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# The Composition of the gcd and Certain Arithmetic Functions

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**Abstract**

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We provide mean value results for sums of the composition of the gcd and arithmetic functions belonging to certain classes. Some applications are also given.

## 1 Introduction

In what follows, $f : \mathbb{N} \longrightarrow \mathbb{C}$ is an arithmetic function with Dirichlet series $F(s)$ and $(a, b)$ is the gcd of $a$ and $b$. The Dirichlet convolution product $f \star g$ of $f$ and $g$ is defined in the usual way. The classical arithmetic functions $\tau$, $\sigma$, $\mu$, $\varphi$, $\omega$ are respectively the number and sum of divisors, the Mvbius function, the Euler totient function and the number of distinct prime factors. Finally, $\gamma$ is the Euler–Mascheroni constant, $[t]$ is the integer part of $t \in \mathbb{R}$ and we set $\psi(t) := t - [t] - 1/2$.

In 1885, E. Cesáro [3] proved the following identity.

**Lemma 1.** *For every positive integer $n$, we have*

$$\sum_{i=1}^{n} f((i,n)) = (f \star \varphi)(n).$$

This follows from

$$\sum_{i=1}^{n} f((i,n)) = \sum_{d|n} f(d) \sum_{\substack{k \le n/d \\ (k,n/d)=1}} 1 = \sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right) = (f \star \varphi)(n).$$

It should be mentioned that such an identity also occurs with some other convolution products where the summation is over some subset of the set of the divisors of $n$. For instance, L. Tóth [13] showed that

$$\sum_{i\in \mathrm{Reg}(n)} f((i,n)) = \sum_{\substack{d\mid n\\ (d,n/d)=1}} f(d)\varphi\left(\frac{n}{d}\right)$$

where the notation $i \in \mathrm{Reg}(n)$ means that $1 \le i \le n$ and there exists an integer $x$ such that $i^2 x \equiv i \pmod{n}$.

Lemma 1 has a lot of interesting applications.

(a) With $f = \mathrm{Id}$ we get

$$\sum_{i=1}^{n}(i,n) = (\mathrm{Id} \star \varphi)(n)$$

which is Pillai's function [11].

(b) With $f = \mu$ we get

$$\sum_{i=1}^{n} \mu((i,n)) = (\mu \star \varphi)(n) \tag{1}$$

and thus the number of primitive Dirichlet characters modulo $n$ is equal to $\sum_{i=1}^{n} \mu((i,n))$. In particular, if $m$ is an odd positive integer then

$$\sum_{i=1}^{2m} \mu((i,2m)) = 0.$$

(c) With $f = \tau$ we have, using $\tau \star \varphi = \sigma$

$$\sum_{i=1}^{n} \tau((i,n)) = \sigma(n) \tag{2}$$

so that

$$\sum_{i=1}^{n} \tau((i,n)) \ll n \log\log n$$

which should be compared to the classical estimate $\sum_{i=1}^{n} \tau(i) \ll n \log n$.

(d) With $f = 2^{\omega}$ we easily get

$$\sum_{i=1}^{n} 2^{\omega((i,n))} = \Psi(n) \tag{3}$$

where $\Psi(n) := (\mu^2 \star \mathrm{Id})(n) = n\prod_{p\mid n}(1+p^{-1})$ is the Dedekind arithmetic function.

(e) Applying Lemma 1 twice with $f = \tau$ and $f = \sigma$ respectively, and using $\tau \star \varphi = \sigma$ and $\sigma \star \varphi = \mathrm{Id} \times \tau$, we obtain

$$\tau(n) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{(i,n)} \tau((i,j,n)).$$

The aim of this paper is to estimate the sums

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} f((i,n)) \right) \tag{4}$$

for $x \geq 1$ sufficiently large and an arithmetic function $f$ verifying certain hypotheses. In section 2, we provide a result for four classes of multiplicative functions and then give some applications in section 3. The aim of section 4 is to provide a refinement of an estimate given in Theorem 8.

# 2 Main result

This section is devoted to the proof of a unified theorem which gives estimates for sums of the type (4). To this end, we first need some specific notation. More precisely, we consider the four following classes of real-valued multiplicative functions.

1. $f \in \mathcal{M}_1(\alpha)$ if there exists a real number $\alpha \geq 0$ such that

$$\sum_{n \leq x} |(f \star \mu)(n)| \ll x(\log x)^{\alpha}. \tag{5}$$

2. $f \in \mathcal{M}_2(\alpha)$ if there exists a real number $\alpha \in [0, 3/2]$ such that, for every positive integer $m$ we have

$$\sum_{n \leq x} |(f \star \mu)(n)| = x \sum_{i=0}^{[\alpha]+m} A_i (\log x)^{\alpha - i} + O\left( \frac{x}{(\log x)^m} \right) \qquad (A_i \in \mathbb{R}). \tag{6}$$

$$\sum_{n \leq x} ((f \star \mu)(n))^2 \ll x(\log x)^{\beta} \qquad (\beta \geq 0). \tag{7}$$

$$f(p^l) - f(p^{l-1}) \text{ is bounded for all } l \geq 1 \text{ and primes } p. \tag{8}$$

$$\text{The sequence } p \longmapsto f(p) - 1 \text{ is ultimately monotone.} \tag{9}$$

3. $f \in \mathcal{M}_3(A)$ if there exist $A > 0$, $B$, $C$, $D \in \mathbb{R}$ and an integrable function $R$ defined on $[1, +\infty)$ such that

$$\sum_{n \leq x} \frac{f(n)}{n} = Ax + B \log x + C\psi(x) + D + R(x)$$

and $R(x) \ll x^{-a}(\log x)^E$ with $a, E \geq 0$.

4. $f \in \mathcal{M}_4(A, \alpha, \beta)$ if there exist $A > 0$, $\alpha \geq 1$ and $\alpha > \beta \geq 0$ such that

$$\sum_{n \leq x} \frac{f(n)}{n} = Ax^{\alpha} + O\left(x^{\beta}\right). \tag{10}$$

Finally, we define $0 \leq \theta_f \leq \frac{1}{2}$ and $\Delta_f \geq 0$ such that

$$\sum_{n \leq \sqrt{x}} \frac{f(n)}{n} \psi\left(\frac{x}{n}\right) \ll x^{\theta_f} (\log x)^{\Delta_f} \tag{11}$$

and we set $\theta := \max(\theta_f, \theta_{\mathrm{Id}})$ and $\Delta := \max(\Delta_f, \Delta_{\mathrm{Id}})$.

*Remark* 2. It is known [5] that one can take

$$\theta_{\mathrm{Id}} = \frac{131}{416} \approx 0.3149\ldots \quad \text{and} \quad \Delta_{\mathrm{Id}} = \frac{26947}{8320} \approx 3.2388\ldots$$

The following result gives further information when $f \in \mathcal{M}_3(A)$ and $f \in \mathcal{M}_4(A, \alpha, \beta)$.

**Lemma 3.** *Let $f \in \mathcal{M}_3(A)$. Then we have for $x$ sufficiently large*

$$\sum_{n \leq x} \frac{f(n)}{n^2} = A \log x + A + G - \frac{B}{x} + \frac{C\psi(x)}{x} + O\left(\frac{(\log x)^E}{x^{a+1}} + \frac{1}{x^2}\right)$$

*where*

$$G := B + C\left(\frac{1}{2} - \gamma\right) + D + \int_1^{\infty} \frac{R(t)\mathrm{d}t}{t^2}. \tag{12}$$

*Let $f \in \mathcal{M}_4(A, \alpha, \beta)$. Then we have for $x$ sufficiently large*

$$\sum_{n \leq x} \frac{f(n)}{n^2} = A\alpha E_{\alpha}(x) + O\left(\mathcal{R}_{\beta}(x)\right)$$

*where*

$$E_{\alpha}(x) := \begin{cases} \dfrac{x^{\alpha-1}}{\alpha - 1}, & \text{if } \alpha > 1; \\ \log x, & \text{if } \alpha = 1 \end{cases} \tag{13}$$

*and*

$$\mathcal{R}_{\beta}(x) := \begin{cases} x^{\beta-1}, & \text{if } \alpha > \beta > 1; \\ \log x, & \text{if } \alpha > \beta = 1; \\ 1, & \text{if } \alpha \geq 1 > \beta \geq 0. \end{cases} \tag{14}$$

*Proof.* Let $f \in \mathcal{M}_3(A)$. Using Abel summation, we get

$$\begin{aligned}\sum_{n\le x}\frac{f(n)}{n^2} &= A+\frac{B\log x}{x}+\frac{C\psi(x)}{x}+\frac{D}{x}+\frac{R(x)}{x}\\ &\quad+\int_1^x\frac{1}{t^2}\left(At+B\log t+C\psi(t)+D+R(t)\right)\mathrm{d}t\\ &= A\log x+A+B+D+\int_1^\infty\frac{R(t)\mathrm{d}t}{t^2}-\frac{B}{x}\\ &\quad+C\left(\frac{\psi(x)}{x}+\int_1^x\frac{\psi(t)\mathrm{d}t}{t^2}\right)+\frac{R(x)}{x}-\int_x^\infty\frac{R(t)\mathrm{d}t}{t^2}.\end{aligned}$$

The estimate

$$\int_1^x\frac{\psi(t)\mathrm{d}t}{t^2}=\frac{1}{2}-\gamma+O\left(\frac{1}{x^2}\right)$$

which can be proven by using Euler-MacLaurin's summation formula, gives

$$\sum_{n\le x}\frac{f(n)}{n^2}=A\log x+A+G-\frac{B}{x}+\frac{C\psi(x)}{x}+O\left(\frac{(\log x)^E}{x^{a+1}}+\frac{1}{x^2}\right).$$

The proof for $f\in\mathcal{M}_4(A,\alpha,\beta)$ is similar and somewhat simpler, so we omit the details. □

Now we can state our main result.

**Theorem 4.** *Let $f$ be a real-valued multiplicative function with Dirichlet series $F(s)$.*

1. *If $f\in\mathcal{M}_1(\alpha)$, then we have for $x$ sufficiently large*

$$\sum_{n\le x}\left(\sum_{i=1}^n f((i,n))\right)=\frac{x^2F(2)}{2\zeta(2)}+O\left\{x\prod_{p\le x}\left(1+\sum_{l=1}^\infty\frac{\left|f\left(p^l\right)-f\left(p^{l-1}\right)\right|}{p^l}\right)+x(\log x)^\alpha\right\}.$$

2. *If $f\in\mathcal{M}_2(\alpha)$, then we have for $x$ sufficiently large*

$$\sum_{n\le x}\left(\sum_{i=1}^n f((i,n))\right)=\frac{x^2F(2)}{2\zeta(2)}+O\left\{x(\log x)^{\frac{2}{3}(\alpha+1)}(\log\log x)^{\frac{4}{3}(\alpha+1)}\right\}.$$

3. *If $f\in\mathcal{M}_3(A)$, then we have for $x$ sufficiently large*

$$\sum_{n\le x}\left(\sum_{i=1}^n f((i,n))\right)=\frac{Ax^2\log x}{2\zeta(2)}+\frac{x^2}{2\zeta(2)}\left\{A\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right)+G\right\}+O\left\{\left(x^{1+\theta}+x^r\right)(\log x)^\Gamma\right\}$$

*where $\theta$ is given in (11), $G$ is given in (12) and*

$$\Gamma:=\max(2,\Delta,E+1)\quad\textit{and}\quad r:=\begin{cases}\frac{1}{2}(3-a), & \text{if } 0\le a<1;\\ 0, & \text{if } a\ge 1.\end{cases}$$

*4. If $f \in \mathcal{M}_4(A, \alpha, \beta)$, then we have for $x$ sufficiently large*

$$\sum_{n\le x}\left(\sum_{i=1}^{n} f((i,n))\right) = \frac{A\alpha x^2 E_\alpha(x)}{2\zeta(\alpha+1)} + O\left\{x\prod_{p\le x}\left(1+\sum_{l=1}^{\infty}\frac{\left|f\left(p^l\right)-f\left(p^{l-1}\right)\right|}{p^l}\right)+x^2\mathcal{R}_\beta(x)\right\}$$

*where $E_\alpha(x)$ is given in (13) and $\mathcal{R}_\beta(x)$ is given in (14).*

*Proof.* Let $f$ be a real-valued multiplicative function with Dirichlet series $F(s)$.

1. Set $g := f \star \mu$. Since $\varphi = \mu \star \mathrm{Id}$, we have, using Lemma 1

$$\begin{aligned}\sum_{n\le x}\left(\sum_{i=1}^{n} f((i,n))\right) &= \sum_{n\le x}(g\star\mathrm{Id})(n) = \sum_{d\le x} g(d)\sum_{k\le x/d} k\\ &= \frac{1}{2}\sum_{d\le x} g(d)\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)\\ &= \frac{1}{2}\sum_{d\le x} g(d)\left\{\frac{x^2}{d^2}-\frac{2x}{d}\psi\left(\frac{x}{d}\right)-\left(\frac{1}{4}-\psi\left(\frac{x}{d}\right)^2\right)\right\}\\ &= \frac{x^2}{2}\sum_{d\le x}\frac{g(d)}{d^2}-x\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)+O\left(\sum_{d\le x}|g(d)|\right).\end{aligned}$$

Using (5) it is easily seen that the series $\sum_{d\ge 1} g(d)d^{-2}$ is absolutely convergent, and hence we have

$$\begin{aligned}\sum_{n\le x}\left(\sum_{i=1}^{n} f((i,n))\right) &= \frac{x^2}{2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2}-x\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)+O\left(\sum_{d\le x}|g(d)|+x^2\sum_{d>x}\frac{|g(d)|}{d^2}\right)\\ &= \frac{x^2F(2)}{2\zeta(2)}-x\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)+O\left(x(\log x)^\alpha+x^2\sum_{d>x}\frac{|g(d)|}{d^2}\right).\end{aligned}$$

Now by Abel summation and (5), we get

$$x^2\sum_{d>x}\frac{|g(d)|}{d^2} = -\sum_{d\le x}|g(d)|+2x^2\int_x^\infty\frac{1}{t^3}\left(\sum_{d\le t}|g(d)|\right)\mathrm{d}t \ll x(\log x)^\alpha$$

and the inequality $|\psi(x/d)| \le 1/2$ gives

$$\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right) \ll \sum_{d\le x}\frac{|g(d)|}{d} \ll \prod_{p\le x}\left(1+\sum_{l=1}^{\infty}\frac{\left|g\left(p^l\right)\right|}{p^l}\right).$$

2. The proof is the same as before except that we are able to treat the sum $\sum_{d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)$ more efficiently. Using (6), (7), (8) and (9) we see that the function $d\longmapsto g(d)d^{-1}$ satisfies the conditions of Theorem 1 of [9] which gives

$$\sum_{d\le xe^{-(\log x)^{1/6}}}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)\ll(\log x)^{\frac{2}{3}(\alpha+1)}(\log\log x)^{\frac{4}{3}(\alpha+1)}$$

and, using (6) and partial summation, we get

$$\sum_{xe^{-(\log x)^{1/6}}<d\le x}\frac{g(d)}{d}\psi\left(\frac{x}{d}\right)\ll\sum_{xe^{-(\log x)^{1/6}}<d\le x}\frac{|g(d)|}{d}\ll(\log x)^{\alpha+1/6}.$$

Note that $\alpha\in[0,3/2]$ implies $(\log x)^{2(\alpha+1)/3}\ge(\log x)^{\alpha+1/6}$.

3. Set $h:=f\star\mathrm{Id}$. Using Dirichlet's hyperbola principle, we have

$$\begin{aligned}
\sum_{n\le x}\frac{h(n)}{n} &= \sum_{n\le x}\sum_{d\mid n}\frac{f(d)}{d} = \sum_{n\le x}\left(\frac{f}{\mathrm{Id}}\star\mathbf{1}\right)(n)\\
&= \sum_{n\le\sqrt{x}}\frac{f(n)}{n}\sum_{m\le x/n}1 + \sum_{n\le\sqrt{x}}\sum_{m\le x/n}\frac{f(m)}{m} - \left[\sqrt{x}\right]\sum_{n\le\sqrt{x}}\frac{f(n)}{n}\\
&= \sum_{n\le\sqrt{x}}\frac{f(n)}{n}\left(\frac{x}{n}-\frac{1}{2}-\psi\left(\frac{x}{n}\right)\right)\\
&+ \sum_{n\le\sqrt{x}}\left\{\frac{Ax}{n}+B\log\frac{x}{n}+C\psi\left(\frac{x}{n}\right)+D+O\left(\left(\frac{n}{x}\right)^{a}(\log x)^{E}\right)\right\}\\
&- \left(\sqrt{x}-\frac{1}{2}-\psi(\sqrt{x})\right)\sum_{n\le\sqrt{x}}\frac{f(n)}{n}\\
&= x\sum_{n\le\sqrt{x}}\frac{f(n)}{n^2} - \sum_{n\le\sqrt{x}}\frac{f(n)}{n}\psi\left(\frac{x}{n}\right) + Ax\sum_{n\le\sqrt{x}}\frac{1}{n} + B\sum_{n\le\sqrt{x}}\log\frac{x}{n}\\
&+ C\sum_{n\le\sqrt{x}}\psi\left(\frac{x}{n}\right) + D\left(\sqrt{x}-\frac{1}{2}-\psi(\sqrt{x})\right) + O\left(x^{(1-a)/2}(\log x)^{E}\right)\\
&- \left(\sqrt{x}-\psi(\sqrt{x})\right)\left(A\sqrt{x}+\frac{B}{2}\log x + C\psi(\sqrt{x}) + D + O\left(x^{-a/2}(\log x)^{E}\right)\right)\\
&= x\sum_{n\le\sqrt{x}}\frac{f(n)}{n^2} + Ax\left(\frac{\log x}{2}+\gamma-\frac{\psi(\sqrt{x})}{\sqrt{x}}+O\left(x^{-1}\right)\right)\\
&+ B\left(\frac{\sqrt{x}\log x}{2}+\sqrt{x}+O\left(\log x\right)\right) - \frac{D}{2} - Ax - \frac{B\sqrt{x}\log x}{2}\\
&+ (A-C)\sqrt{x}\psi(\sqrt{x}) + \frac{B}{2}\psi(\sqrt{x})\log x + C\psi(\sqrt{x})^2\\
&+ O\left(x^{\theta}(\log x)^{\Delta} + x^{(1-a)/2}(\log x)^{E}\right)\\
&= x\sum_{n\le\sqrt{x}}\frac{f(n)}{n^2} + \frac{Ax\log x}{2} + Ax\left(\gamma-1\right) + B\sqrt{x} - C\sqrt{x}\psi(\sqrt{x})\\
&+ O\left(x^{\theta}(\log x)^{\Delta} + x^{(1-a)/2}(\log x)^{E} + \log x\right).
\end{aligned}$$

Now by Lemma 3 we get

$$\begin{aligned}
\sum_{n\le x}\frac{h(n)}{n} &= x\left\{\frac{A\log x}{2}+A+G-\frac{B}{\sqrt{x}}+\frac{C\psi(\sqrt{x})}{\sqrt{x}}+O\left(\frac{(\log x)^{E}}{x^{(a+1)/2}}+\frac{1}{x}\right)\right\}\\
&+ \frac{Ax\log x}{2} + Ax\left(\gamma-1\right) + B\sqrt{x} - C\sqrt{x}\psi(\sqrt{x})\\
&+ O\left(x^{\theta}(\log x)^{\Delta} + x^{(1-a)/2}(\log x)^{E} + \log x\right)\\
&= Ax\log x + \left(A\gamma+G\right)x + O\left(x^{\theta}(\log x)^{\Delta} + x^{(1-a)/2}(\log x)^{E} + \log x\right).
\end{aligned}$$

An Abel summation then gives

$$\sum_{n\le x} h(n) = \frac{Ax^2\log x}{2} + \frac{x^2}{4}\left(A(2\gamma+1)+2G\right) + O\left(x^{1+\theta}(\log x)^{\Delta} + x^{(3-a)/2}(\log x)^{E} + x\log x\right)$$

and hence

$$\begin{aligned}
\sum_{n\le x}\left(\sum_{i=1}^{n} f((i,n))\right) &= \sum_{d\le x}\mu(d)\sum_{k\le x/d} h(k)\\
&= \frac{Ax^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}\log\frac{x}{d} + \frac{x^2}{4}\left(A(2\gamma+1)+2G\right)\sum_{d\le x}\frac{\mu(d)}{d^2}\\
&+ O\left(x^{1+\theta}(\log x)^{\Delta} + x^{(3-a)/2}(\log x)^{E}\sum_{d\le x}\frac{1}{d^{(3-a)/2}} + x(\log x)^2\right)\\
&= \frac{Ax^2\log x}{2\zeta(2)} - \frac{A\zeta'(2)}{2\zeta(2)^2}x^2 + \frac{x^2}{4\zeta(2)}\left(A(2\gamma+1)+2G\right)\\
&+ O\left(\left(x^{1+\theta}+x^{r}\right)(\log x)^{\Gamma}\right)
\end{aligned}$$

which is the asserted result.

4. The proof is similar to the points 1 and 2. We use $g := f\star\mu$ and we have as above

$$\begin{aligned}
\sum_{n\le x}\left(\sum_{i=1}^{n} f((i,n))\right) &= \sum_{d\le x} g(d)\sum_{k\le x/d} k\\
&= \frac{1}{2}\sum_{d\le x} g(d)\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)\\
&= \frac{x^2}{2}\sum_{d\le x}\frac{g(d)}{d^2} + O\left(x\sum_{d\le x}\frac{|g(d)|}{d}\right)\\
&= \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}\sum_{k\le x/d}\frac{f(k)}{k^2} + O\left(x\sum_{d\le x}\frac{|g(d)|}{d}\right)
\end{aligned}$$

and using Lemma 3 gives the desired result.

The proof of Theorem 4 is complete. □

# 3 Applications

We first introduce some additional notation. The functions $\mu, \tau, \sigma,\ \varphi,$ Id and $\mathbf{1}$ have their usual meanings and we add the following multiplicative functions.

- $\beta(n)$ is the number of square-full divisors of $n$.
- $a(n)$ is the number of non-isomorphic abelian groups of order $n$.
- $\tau^{(e)}(n)$ and $\sigma^{(e)}(n)$ are respectively the number and the sum of exponential divisors of $n$.
- If $k \geq 2$ is any fixed integer, $\mu_k$ is the characteristic function of the set of $k$-free integers, $\tau_k$ is the $k$-th Piltz divisor function defined by $\tau_k = \underbrace{\mathbf{1} \star \cdots \star \mathbf{1}}_{k \text{ times}}$ with $\tau_2 = \tau$, $\tau_{(k)}(n)$ is the number of $k$-free divisors of $n$ with $\tau_{(2)} = 2^\omega$ and $\gamma_k(n)$ is the greatest $k$-free divisor of $n$.
- If $\mathbb{K}/\mathbb{Q}$ is any fixed number field of degree $d \geq 2$, $\nu_{\mathbb{K}}(n)$ is the number of nonzero integral ideals of norm $n$. The Dedekind zeta-function of $\mathbb{K}$ is denoted by $\zeta_{\mathbb{K}}$.

The following lemma gives the distribution of these functions into the classes $\mathcal{M}_i$ .

**Lemma 5.** *Let $k \geq 2$ be a fixed integer. We have the following distribution.*

| $\mathcal{M}_1(\alpha)$ | $\mathcal{M}_2(\alpha)$ | $\mathcal{M}_3(A)$ | $\mathcal{M}_4(A, \alpha, \beta)$ |
|---|---|---|---|
| $\beta \in \mathcal{M}_1(0)$ | $\tau \in \mathcal{M}_2(0)$ | $\varphi \in \mathcal{M}_3\left(\zeta(2)^{-1}\right)$ | $\gamma_k \in \mathcal{M}_4\left(A_k, 1, \frac{1}{k}\right)$ |
| $\tau^{(e)} \in \mathcal{M}_1(0)$ | $\tau_{(k)} \in \mathcal{M}_2(0)$ | $\sigma \in \mathcal{M}_3(\zeta(2))$ | |
| $\mu_k \in \mathcal{M}_1(0)$ | $\mu \in \mathcal{M}_2(1)$ | $\sigma^{(e)} \in \mathcal{M}_3(2\kappa)$ | |
| $a \in \mathcal{M}_1(0)$ | | | |

*where $\kappa \approx 0.568$ and*

$$A_k := \prod_p \left(1 - \frac{1}{p^{k-1}(p+1)}\right). \tag{15}$$

*Proof.* In the sequel, $P(n)$ is the number of unrestricted partitions of $n$.

1. For the class $\mathcal{M}_1(\alpha)$, use

$$(\beta \star \mu)(n) = \begin{cases} 1, & \text{if } n \text{ is square-full;} \\ 0, & \text{otherwise} \end{cases}$$

$$(\mu_k \star \mu)(n) = \begin{cases} (-1)^{\omega(m)}, & \text{if } n = m^k \text{ and } \mu_2(m) = 1; \\ 0, & \text{otherwise} \end{cases}$$

and for the function $a$ we have $|(a\star\mu)(p)| = P(1)-1 = 0$ and for all integers $l\geq 2$ we have $\left|(a\star\mu)\left(p^l\right)\right| = |P(l)-P(l-1)| < 2\times 5^{l/4}$ (see [7] for instance) so that the function $|a\star\mu|$ satisfies Wirsing's conditions (i.e. $0\leq f(p^l)\leq \lambda_1\lambda_2^l$ for some real numbers $\lambda_1>0$ and $0\leq\lambda_2<2$) and hence

$$\sum_{n\leq x}|(a\star\mu)(n)| \ll \frac{x}{\log x}\exp\left(\sum_{p\leq x}\frac{|(a\star\mu)(p)|}{p}\right) \ll \frac{x}{\log x}.$$

For the function $\tau^{(e)}$ we use $(\tau^{(e)}\star\mu)(p) = \tau(1)-1 = 0$ and for all integer $l\geq 2$ we have $(\tau^{(e)}\star\mu)\left(p^l\right) = \tau(l)-\tau(l-1)$ (see [16]) so that

$$\sum_{n\leq x}\left|(\tau^{(e)}\star\mu)(n)\right| \ll \frac{x}{\log x}.$$

2. For the class $\mathcal{M}_2(\alpha)$, we use the fact that $\tau\star\mu = \mathbf{1}$ and $\tau_{(k)}\star\mu = \mu_k$ which proves the result for $\tau$ and $\tau_{(k)}$. The function $\mu$ needs more work. First we have

$$(\mu\star\mu)(n) = \begin{cases}(-2)^{\omega(a)}, & \text{if } n=ab^2 \text{ with } (a,b)=1 \text{ and } \mu_2(a)=\mu_2(b)=1;\\ 0, & \text{otherwise}\end{cases}$$

so that the conditions (7), (8) and (9) are easily checked. We now prove the following identity

$$\sum_{n\leq x}|(\mu\star\mu)(n)| = A_0 x\log x + A_1 x + O\left(x^{1/2}(\log x)^3\right) \tag{16}$$

where $A_0 = \prod_p(1-2p^{-2}+p^{-4}) \approx 0.3695\ldots$ and $A_1 = A_0\left(2\gamma-1+4\sum_p\frac{\log p}{p^2-1}\right)$ which implies condition (6).
To do this we first set $f(n) := |(\mu\star\mu)(n)|$ which is multiplicative with Dirichlet series $F(s) = \zeta(s)^2H(s)$, where $H(s) := \prod_p(1-2p^{-2s}+p^{-4s})$ is absolutely convergent in the half-plane $\sigma>1/2$. Moreover, if we set $H(s) := \sum_{n=1}^{\infty}h(n)n^{-s}$, then we have from the Euler product

$$h(n) = \begin{cases}(-2)^{\omega(a)}, & \text{if } n=a^2b^4 \text{ with } (a,b)=1 \text{ and } \mu_2(a)=\mu_2(b)=1;\\ 0, & \text{otherwise}\end{cases}$$

so that

$$\sum_{n\leq x}\frac{|h(n)|}{n^{1/2}} \leq \sum_{a\leq x^{1/2}}\frac{2^{\omega(a)}}{a}\sum_{b\leq(x/a^2)^{1/4}}\frac{1}{b^2} \ll (\log x)^2.$$

Now we are able to show (16). From the factorization $F(s) = \zeta(s)^2H(s)$, we infer that

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{n\le x} (\tau\star h)(n) = \sum_{d\le x} h(d) \sum_{k\le x/d} \tau(k)\\ &= \sum_{d\le x} h(d)\left\{\frac{x}{d}\log\frac{x}{d} + \frac{x}{d}(2\gamma-1) + O\left(\left(\frac{x}{d}\right)^{1/2}\right)\right\}\\ &= x(\log x + 2\gamma - 1)\sum_{d\le x}\frac{h(d)}{d} - x\sum_{d\le x}\frac{h(d)\log d}{d} + O\left\{x^{1/2}\sum_{d\le x}\frac{|h(d)|}{d^{1/2}}\right\}\\ &= H(1)x\log x + x\left\{H(1)(2\gamma-1) + H'(1)\right\} + O\left(x^{1/2}(\log x)^2\right)\\ &+ O\left(x\log x\sum_{d>x}\frac{|h(d)|}{d} + x\sum_{d>x}\frac{|h(d)|\log d}{d}\right)\end{aligned}$$

and we conclude the proof by using Abel summation to get

$$\sum_{d>x}\frac{|h(d)|}{d} \ll x^{-1/2}(\log x)^2 \quad \text{and} \quad \sum_{d>x}\frac{|h(d)|\log d}{d} \ll x^{-1/2}(\log x)^3.$$

3. For the class $\mathcal{M}_3(A)$, we first have the well known estimates

$$\begin{aligned}\sum_{n\le x}\frac{\varphi(n)}{n} &= \frac{x}{\zeta(2)} + O\left((\log x)^{2/3}(\log\log x)^{4/3}\right).\\ \sum_{n\le x}\frac{\sigma(n)}{n} &= x\zeta(2) - \frac{\log x}{2} + O\left((\log x)^{2/3}\right).\end{aligned}$$

For the function $\sigma^{(e)}$ one can deduce from the results proven in [9, 10] that

$$\sum_{n\le x}\frac{\sigma^{(e)}(n)}{n} = 2\kappa x + O\left((\log x)^{5/3}\right).$$

4. For the class $\mathcal{M}_4(A,\alpha,\beta)$, use (see [12])

$$\sum_{n\le x}\frac{\gamma_k(n)}{n} = A_k x + O\left(x^{1/k}\right)$$

where $A_k$ is given in (15).

The proof is complete. □

For the function $\nu_{\mathbb{K}}$, we have the following result in the case of Galois extensions.

**Lemma 6.** *Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of degree $d \geq 2$. Then, for every positive integer $n$, we have*

$$|(\nu_{\mathbb{K}} \star \mu)(n)| \leq \tau_{d-1}(n)$$

*so that $\nu_{\mathbb{K}} \in \mathcal{M}_1(d-2)$.*

*Proof.* The result is obvious for $n = 1$ and, by multiplicativity, it suffices to prove the inequality for prime powers. Let $p$ be any prime number and $l \geq 1$ be any integer. Since $\mathbb{K}/\mathbb{Q}$ is Galois, all prime ideals above $p$ have the same residual degree denoted by $f_p$ and we set $g_p$ to be the number of those prime ideals. If $f_p = 1$ then $\nu_{\mathbb{K}}\left(p^m\right) = \tau_{g_p}\left(p^m\right)$ for all integer $m \geq 0$ so that

$$\left(\nu_{\mathbb{K}} \star \mu\right)\left(p^l\right) = \nu_{\mathbb{K}}\left(p^l\right) - \nu_{\mathbb{K}}\left(p^{l-1}\right) = \tau_{g_p}\left(p^l\right) - \tau_{g_p}\left(p^{l-1}\right) = \left(\tau_{g_p} \star \mu\right)\left(p^l\right) = \tau_{g_p - 1}\left(p^l\right).$$

If $f_p \geq 2$ and since $\mathbb{K}/\mathbb{Q}$ is Galois, we have

$$\nu_{\mathbb{K}}\left(p^l\right) = \begin{cases} \binom{g_p + l/f_p - 1}{l/f_p}, & \text{if } f_p \mid l; \\ \\ 0, & \text{otherwise;} \end{cases}$$

so that

$$\left(\nu_{\mathbb{K}} \star \mu\right)\left(p^l\right) = \begin{cases} -1, & \text{if } l = 1; \\ \\ \binom{g_p + l/f_p - 1}{l/f_p}, & \text{if } l \geq 2 \text{ and } f_p \mid l \text{ and } f_p \nmid (l-1); \\ \\ -\binom{g_p + (l-1)/f_p - 1}{(l-1)/f_p}, & \text{if } l \geq 2 \text{ and } f_p \nmid l \text{ and } f_p \mid (l-1); \\ \\ 0, & \text{otherwise.} \end{cases}$$

We have the trivial inequality $g_p \leq d$ and if $l, f_p \geq 2$ then $l/f_p \leq l/2 \leq l-1$ so that, using the fact that if $g \geq 1$ and $0 \leq x \leq y$ then $\binom{g+x-1}{x} \leq \binom{g+y-1}{y}$, we have in every case

$$\left|\left(\nu_{\mathbb{K}} \star \mu\right)\left(p^l\right)\right| \leq \binom{g_p + l - 2}{l} \leq \binom{d + l - 2}{l} = \tau_{d-1}\left(p^l\right)$$

which concludes the proof. □

*Remark* 7. One can also have the same inequality in some nonnormal cases. For instance, let $\mathbb{K}_3$ be a cubic field with negative discriminant, so that $\mathbb{K}_3$ is not Galois. The factorization of prime numbers into prime ideals is nevertheless well known and one can prove [1] that we have in fact the following situation.

| Factorization of $(p)$ | $l$ | $(\nu_{\mathbb{K}_3} \star \mu)\left(p^l\right)$ |
|---|---|---|
| Completely split | any | $l+1$ |
| inert | $l \equiv 0 \pmod 3$ | 1 |
| inert | $l \equiv 1 \pmod 3$ | $-1$ |
| inert | $l \equiv -1 \pmod 3$ | 0 |
| split | $l \equiv 0 \pmod 2$ | 1 |
| split | $l \equiv 1 \pmod 2$ | 0 |
| ramified | any | 1 |
| Completely ramified | any | 0 |

so that we also have in this case $\left|(\nu_{\mathbb{K}_3} \star \mu)\left(p^l\right)\right| \leq \tau\left(p^l\right)$ and hence $\nu_{\mathbb{K}_3} \in \mathcal{M}_1(1)$.

Now collecting all those results with Theorem 4 we obtain the following estimates.

**Theorem 8.** *For $x$ sufficiently large, we have*

*(i)*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\beta((i,n))\right)=\frac{x^2\zeta(4)\zeta(6)}{2\zeta(12)}+O\left(x\right).$$

*(ii)*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}a((i,n))\right)=\frac{x^2}{2}\prod_{k=2}^{\infty}\zeta(2k)+O\left(x\right).$$

*(iii) Let $k\geq 2$ be a fixed integer. Then we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\mu_k((i,n))\right)=\frac{x^2}{2\zeta(2k)}+O\left(x\right).$$

*(iv) If $\widetilde{\tau}(l):=\tau(l)-\tau(l-1)-\tau(l-2)+\tau(l-3)$ for $l\geq 5$ then we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\tau^{(e)}((i,n))\right)=\frac{x^2\zeta(4)}{2}\prod_{p}\left(1+\sum_{l=5}^{\infty}\frac{\widetilde{\tau}(l)}{p^{2l}}\right)+O\left(x\right).$$

*(v) Let $\Psi$ be the Dedekind arithmetical function. Then we have*

$$\sum_{n\leq x}\Psi(n)=\frac{x^2\zeta(2)}{2\zeta(4)}+O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

*More generally, let $k\geq 2$ be a fixed integer. Then we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\tau_{(k)}((i,n))\right)=\frac{x^2\zeta(2)}{2\zeta(2k)}+O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

*(vi)*

$$\sum_{n\leq x}\sigma(n)=\frac{x^2\zeta(2)}{2}+O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

*(vii) Let $\mathcal{N}(n)$ be the number of primitive Dirichlet characters modulo $n$. Then we have*

$$\sum_{n\leq x}\mathcal{N}(n)=\frac{x^2}{2\zeta(2)^2}+O\left(x(\log x)^{4/3}(\log\log x)^{8/3}\right).$$

*(viii) Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of degree $d\geq 2$. Then we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right)=\frac{x^2\zeta_{\mathbb{K}}(2)}{2\zeta(2)}+O\left(x(\log x)^{d-1}\right).$$

*(ix) Let $\mathbb{K}_3/\mathbb{Q}$ be a cubic field with negative discriminant. Let $\mathbb{K}_6/\mathbb{Q}$ be a normal closure of $\mathbb{K}_3$ and $L(s,\psi,\mathbb{K}_6/\mathbb{Q})$ be the Artin L-function associated to the character $\psi$ of a two-dimensional finite representation of $\mathrm{Gal}\,(\mathbb{K}_6/\mathbb{Q})\simeq\mathcal{S}_3$. Then we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}_3}((i,n))\right)=\frac{x^2L(2,\psi,\mathbb{K}_6/\mathbb{Q})}{2}+O\left(x(\log x)^2\right).$$

*(x)*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\varphi((i,n))\right)=\frac{x^2\log x}{2\zeta(2)^2}+\frac{x^2}{2\zeta(2)^2}\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}+C_\varphi\right)+O\left(x^{3/2}(\log x)^{26947/8320}\right)$$

*where $C_\varphi\in\mathbb{R}$.*

*(xi)*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}(i,n)\right)=\frac{x^2\log x}{2\zeta(2)}+\frac{x^2}{2\zeta(2)}\left(2\gamma-\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(x^{547/416}(\log x)^{26947/8320}\right).$$

*(xii)*

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\sigma^{(e)}((i,n))\right)=\frac{\kappa x^2\log x}{\zeta(2)}+\frac{x^2}{\zeta(2)}\left\{\kappa\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right)+C_{\sigma^{(e)}}\right\}+O\left(x^{3/2}(\log x)^{26947/8320}\right)$$

*where* $\kappa\approx 0.568$ *and* $C_{\sigma^{(e)}}\in\mathbb{R}$.

*(xiii)*

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\sigma((i,n))\right)=\frac{x^2\log x}{2}+\frac{x^2}{2}\left(\gamma+\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}+C_{\sigma}\right)+O\left(x^{3/2}(\log x)^{26947/8320}\right)$$

*where* $C_\sigma\in\mathbb{R}$.

*(xiv) Let* $k\ge 2$ *be a fixed integer. Then we have*

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\gamma_k((i,n))\right)=\frac{A_k x^2\log x}{2\zeta(2)}+O\left(x^2\right)$$

*where* $A_k$ *is given in* (15).

*Remark* 9. The first estimate in (v) and estimate (vi) are slightly weaker than the result obtained by Walfisz [15] who proved that one can remove the factor $(\log\log x)^{4/3}$. This is due to the use of the Walfisz–Pétermann's result in its whole generality which does not take account of these particular cases. For instance with (vi), we have here $\tau\star\mu=\mathbf{1}$ and Walfisz showed precisely that (see also section 4)

$$\sum_{n\le x}\frac{\mathbf{1}(n)}{n}\psi\left(\frac{x}{n}\right)=\sum_{n\le x}\frac{1}{n}\psi\left(\frac{x}{n}\right)\ll(\log x)^{2/3}.$$

Using this estimate gives Walfisz's result. More generally, it can be shown by induction that, for every integer $k\ge 1$, we have (see [8] for instance)

$$\sum_{n\le x}\frac{\tau_k(n)}{n}\psi\left(\frac{x}{n}\right)\ll(\log x)^{k-1/3}.$$

Using the method of Theorem 4 we get for $k\ge 2$

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\tau_k((i,n))\right)=\frac{x^2\zeta(2)^{k-1}}{2}+O\left(x(\log x)^{k-4/3}\right). \tag{17}$$

*Remark* 10. Estimate (xi) has first been obtained in [4] and later rediscovered in [2].

# 4 Quadratic fields

Let $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ be a quadratic field of discriminant $D$ and set $d = |D|$. $\chi$ is the primitive Dirichlet character associated to $\mathbb{K}$ so that $\chi(\cdot) = (d/\cdot)$ where $(a/b)$ is a Kronecker symbol. Finally let $L(s,\chi)$ be the $L$-function associated to $\chi$. It is known that for $\sigma > 1$, we have the factorization $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi)$, and hence using estimate (viii) of Theorem 8 we obtain

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right) = \frac{x^2L(2,\chi)}{2} + O\left(x\log x\right).$$

The purpose of this section is to show that the error term can be improved to

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right) = \frac{x^2L(2,\chi)}{2} + O\left(d^{1/2}x(\log x)^{2/3}\right). \tag{18}$$

The identity $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi)$ implies that $\nu_{\mathbb{K}} \star \mu = \chi$ and thus it is easy to verify that $\nu_{\mathbb{K}}$ satisfies hypotheses (6) and (7) of the class $\mathcal{M}_2(\alpha)$ with $\alpha = 0$. We also have $\left|\left(\nu_{\mathbb{K}} \star \mu\right)\left(p^l\right)\right| = \left|\chi\left(p^l\right)\right| \leq 1$ of hypothesis (8). In fact, the only condition which fails is that $p \longmapsto \left(\nu_{\mathbb{K}} \star \mu\right)(p) = \chi(p)$ is not ultimately monotone since $\chi(p) = 1,\ 0,\ -1$ depending on whether $p$ completely splits, is ramified or is inert in $\mathbb{K}$. The following result is a first step into the direction of (18).

**Lemma 11.** *For every real number $x \geq 1$ sufficiently large and every real number $T$ such that $1 \leq T \leq x$, we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right) = \frac{x^2L(2,\chi)}{2} - x\sum_{n\leq T}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) + O\left(x^2T^{-2}d^{1/2}\log d + T\right).$$

*In particular, we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right) = \frac{x^2L(2,\chi)}{2} - x\sum_{n\leq x^{1/2}}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) + O\left(xd^{1/2}\log d\right).$$

*Proof.* Using Dirichlet's hyperbola principle, we get

$$\begin{aligned}
\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right) &= \sum_{n\leq x}\left(\chi\star\mathrm{Id}\right)(n)\\
&= \sum_{n\leq T}\chi(n)\sum_{k\leq x/n}k + \sum_{n\leq x/T}n\sum_{k\leq x/n}\chi(k) - \sum_{n\leq T}\chi(n)\sum_{n\leq x/T}n\\
&= \frac{1}{2}\sum_{n\leq T}\chi(n)\left[\frac{x}{n}\right]\left(\left[\frac{x}{n}\right]+1\right) + O\left\{\sum_{n\leq x/T}n\left|\sum_{k\leq x/n}\chi(k)\right|\right\}\\
&\quad + O\left\{\left(\sum_{n\leq x/T}n\right)\left|\sum_{n\leq T}\chi(n)\right|\right\}
\end{aligned}$$

and the use of the Pólya–Vinogradov inequality and the estimate

$$\left[\frac{x}{n}\right]\left(\left[\frac{x}{n}\right]+1\right)=\frac{x^2}{n^2}-\frac{2x}{n}\psi\left(\frac{x}{n}\right)+O(1)$$

give

$$\sum_{n\le x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right)=\frac{x^2}{2}\sum_{n\le T}\frac{\chi(n)}{n^2}-x\sum_{n\le T}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right)+O\left(x^2T^{-2}d^{1/2}\log d+T\right).$$

We conclude the proof by noticing that

$$\sum_{n\le T}\frac{\chi(n)}{n^2}=L(2,\chi)-\sum_{n>T}\frac{\chi(n)}{n^2}$$

and we get by Abel summation and the Pólya–Vinogradov inequality the estimate

$$\left|\sum_{n>T}\frac{\chi(n)}{n^2}\right|\le\frac{4d^{1/2}\log d}{T^2}$$

giving the asserted result. □

*Remark* 12. The choice of $T=x^{1/2}$ is obviously not the best possible since $T=x^{2/3}$ provides an error-term of the form $O\left(x^{2/3}d^{1/2}\log d\right)$, but it will be sufficient for our purpose.

Using Lemma 11, we can see that (18) follows at once from the estimate

$$\sum_{n\le x^{1/2}}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right)\ll d^{1/2}(\log x)^{2/3}\tag{19}$$

where $\chi$ is any primitive real Dirichlet character of modulus $d\ge 2$. For $x$ large we set $w(x):=\exp\left(c(\log x)^{2/3}\right)$ where $c>0$ is an absolute constant. Since

$$\sum_{n\le w(x)}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right)\ll\sum_{n\le w(x)}\frac{1}{n}\ll(\log x)^{2/3}$$

it is sufficient to prove

$$\sum_{w(x)<n\le x^{1/2}}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right)\ll d^{1/2}(\log x)^{2/3}\tag{20}$$

so that we set $N\ge 1$ to be a large integer satisfying $w(x)<N\le x^{1/2}$ and consider sums of the type

$$S_N(\chi):=\sum_{N<n\le N_1}\chi(n)\psi\left(\frac{x}{n}\right)\tag{21}$$

with $N_1\ge 1$ integer such that $N<N_1\le 2N$.

The proof of (20) uses ideas developed by Walfisz in [15] and exploited by Pétermann [9] and Pétermann & Wu [10]. The first step to estimate (21) is to use an approximation of the function $\psi$ by trigonometric polynomials as it was shown by Vaaler [14].

**Lemma 13.** *For all real number $x \geq 1$ and all integer $H \geq 1$ we have*

$$\psi(x) = -\sum_{0<|h|\leq H} \Phi\left(\frac{h}{H+1}\right)\frac{e(hx)}{2\pi ih} + \mathcal{R}_H(x)$$

*where $\Phi(t) := \pi t\,(1-|t|)\cot(\pi t) + |t|$ for $0 < |t| < 1$ and*

$$|\mathcal{R}_H(x)| \leq \frac{1}{2H+2}\sum_{|h|\leq H}\left(1-\frac{|h|}{H+1}\right)e(hx).$$

*Moreover, we have $0 < \Phi(t) < 1$ for $0 < |t| < 1$.*

Note that

$$\sum_{|h|\leq H}\left(1-\frac{|h|}{H+1}\right)e(hx) = \frac{1}{H+1}\left|\sum_{h=0}^{H} e(hx)\right|^2$$

so that the sum in the error-term is a nonnegative real number. Using this useful tool by multiplying by $\chi(n)$ and summing over $(N, N_1]$ we get

$$S_N(\chi) = -\frac{1}{2\pi i}\sum_{0<|h|\leq H}\Phi\left(\frac{h}{H+1}\right)\frac{1}{h}\sum_{N<n\leq N_1}\chi(n)e\left(\frac{hx}{n}\right) + \sum_{N<n\leq N_1}\chi(n)\mathcal{R}_H\left(\frac{x}{n}\right)$$

with

$$\begin{aligned}\left|\sum_{N<n\leq N_1}\chi(n)\mathcal{R}_H\left(\frac{x}{n}\right)\right| &\leq \sum_{N<n\leq N_1}\left|\mathcal{R}_H\left(\frac{x}{n}\right)\right| \\ &\leq \frac{1}{2H+2}\sum_{|h|\leq H}\left(1-\frac{|h|}{H+1}\right)\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right) \\ &= \frac{N}{2H+2} + \frac{1}{2H+2}\sum_{0<|h|\leq H}\left(1-\frac{|h|}{H+1}\right)\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right) \\ &\leq \frac{N}{2H+2} + \frac{1}{H+1}\sum_{1\leq h\leq H}\left|\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)\right|\end{aligned}$$

so that

$$\begin{aligned}S_N(\chi) = &-\frac{1}{2\pi i}\sum_{0<|h|\leq H}\Phi\left(\frac{h}{H+1}\right)\frac{1}{h}\sum_{N<n\leq N_1}\chi(n)e\left(\frac{hx}{n}\right) \\ &+ O\left\{NH^{-1} + H^{-1}\sum_{1\leq h\leq H}\left|\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)\right|\right\}.\end{aligned}$$

Since $\chi$ is primitive, we can expand it as a linear combination of additive characters using Gauss sums, which gives

$$S_N(\chi) = -\frac{1}{2\pi i\tau(\overline{\chi})}\sum_{a \bmod d}\overline{\chi}(a)\sum_{0<|h|\le H}\Phi\left(\frac{h}{H+1}\right)\frac{1}{h}\sum_{N<n\le N_1}e\left(\frac{hx}{n}+\frac{an}{d}\right)$$
$$+O\left\{NH^{-1}+H^{-1}\sum_{1\le h\le H}\left|\sum_{N<n\le N_1}e\left(\frac{hx}{n}\right)\right|\right\}$$

where $\tau(\overline{\chi})$ is the Gauss sum associated to $\overline{\chi}$. Since $\chi$ is primitive, we have $|\tau(\overline{\chi})| = d^{1/2}$ so that

$$S_N(\chi) \ll NH^{-1}+d^{-1/2}\sum_{a \bmod d}\sum_{1\le h\le H}\frac{1}{h}\left|\sum_{N<n\le N_1}e\left(\frac{hx}{n}+\frac{an}{d}\right)\right|+H^{-1}\sum_{1\le h\le H}\left|\sum_{N<n\le N_1}e\left(\frac{hx}{n}\right)\right|.$$

The second step is given by the following lemma which lies at the heart of Walfisz's method.

**Lemma 14.** *Suppose that $e^{200} \le N < N_1 \le 2N$ and $T \ge N^2$ and let $\alpha \in \mathbb{R}$. Then there exists $c_0 > 0$ such that uniformly in $\alpha$ we have*

$$\sum_{N<n\le N_1}e\left(\frac{T}{n}+\alpha n\right) \ll N\exp\left\{-c_0\frac{(\log N)^3}{(\log TN^{-1})^2}\right\}.$$

*Proof.* Set $G_\alpha(y) := T/y + \alpha y$ for $N \le y \le 2N$. The case $\alpha = 0$ is Lemma 2.5 of [10]. The proof of this result uses a general theorem obtained by Karatsuba (see [6] Theorem 1, [9] Lemma C or [10] Lemma 2.4) which requires conditions on derivatives of orders $\ge 2$ of $G_\alpha$. Since $G_\alpha^{(j)}(y) = G_0^{(j)}(y)$ for all integer $j \ge 2$, we can see that the linear phase $e(\alpha n)$ does not affect Karatsuba's result and thus we can closely follow the proof of Lemma 2.5 of [10] giving the asserted estimate. It should be mentioned that the condition $T \ge N^2$ ensures that Karatsuba's theorem is used with derivatives of $G_\alpha$ having orders $\ge 2$. □

Applying Lemma 14 we obtain with $e^{200} \le N < N_1 \le 2N$ and $N \le x^{1/2}$

$$S_N(\chi) \ll NH^{-1} + Nd^{1/2}\exp\left\{-c_0\frac{(\log N)^3}{(\log HxN^{-1})^2}\right\}\log H$$

and choosing $H = [\exp\{(\log N)^3/(\log x)^2\}]$ gives for $e^{c(\log x)^{2/3}} \le N \le x^{1/2}$

$$S_N(\chi) \ll \frac{Nd^{1/2}(\log N)^3}{(\log x)^2}\exp\left\{-c_1(\log N)^3/(\log x)^2\right\}$$

with some absolute constants $c, c_1 > 0$ depending only on $c_0$ and where we have used the bounds $N \ge e^{c(\log x)^{2/3}}$ and $H \le x^{1/8}$. An application of Abel summation yields

$$\sum_{N<n\le N_1}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) \ll \frac{d^{1/2}(\log N)^3}{(\log x)^2}\exp\left\{-c_1(\log N)^3/(\log x)^2\right\}$$

for $e^{c(\log x)^{2/3}} \leq N \leq x^{1/2}$ and a similar argument to that used in the proof of Lemma 2.3 of [10] finally gives

$$\sum_{w(x)<n\leq x^{1/2}} \frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) \ll d^{1/2}(\log x)^{2/3}$$

which completes the proof of (20) and (19). The following result has thus been proved.

**Theorem 15.** *Let $\mathbb{K}/\mathbb{Q}$ be a quadratic field of discriminant $D$ and let $\chi$ be the quadratic Dirichlet character associated to $\mathbb{K}$. For every real number $x \geq \exp\left((\log|D|)^{3/2}\right)$ sufficiently large, we have*

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right) = \frac{x^2L(2,\chi)}{2} + O\left(|D|^{1/2}x(\log x)^{2/3}\right).$$

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