mathbb{Q}$ be a quadratic field of discriminant D and let χ be the quadratic Dirichlet character associated to $\mathbb{K}$. For every real number $x \ge \exp\left((\log|D|)^{3/2}\right)$ sufficiently large, we have*

$$\sum_{n\le x}\left(\sum_{i=1}^{n} \nu_{\mathbb{K}}(\gcd(i,n))\right) = \frac{x^2 L(2,\chi)}{2} + O\left(|D|^{1/2} x (\log x)^{2/3}\right).$$

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