# The Composition of the gcd and Certain Arithmetic Functions

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**Abstract**

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We provide mean value results for sums of the composition of the gcd and arithmetic functions belonging to certain classes. Some applications are also given.

## 1 Introduction

In what follows, $f : \mathbb{N} \longrightarrow \mathbb{C}$ is an arithmetic function with Dirichlet series $F(s)$ and $(a,b)$ is the gcd of $a$ and $b$. The Dirichlet convolution product $f \star g$ of $f$ and $g$ is defined in the usual way. The classical arithmetic functions $\tau$, $\sigma$, $\mu$, $\varphi$, $\omega$ are respectively the number and sum of divisors, the Mvbius function, the Euler totient function and the number of distinct prime factors. Finally, $\gamma$ is the Euler–Mascheroni constant, $[t]$ is the integer part of $t \in \mathbb{R}$ and we set $\psi(t) := t - [t] - 1/2$.

In 1885, E. Cesáro [3] proved the following identity.

**Lemma 1.** *For every positive integer $n$, we have*

$$\sum_{i=1}^{n} f((i,n)) = (f \star \varphi)(n).$$

This follows from

$$\sum_{i=1}^{n} f((i,n)) = \sum_{d|n} f(d) \sum_{\substack{k \leq n/d \\ (k,n/d)=1}} 1 = \sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right) = (f \star \varphi)(n).$$

It should be mentioned that such an identity also occurs with some other convolution products where the summation is over some subset of the set of the divisors of $n$. For instance, L. Tóth [13] showed that

$$\sum_{i \in \text{Reg}(n)} f((i,n)) = \sum_{\substack{d|n \\ (d,n/d)=1}} f(d)\varphi\left(\frac{n}{d}\right)$$

where the notation $i \in \text{Reg}(n)$ means that $1 \le i \le n$ and there exists an integer $x$ such that $i^2 x \equiv i \pmod{n}$.

Lemma 1 has a lot of interesting applications.

(a) With $f = \text{Id}$ we get

$$\sum_{i=1}^{n}(i,n) = (\text{Id} \star \varphi)(n)$$

which is Pillai's function [11].

(b) With $f = \mu$ we get

$$\sum_{i=1}^{n}\mu((i,n)) = (\mu \star \varphi)(n) \tag{1}$$

and thus the number of primitive Dirichlet characters modulo $n$ is equal to $\sum_{i=1}^{n}\mu((i,n))$. In particular, if $m$ is an odd positive integer then

$$\sum_{i=1}^{2m}\mu((i,2m)) = 0.$$

(c) With $f = \tau$ we have, using $\tau \star \varphi = \sigma$

$$\sum_{i=1}^{n}\tau((i,n)) = \sigma(n) \tag{2}$$

so that

$$\sum_{i=1}^{n}\tau((i,n)) \ll n \log\log n$$

which should be compared to the classical estimate $\sum_{i=1}^{n}\tau(i) \ll n\log n$.

(d) With $f = 2^{\omega}$ we easily get

$$\sum_{i=1}^{n}2^{\omega((i,n))} = \Psi(n) \tag{3}$$

where $\Psi(n) := (\mu^2 \star \text{Id})(n) = n\prod_{p|n}(1+p^{-1})$ is the Dedekind arithmetic function.

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(e) Applying Lemma 1 twice with $f = \tau$ and $f = \sigma$ respectively, and using $\tau \star \varphi = \sigma$ and $\sigma \star \varphi = \mathrm{Id} \times \tau$, we obtain

$$\tau(n) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{(i,n)} \tau((i,j,n)).$$

The aim of this paper is to estimate the sums

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} f((i,n)) \right) \tag{4}$$

for $x \geq 1$ sufficiently large and an arithmetic function $f$ verifying certain hypotheses. In section 2, we provide a result for four classes of multiplicative functions and then give some applications in section 3. The aim of section 4 is to provide a refinement of an estimate given in Theorem 8.

## 2    Main result

This section is devoted to the proof of a unified theorem which gives estimates for sums of the type (4). To this end, we first need some specific notation. More precisely, we consider the four following classes of real-valued multiplicative functions.

1. $f \in \mathcal{M}_1(\alpha)$ if there exists a real number $\alpha \geq 0$ such that

$$\sum_{n \leq x} |(f \star \mu)(n)| \ll x (\log x)^{\alpha}. \tag{5}$$

2. $f \in \mathcal{M}_2(\alpha)$ if there exists a real number $\alpha \in [0, 3/2]$ such that, for every positive integer $m$ we have

$$\sum_{n \leq x} |(f \star \mu)(n)| = x \sum_{i=0}^{[\alpha]+m} A_i (\log x)^{\alpha-i} + O\left( \frac{x}{(\log x)^m} \right) \qquad (A_i \in \mathbb{R}). \tag{6}$$

$$\sum_{n \leq x} ((f \star \mu)(n))^2 \ll x (\log x)^{\beta} \qquad (\beta \geq 0). \tag{7}$$

$$f(p^l) - f(p^{l-1}) \text{ is bounded for all } l \geq 1 \text{ and primes } p. \tag{8}$$

The sequence $p \longmapsto f(p) - 1$ is ultimately monotone. $\tag{9}$

3. $f \in \mathcal{M}_3(A)$ if there exist $A > 0$, $B$, $C$, $D \in \mathbb{R}$ and an integrable function $R$ defined on $[1, +\infty)$ such that

$$\sum_{n \leq x} \frac{f(n)}{n} = Ax + B \log x + C\psi(x) + D + R(x)$$

and $R(x) \ll x^{-a} (\log x)^E$ with $a, E \geq 0$.

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4. $f \in \mathcal{M}_4(A, \alpha, \beta)$ if there exist $A > 0$, $\alpha \geq 1$ and $\alpha > \beta \geq 0$ such that

$$\sum_{n \leq x} \frac{f(n)}{n} = Ax^\alpha + O\left(x^\beta\right).$$

(10)

Finally, we define $0 \leq \theta_f \leq \frac{1}{2}$ and $\Delta_f \geq 0$ such that

$$\sum_{n \leq \sqrt{x}} \frac{f(n)}{n} \psi\left(\frac{x}{n}\right) \ll x^{\theta_f} (\log x)^{\Delta_f}$$

(11)

and we set $\theta := \max\left(\theta_f, \theta_{\mathrm{Id}}\right)$ and $\Delta := \max\left(\Delta_f, \Delta_{\mathrm{Id}}\right)$.

*Remark* 2. It is known [5] that one can take

$$\theta_{\mathrm{Id}} = \frac{131}{416} \approx 0.3149\ldots \quad \text{and} \quad \Delta_{\mathrm{Id}} = \frac{26947}{8320} \approx 3.2388\ldots$$

The following result gives further information when $f \in \mathcal{M}_3(A)$ and $f \in \mathcal{M}_4(A, \alpha, \beta)$.

**Lemma 3.** *Let* $f \in \mathcal{M}_3(A)$. *Then we have for* $x$ *sufficiently large*

$$\sum_{n \leq x} \frac{f(n)}{n^2} = A \log x + A + G - \frac{B}{x} + \frac{C\psi(x)}{x} + O\left(\frac{(\log x)^E}{x^{a+1}} + \frac{1}{x^2}\right)$$

*where*

$$G := B + C\left(\frac{1}{2} - \gamma\right) + D + \int_1^\infty \frac{R(t)\mathrm{d}t}{t^2}.$$

(12)

*Let* $f \in \mathcal{M}_4(A, \alpha, \beta)$. *Then we have for* $x$ *sufficiently large*

$$\sum_{n \leq x} \frac{f(n)}{n^2} = A\alpha E_\alpha(x) + O\left(\mathcal{R}_\beta(x)\right)$$

*where*

$$E_\alpha(x) := \begin{cases} \dfrac{x^{\alpha-1}}{\alpha - 1}, & \text{if } \alpha > 1; \\\\ \log x, & \text{if } \alpha = 1 \end{cases}$$

(13)

*and*

$$\mathcal{R}_\beta(x) := \begin{cases} x^{\beta-1}, & \text{if } \alpha > \beta > 1; \\ \log x, & \text{if } \alpha > \beta = 1; \\ 1, & \text{if } \alpha \geq 1 > \beta \geq 0. \end{cases}$$

(14)

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*Proof.* Let $f \in \mathcal{M}_3(A)$. Using Abel summation, we get

$$\sum_{n \leq x} \frac{f(n)}{n^2} = A + \frac{B \log x}{x} + \frac{C\psi(x)}{x} + \frac{D}{x} + \frac{R(x)}{x}$$

$$+ \int_1^x \frac{1}{t^2} \left( At + B \log t + C\psi(t) + D + R(t) \right) \mathrm{d}t$$

$$= A \log x + A + B + D + \int_1^\infty \frac{R(t)\mathrm{d}t}{t^2} - \frac{B}{x}$$

$$+ C \left( \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)\mathrm{d}t}{t^2} \right) + \frac{R(x)}{x} - \int_x^\infty \frac{R(t)\mathrm{d}t}{t^2}.$$

The estimate

$$\int_1^x \frac{\psi(t)\mathrm{d}t}{t^2} = \frac{1}{2} - \gamma + O\left( \frac{1}{x^2} \right)$$

which can be proven by using Euler-MacLaurin's summation formula, gives

$$\sum_{n \leq x} \frac{f(n)}{n^2} = A \log x + A + G - \frac{B}{x} + \frac{C\psi(x)}{x} + O\left( \frac{(\log x)^E}{x^{a+1}} + \frac{1}{x^2} \right).$$

The proof for $f \in \mathcal{M}_4(A, \alpha, \beta)$ is similar and somewhat simpler, so we omit the details. $\square$

Now we can state our main result.

**Theorem 4.** *Let $f$ be a real-valued multiplicative function with Dirichlet series $F(s)$.*

1. *If $f \in \mathcal{M}_1(\alpha)$, then we have for $x$ sufficiently large*

$$\sum_{n \leq x} \left( \sum_{i=1}^n f((i,n)) \right) = \frac{x^2 F(2)}{2\zeta(2)} + O\left\{ x \prod_{p \leq x} \left( 1 + \sum_{l=1}^\infty \frac{|f(p^l) - f(p^{l-1})|}{p^l} \right) + x(\log x)^\alpha \right\}.$$

2. *If $f \in \mathcal{M}_2(\alpha)$, then we have for $x$ sufficiently large*

$$\sum_{n \leq x} \left( \sum_{i=1}^n f((i,n)) \right) = \frac{x^2 F(2)}{2\zeta(2)} + O\left\{ x(\log x)^{\frac{2}{3}(\alpha+1)} (\log \log x)^{\frac{4}{3}(\alpha+1)} \right\}.$$

3. *If $f \in \mathcal{M}_3(A)$, then we have for $x$ sufficiently large*

$$\sum_{n \leq x} \left( \sum_{i=1}^n f((i,n)) \right) = \frac{Ax^2 \log x}{2\zeta(2)} + \frac{x^2}{2\zeta(2)} \left\{ A\left( \gamma + \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + G \right\} + O\left\{ \left( x^{1+\theta} + x^r \right) (\log x)^\Gamma \right\}$$

*where $\theta$ is given in* (11)*, $G$ is given in* (12) *and*

$$\Gamma := \max(2, \Delta, E+1) \quad and \quad r := \begin{cases} \frac{1}{2}(3-a), & \text{if } 0 \leq a < 1; \\ 0, & \text{if } a \geq 1. \end{cases}$$

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4. If $f \in \mathcal{M}_4(A, \alpha, \beta)$, then we have for $x$ sufficiently large

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} f((i, n)) \right) = \frac{A\alpha x^2 E_\alpha(x)}{2\zeta(\alpha + 1)} + O\left\{ x \prod_{p \leq x} \left( 1 + \sum_{l=1}^{\infty} \frac{\left| f\left(p^l\right) - f\left(p^{l-1}\right) \right|}{p^l} \right) + x^2 \mathcal{R}_\beta(x) \right\}$$

where $E_\alpha(x)$ is given in (13) and $\mathcal{R}_\beta(x)$ is given in (14).

*Proof.* Let $f$ be a real-valued multiplicative function with Dirichlet series $F(s)$.

1. Set $g := f \star \mu$. Since $\varphi = \mu \star \mathrm{Id}$, we have, using Lemma 1

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} f((i, n)) \right) = \sum_{n \leq x} (g \star \mathrm{Id})(n) = \sum_{d \leq x} g(d) \sum_{k \leq x/d} k$$

$$= \frac{1}{2} \sum_{d \leq x} g(d) \left[ \frac{x}{d} \right] \left( \left[ \frac{x}{d} \right] + 1 \right)$$

$$= \frac{1}{2} \sum_{d \leq x} g(d) \left\{ \frac{x^2}{d^2} - \frac{2x}{d} \psi \left( \frac{x}{d} \right) - \left( \frac{1}{4} - \psi \left( \frac{x}{d} \right)^2 \right) \right\}$$

$$= \frac{x^2}{2} \sum_{d \leq x} \frac{g(d)}{d^2} - x \sum_{d \leq x} \frac{g(d)}{d} \psi \left( \frac{x}{d} \right) + O\left( \sum_{d \leq x} |g(d)| \right).$$

Using (5) it is easily seen that the series $\sum_{d \geq 1} g(d) d^{-2}$ is absolutely convergent, and hence we have

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} f((i, n)) \right) = \frac{x^2}{2} \sum_{d=1}^{\infty} \frac{g(d)}{d^2} - x \sum_{d \leq x} \frac{g(d)}{d} \psi \left( \frac{x}{d} \right) + O\left( \sum_{d \leq x} |g(d)| + x^2 \sum_{d > x} \frac{|g(d)|}{d^2} \right)$$

$$= \frac{x^2 F(2)}{2\zeta(2)} - x \sum_{d \leq x} \frac{g(d)}{d} \psi \left( \frac{x}{d} \right) + O\left( x(\log x)^\alpha + x^2 \sum_{d > x} \frac{|g(d)|}{d^2} \right).$$

Now by Abel summation and (5), we get

$$x^2 \sum_{d > x} \frac{|g(d)|}{d^2} = -\sum_{d \leq x} |g(d)| + 2x^2 \int_x^\infty \frac{1}{t^3} \left( \sum_{d \leq t} |g(d)| \right) dt \ll x(\log x)^\alpha$$

and the inequality $|\psi(x/d)| \leq 1/2$ gives

$$\sum_{d \leq x} \frac{g(d)}{d} \psi \left( \frac{x}{d} \right) \ll \sum_{d \leq x} \frac{|g(d)|}{d} \ll \prod_{p \leq x} \left( 1 + \sum_{l=1}^{\infty} \frac{\left| g\left(p^l\right) \right|}{p^l} \right).$$

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2. The proof is the same as before except that we are able to treat the sum $\sum_{d \leq x} \frac{g(d)}{d} \psi\left(\frac{x}{d}\right)$ more efficiently. Using (6), (7), (8) and (9) we see that the function $d \longmapsto g(d)d^{-1}$ satisfies the conditions of Theorem 1 of [9] which gives

$$\sum_{d \leq xe^{-(\log x)^{1/6}}} \frac{g(d)}{d} \psi\left(\frac{x}{d}\right) \ll (\log x)^{\frac{2}{3}(\alpha+1)} (\log\log x)^{\frac{4}{3}(\alpha+1)}$$

and, using (6) and partial summation, we get

$$\sum_{xe^{-(\log x)^{1/6}} < d \leq x} \frac{g(d)}{d} \psi\left(\frac{x}{d}\right) \ll \sum_{xe^{-(\log x)^{1/6}} < d \leq x} \frac{|g(d)|}{d} \ll (\log x)^{\alpha+1/6}.$$

Note that $\alpha \in [0, 3/2]$ implies $(\log x)^{2(\alpha+1)/3} \geq (\log x)^{\alpha+1/6}$.

3. Set $h := f \star \mathrm{Id}$. Using Dirichlet's hyperbola principle, we have

$$\sum_{n \le x} \frac{h(n)}{n} = \sum_{n \le x} \sum_{d \mid n} \frac{f(d)}{d} = \sum_{n \le x} \left( \frac{f}{\mathrm{Id}} \star \mathbf{1} \right)(n)$$

$$= \sum_{n \le \sqrt{x}} \frac{f(n)}{n} \sum_{m \le x/n} 1 + \sum_{n \le \sqrt{x}} \sum_{m \le x/n} \frac{f(m)}{m} - \left[ \sqrt{x} \right] \sum_{n \le \sqrt{x}} \frac{f(n)}{n}$$

$$= \sum_{n \le \sqrt{x}} \frac{f(n)}{n} \left( \frac{x}{n} - \frac{1}{2} - \psi\left( \frac{x}{n} \right) \right)$$

$$+ \sum_{n \le \sqrt{x}} \left\{ \frac{Ax}{n} + B \log \frac{x}{n} + C\psi\left( \frac{x}{n} \right) + D + O\left( \left( \frac{n}{x} \right)^a (\log x)^E \right) \right\}$$

$$- \left( \sqrt{x} - \frac{1}{2} - \psi(\sqrt{x}) \right) \sum_{n \le \sqrt{x}} \frac{f(n)}{n}$$

$$= x \sum_{n \le \sqrt{x}} \frac{f(n)}{n^2} - \sum_{n \le \sqrt{x}} \frac{f(n)}{n} \psi\left( \frac{x}{n} \right) + Ax \sum_{n \le \sqrt{x}} \frac{1}{n} + B \sum_{n \le \sqrt{x}} \log \frac{x}{n}$$

$$+ C \sum_{n \le \sqrt{x}} \psi\left( \frac{x}{n} \right) + D \left( \sqrt{x} - \frac{1}{2} - \psi(\sqrt{x}) \right) + O\left( x^{(1-a)/2} (\log x)^E \right)$$

$$- \left( \sqrt{x} - \psi(\sqrt{x}) \right) \left( A\sqrt{x} + \frac{B}{2} \log x + C\psi(\sqrt{x}) + D + O\left( x^{-a/2} (\log x)^E \right) \right)$$

$$= x \sum_{n \le \sqrt{x}} \frac{f(n)}{n^2} + Ax \left( \frac{\log x}{2} + \gamma - \frac{\psi(\sqrt{x})}{\sqrt{x}} + O\left( x^{-1} \right) \right)$$

$$+ B \left( \frac{\sqrt{x} \log x}{2} + \sqrt{x} + O\left( \log x \right) \right) - \frac{D}{2} - Ax - \frac{B\sqrt{x} \log x}{2}$$

$$+ (A - C)\sqrt{x}\psi(\sqrt{x}) + \frac{B}{2}\psi(\sqrt{x}) \log x + C\psi(\sqrt{x})^2$$

$$+ O\left( x^\theta (\log x)^\Delta + x^{(1-a)/2}(\log x)^E \right)$$

$$= x \sum_{n \le \sqrt{x}} \frac{f(n)}{n^2} + \frac{Ax \log x}{2} + Ax(\gamma - 1) + B\sqrt{x} - C\sqrt{x}\psi(\sqrt{x})$$

$$+ O\left( x^\theta (\log x)^\Delta + x^{(1-a)/2}(\log x)^E + \log x \right).$$

Now by Lemma 3 we get

$$\sum_{n \le x} \frac{h(n)}{n} = x \left\{ \frac{A \log x}{2} + A + G - \frac{B}{\sqrt{x}} + \frac{C\psi(\sqrt{x})}{\sqrt{x}} + O\left( \frac{(\log x)^E}{x^{(a+1)/2}} + \frac{1}{x} \right) \right\}$$

$$+ \frac{Ax \log x}{2} + Ax(\gamma - 1) + B\sqrt{x} - C\sqrt{x}\psi(\sqrt{x})$$

$$+ O\left( x^\theta (\log x)^\Delta + x^{(1-a)/2}(\log x)^E + \log x \right)$$

$$= Ax \log x + (A\gamma + G)x + O\left( x^\theta (\log x)^\Delta + x^{(1-a)/2}(\log x)^E + \log x \right).$$

An Abel summation then gives

$$\sum_{n \leq x} h(n) = \frac{Ax^2 \log x}{2} + \frac{x^2}{4} \left(A(2\gamma + 1) + 2G\right) + O\left(x^{1+\theta}(\log x)^\Delta + x^{(3-a)/2}(\log x)^E + x \log x\right)$$

and hence

$$\sum_{n \leq x} \left(\sum_{i=1}^n f((i,n))\right) = \sum_{d \leq x} \mu(d) \sum_{k \leq x/d} h(k)$$

$$= \frac{Ax^2}{2} \sum_{d \leq x} \frac{\mu(d)}{d^2} \log \frac{x}{d} + \frac{x^2}{4} \left(A(2\gamma + 1) + 2G\right) \sum_{d \leq x} \frac{\mu(d)}{d^2}$$

$$+ O\left(x^{1+\theta}(\log x)^\Delta + x^{(3-a)/2}(\log x)^E \sum_{d \leq x} \frac{1}{d^{(3-a)/2}} + x(\log x)^2\right)$$

$$= \frac{Ax^2 \log x}{2\zeta(2)} - \frac{A\zeta'(2)}{2\zeta(2)^2}x^2 + \frac{x^2}{4\zeta(2)} \left(A(2\gamma + 1) + 2G\right)$$

$$+ O\left(\left(x^{1+\theta} + x^r\right)(\log x)^\Gamma\right)$$

which is the asserted result.

4. The proof is similar to the points 1 and 2. We use $g := f \star \mu$ and we have as above

$$\sum_{n \leq x} \left(\sum_{i=1}^n f((i,n))\right) = \sum_{d \leq x} g(d) \sum_{k \leq x/d} k$$

$$= \frac{1}{2} \sum_{d \leq x} g(d) \left[\frac{x}{d}\right] \left(\left[\frac{x}{d}\right] + 1\right)$$

$$= \frac{x^2}{2} \sum_{d \leq x} \frac{g(d)}{d^2} + O\left(x \sum_{d \leq x} \frac{|g(d)|}{d}\right)$$

$$= \frac{x^2}{2} \sum_{d \leq x} \frac{\mu(d)}{d^2} \sum_{k \leq x/d} \frac{f(k)}{k^2} + O\left(x \sum_{d \leq x} \frac{|g(d)|}{d}\right)$$

and using Lemma 3 gives the desired result.

The proof of Theorem 4 is complete. □

# 3 Applications

We first introduce some additional notation. The functions $\mu, \tau, \sigma, \ \varphi,$ Id and $\mathbf{1}$ have their usual meanings and we add the following multiplicative functions.

- $\beta(n)$ is the number of square-full divisors of $n$.

- $a(n)$ is the number of non-isomorphic abelian groups of order $n$.

- $\tau^{(e)}(n)$ and $\sigma^{(e)}(n)$ are respectively the number and the sum of exponential divisors of $n$.

- If $k \geq 2$ is any fixed integer, $\mu_k$ is the characteristic function of the set of $k$-free integers, $\tau_k$ is the $k$-th Piltz divisor function defined by $\tau_k = \underbrace{\mathbf{1} \star \cdots \star \mathbf{1}}_{k \text{ times}}$ with $\tau_2 = \tau$, $\tau_{(k)}(n)$ is the number of $k$-free divisors of $n$ with $\tau_{(2)} = 2^\omega$ and $\gamma_k(n)$ is the greatest $k$-free divisor of $n$.

- If $\mathbb{K}/\mathbb{Q}$ is any fixed number field of degree $d \geq 2$, $\nu_{\mathbb{K}}(n)$ is the number of nonzero integral ideals of norm $n$. The Dedekind zeta-function of $\mathbb{K}$ is denoted by $\zeta_{\mathbb{K}}$.

The following lemma gives the distribution of these functions into the classes $\mathcal{M}_i$ .

**Lemma 5.** *Let $k \geq 2$ be a fixed integer. We have the following distribution.*

| $\mathcal{M}_1(\alpha)$ | $\mathcal{M}_2(\alpha)$ | $\mathcal{M}_3(A)$ | $\mathcal{M}_4(A, \alpha, \beta)$ |
|---|---|---|---|
| $\beta \in \mathcal{M}_1(0)$ | $\tau \in \mathcal{M}_2(0)$ | $\varphi \in \mathcal{M}_3\left(\zeta(2)^{-1}\right)$ | $\gamma_k \in \mathcal{M}_4\left(A_k, 1, \frac{1}{k}\right)$ |
| $\tau^{(e)} \in \mathcal{M}_1(0)$ | $\tau_{(k)} \in \mathcal{M}_2(0)$ | $\sigma \in \mathcal{M}_3(\zeta(2))$ | |
| $\mu_k \in \mathcal{M}_1(0)$ | $\mu \in \mathcal{M}_2(1)$ | $\sigma^{(e)} \in \mathcal{M}_3(2\kappa)$ | |
| $a \in \mathcal{M}_1(0)$ | | | |

*where $\kappa \approx 0.568$ and*

$$A_k := \prod_p \left(1 - \frac{1}{p^{k-1}(p+1)}\right). \tag{15}$$

*Proof.* In the sequel, $P(n)$ is the number of unrestricted partitions of $n$.

1. For the class $\mathcal{M}_1(\alpha)$, use

$$(\beta \star \mu)(n) = \begin{cases} 1, & \text{if } n \text{ is square-full}; \\ 0, & \text{otherwise} \end{cases}$$

$$(\mu_k \star \mu)(n) = \begin{cases} (-1)^{\omega(m)}, & \text{if } n = m^k \text{ and } \mu_2(m) = 1; \\ 0, & \text{otherwise} \end{cases}$$

and for the function $a$ we have $|(a \star \mu)(p)| = P(1) - 1 = 0$ and for all integers $l \geq 2$ we have $|(a \star \mu)(p^l)| = |P(l) - P(l-1)| < 2 \times 5^{l/4}$ (see [7] for instance) so that the function $|a \star \mu|$ satisfies Wirsing's conditions (i.e. $0 \leq f(p^l) \leq \lambda_1 \lambda_2^l$ for some real numbers $\lambda_1 > 0$ and $0 \leq \lambda_2 < 2$) and hence

$$\sum_{n \leq x} |(a \star \mu)(n)| \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|(a \star \mu)(p)|}{p}\right) \ll \frac{x}{\log x}.$$

For the function $\tau^{(e)}$ we use $(\tau^{(e)} \star \mu)(p) = \tau(1) - 1 = 0$ and for all integer $l \geq 2$ we have $(\tau^{(e)} \star \mu)(p^l) = \tau(l) - \tau(l-1)$ (see [16]) so that

$$\sum_{n \leq x} |(\tau^{(e)} \star \mu)(n)| \ll \frac{x}{\log x}.$$

2. For the class $\mathcal{M}_2(\alpha)$, we use the fact that $\tau \star \mu = \mathbf{1}$ and $\tau_{(k)} \star \mu = \mu_k$ which proves the result for $\tau$ and $\tau_{(k)}$. The function $\mu$ needs more work. First we have

$$(\mu \star \mu)(n) = \begin{cases} (-2)^{\omega(a)}, & \text{if } n = ab^2 \text{ with } (a,b) = 1 \text{ and } \mu_2(a) = \mu_2(b) = 1; \\ 0, & \text{otherwise} \end{cases}$$

so that the conditions (7), (8) and (9) are easily checked. We now prove the following identity

$$\sum_{n \leq x} |(\mu \star \mu)(n)| = A_0 x \log x + A_1 x + O\left(x^{1/2}(\log x)^3\right) \tag{16}$$

where $A_0 = \prod_p (1 - 2p^{-2} + p^{-4}) \approx 0.3695\ldots$ and $A_1 = A_0\left(2\gamma - 1 + 4\sum_p \frac{\log p}{p^2 - 1}\right)$ which implies condition (6).

To do this we first set $f(n) := |(\mu \star \mu)(n)|$ which is multiplicative with Dirichlet series $F(s) = \zeta(s)^2 H(s)$, where $H(s) := \prod_p (1 - 2p^{-2s} + p^{-4s})$ is absolutely convergent in the half-plane $\sigma > 1/2$. Moreover, if we set $H(s) := \sum_{n=1}^{\infty} h(n)n^{-s}$, then we have from the Euler product

$$h(n) = \begin{cases} (-2)^{\omega(a)}, & \text{if } n = a^2 b^4 \text{ with } (a,b) = 1 \text{ and } \mu_2(a) = \mu_2(b) = 1; \\ 0, & \text{otherwise} \end{cases}$$

so that

$$\sum_{n \leq x} \frac{|h(n)|}{n^{1/2}} \leq \sum_{a \leq x^{1/2}} \frac{2^{\omega(a)}}{a} \sum_{b \leq (x/a^2)^{1/4}} \frac{1}{b^2} \ll (\log x)^2.$$

Now we are able to show (16). From the factorization $F(s) = \zeta(s)^2 H(s)$, we infer that

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} (\tau \star h)(n) = \sum_{d \leq x} h(d) \sum_{k \leq x/d} \tau(k)$$

$$= \sum_{d \leq x} h(d) \left\{ \frac{x}{d} \log \frac{x}{d} + \frac{x}{d}(2\gamma - 1) + O\left(\left(\frac{x}{d}\right)^{1/2}\right) \right\}$$

$$= x(\log x + 2\gamma - 1) \sum_{d \leq x} \frac{h(d)}{d} - x \sum_{d \leq x} \frac{h(d) \log d}{d} + O\left\{ x^{1/2} \sum_{d \leq x} \frac{|h(d)|}{d^{1/2}} \right\}$$

$$= H(1)x \log x + x \left\{ H(1)(2\gamma - 1) + H'(1) \right\} + O\left( x^{1/2} (\log x)^2 \right)$$

$$+ O\left( x \log x \sum_{d > x} \frac{|h(d)|}{d} + x \sum_{d > x} \frac{|h(d)| \log d}{d} \right)$$

and we conclude the proof by using Abel summation to get

$$\sum_{d > x} \frac{|h(d)|}{d} \ll x^{-1/2} (\log x)^2 \quad \text{and} \quad \sum_{d > x} \frac{|h(d)| \log d}{d} \ll x^{-1/2} (\log x)^3.$$

3. For the class $\mathcal{M}_3(A)$, we first have the well known estimates

$$\sum_{n \leq x} \frac{\varphi(n)}{n} = \frac{x}{\zeta(2)} + O\left( (\log x)^{2/3} (\log \log x)^{4/3} \right).$$

$$\sum_{n \leq x} \frac{\sigma(n)}{n} = x\zeta(2) - \frac{\log x}{2} + O\left( (\log x)^{2/3} \right).$$

For the function $\sigma^{(e)}$ one can deduce from the results proven in [9, 10] that

$$\sum_{n \leq x} \frac{\sigma^{(e)}(n)}{n} = 2\kappa x + O\left( (\log x)^{5/3} \right).$$

4. For the class $\mathcal{M}_4(A, \alpha, \beta)$, use (see [12])

$$\sum_{n \leq x} \frac{\gamma_k(n)}{n} = A_k x + O\left( x^{1/k} \right)$$

where $A_k$ is given in (15).

The proof is complete. $\square$

For the function $\nu_{\mathbb{K}}$, we have the following result in the case of Galois extensions.

**Lemma 6.** *Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of degree $d \geq 2$. Then, for every positive integer $n$, we have*

$$|(\nu_\mathbb{K} \star \mu)(n)| \leq \tau_{d-1}(n)$$

*so that $\nu_\mathbb{K} \in \mathcal{M}_1(d-2)$.*

*Proof.* The result is obvious for $n = 1$ and, by multiplicativity, it suffices to prove the inequality for prime powers. Let $p$ be any prime number and $l \geq 1$ be any integer. Since $\mathbb{K}/\mathbb{Q}$ is Galois, all prime ideals above $p$ have the same residual degree denoted by $f_p$ and we set $g_p$ to be the number of those prime ideals. If $f_p = 1$ then $\nu_\mathbb{K}(p^m) = \tau_{g_p}(p^m)$ for all integer $m \geq 0$ so that

$$(\nu_\mathbb{K} \star \mu)\left(p^l\right) = \nu_\mathbb{K}\left(p^l\right) - \nu_\mathbb{K}\left(p^{l-1}\right) = \tau_{g_p}\left(p^l\right) - \tau_{g_p}\left(p^{l-1}\right) = \left(\tau_{g_p} \star \mu\right)\left(p^l\right) = \tau_{g_p-1}\left(p^l\right).$$

If $f_p \geq 2$ and since $\mathbb{K}/\mathbb{Q}$ is Galois, we have

$$\nu_\mathbb{K}\left(p^l\right) = \begin{cases} \dbinom{g_p + l/f_p - 1}{l/f_p}, & \text{if } f_p \mid l; \\ \\ 0, & \text{otherwise;} \end{cases}$$

so that

$$(\nu_\mathbb{K} \star \mu)\left(p^l\right) = \begin{cases} -1, & \text{if } l = 1; \\ \\ \dbinom{g_p + l/f_p - 1}{l/f_p}, & \text{if } l \geq 2 \text{ and } f_p \mid l \text{ and } f_p \nmid (l-1); \\ \\ -\dbinom{g_p + (l-1)/f_p - 1}{(l-1)/f_p}, & \text{if } l \geq 2 \text{ and } f_p \nmid l \text{ and } f_p \mid (l-1); \\ \\ 0, & \text{otherwise.} \end{cases}$$

We have the trivial inequality $g_p \leq d$ and if $l, f_p \geq 2$ then $l/f_p \leq l/2 \leq l-1$ so that, using the fact that if $g \geq 1$ and $0 \leq x \leq y$ then $\binom{g+x-1}{x} \leq \binom{g+y-1}{y}$, we have in every case

$$\left|(\nu_\mathbb{K} \star \mu)\left(p^l\right)\right| \leq \binom{g_p + l - 2}{l} \leq \binom{d + l - 2}{l} = \tau_{d-1}\left(p^l\right)$$

which concludes the proof. $\qquad \square$

*Remark* 7. One can also have the same inequality in some nonnormal cases. For instance, let $\mathbb{K}_3$ be a cubic field with negative discriminant, so that $\mathbb{K}_3$ is not Galois. The factorization of prime numbers into prime ideals is nevertheless well known and one can prove [1] that we have in fact the following situation.

| Factorization of $(p)$ | $l$ | $(\nu_{\mathbb{K}_3} \star \mu)\left(p^l\right)$ |
|---|---|---|
| Completely split | any | $l+1$ |
| inert | $l \equiv 0 \pmod{3}$ | $1$ |
| inert | $l \equiv 1 \pmod{3}$ | $-1$ |
| inert | $l \equiv -1 \pmod{3}$ | $0$ |
| split | $l \equiv 0 \pmod{2}$ | $1$ |
| split | $l \equiv 1 \pmod{2}$ | $0$ |
| ramified | any | $1$ |
| Completely ramified | any | $0$ |

so that we also have in this case $\left|(\nu_{\mathbb{K}_3} \star \mu)\left(p^l\right)\right| \leq \tau\left(p^l\right)$ and hence $\nu_{\mathbb{K}_3} \in \mathcal{M}_1(1)$.

Now collecting all those results with Theorem 4 we obtain the following estimates.

**Theorem 8.** *For $x$ sufficiently large, we have*

*(i)*
$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \beta((i,n))\right) = \frac{x^2 \zeta(4)\zeta(6)}{2\zeta(12)} + O(x).$$

*(ii)*
$$\sum_{n \leq x}\left(\sum_{i=1}^{n} a((i,n))\right) = \frac{x^2}{2} \prod_{k=2}^{\infty} \zeta(2k) + O(x).$$

*(iii) Let $k \geq 2$ be a fixed integer. Then we have*
$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \mu_k((i,n))\right) = \frac{x^2}{2\zeta(2k)} + O(x).$$

*(iv) If $\widetilde{\tau}(l) := \tau(l) - \tau(l-1) - \tau(l-2) + \tau(l-3)$ for $l \geq 5$ then we have*
$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \tau^{(e)}((i,n))\right) = \frac{x^2 \zeta(4)}{2} \prod_{p}\left(1 + \sum_{l=5}^{\infty} \frac{\widetilde{\tau}(l)}{p^{2l}}\right) + O(x).$$

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(v) Let $\Psi$ be the Dedekind arithmetical function. Then we have

$$\sum_{n \leq x} \Psi(n) = \frac{x^2 \zeta(2)}{2\zeta(4)} + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

More generally, let $k \geq 2$ be a fixed integer. Then we have

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \tau_{(k)}((i,n))\right) = \frac{x^2 \zeta(2)}{2\zeta(2k)} + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

(vi)

$$\sum_{n \leq x} \sigma(n) = \frac{x^2 \zeta(2)}{2} + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).$$

(vii) Let $\mathcal{N}(n)$ be the number of primitive Dirichlet characters modulo $n$. Then we have

$$\sum_{n \leq x} \mathcal{N}(n) = \frac{x^2}{2\zeta(2)^2} + O\left(x(\log x)^{4/3}(\log\log x)^{8/3}\right).$$

(viii) Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of degree $d \geq 2$. Then we have

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \nu_{\mathbb{K}}((i,n))\right) = \frac{x^2 \zeta_{\mathbb{K}}(2)}{2\zeta(2)} + O\left(x(\log x)^{d-1}\right).$$

(ix) Let $\mathbb{K}_3/\mathbb{Q}$ be a cubic field with negative discriminant. Let $\mathbb{K}_6/\mathbb{Q}$ be a normal closure of $\mathbb{K}_3$ and $L(s,\psi,\mathbb{K}_6/\mathbb{Q})$ be the Artin L-function associated to the character $\psi$ of a two-dimensional finite representation of $\mathrm{Gal}\,(\mathbb{K}_6/\mathbb{Q}) \simeq \mathcal{S}_3$. Then we have

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \nu_{\mathbb{K}_3}((i,n))\right) = \frac{x^2 L(2,\psi,\mathbb{K}_6/\mathbb{Q})}{2} + O\left(x(\log x)^2\right).$$

(x)

$$\sum_{n \leq x}\left(\sum_{i=1}^{n} \varphi((i,n))\right) = \frac{x^2 \log x}{2\zeta(2)^2} + \frac{x^2}{2\zeta(2)^2}\left(\gamma + \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} + C_\varphi\right) + O\left(x^{3/2}(\log x)^{26947/8320}\right)$$

where $C_\varphi \in \mathbb{R}$.

(xi)

$$\sum_{n \leq x}\left(\sum_{i=1}^{n}(i,n)\right) = \frac{x^2 \log x}{2\zeta(2)} + \frac{x^2}{2\zeta(2)}\left(2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + O\left(x^{547/416}(\log x)^{26947/8320}\right).$$

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*(xii)*

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \sigma^{(e)}((i,n)) \right) = \frac{\kappa x^2 \log x}{\zeta(2)} + \frac{x^2}{\zeta(2)} \left\{ \kappa \left( \gamma + \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + C_{\sigma^{(e)}} \right\} + O\left( x^{3/2} (\log x)^{26947/8320} \right)$$

where $\kappa \approx 0.568$ and $C_{\sigma^{(e)}} \in \mathbb{R}$.

*(xiii)*

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \sigma((i,n)) \right) = \frac{x^2 \log x}{2} + \frac{x^2}{2} \left( \gamma + \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} + C_{\sigma} \right) + O\left( x^{3/2} (\log x)^{26947/8320} \right)$$

where $C_{\sigma} \in \mathbb{R}$.

*(xiv)* Let $k \geq 2$ be a fixed integer. Then we have

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \gamma_k((i,n)) \right) = \frac{A_k x^2 \log x}{2\zeta(2)} + O\left( x^2 \right)$$

where $A_k$ is given in (15).

*Remark* 9. The first estimate in (v) and estimate (vi) are slightly weaker than the result obtained by Walfisz [15] who proved that one can remove the factor $(\log \log x)^{4/3}$. This is due to the use of the Walfisz–Pétermann's result in its whole generality which does not take account of these particular cases. For instance with (vi), we have here $\tau \star \mu = \mathbf{1}$ and Walfisz showed precisely that (see also section 4)

$$\sum_{n \leq x} \frac{\mathbf{1}(n)}{n} \psi \left( \frac{x}{n} \right) = \sum_{n \leq x} \frac{1}{n} \psi \left( \frac{x}{n} \right) \ll (\log x)^{2/3}.$$

Using this estimate gives Walfisz's result. More generally, it can be shown by induction that, for every integer $k \geq 1$, we have (see [8] for instance)

$$\sum_{n \leq x} \frac{\tau_k(n)}{n} \psi \left( \frac{x}{n} \right) \ll (\log x)^{k-1/3}.$$

Using the method of Theorem 4 we get for $k \geq 2$

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \tau_k((i,n)) \right) = \frac{x^2 \zeta(2)^{k-1}}{2} + O\left( x(\log x)^{k-4/3} \right). \tag{17}$$

*Remark* 10. Estimate (xi) has first been obtained in [4] and later rediscovered in [2].

# 4  Quadratic fields

Let $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ be a quadratic field of discriminant $D$ and set $d = |D|$. $\chi$ is the primitive Dirichlet character associated to $\mathbb{K}$ so that $\chi(\cdot) = (d/\cdot)$ where $(a/b)$ is a Kronecker symbol. Finally let $L(s, \chi)$ be the $L$-function associated to $\chi$. It is known that for $\sigma > 1$, we have the factorization $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s, \chi)$, and hence using estimate (viii) of Theorem 8 we obtain

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \nu_{\mathbb{K}}((i, n)) \right) = \frac{x^2 L(2, \chi)}{2} + O\left( x \log x \right).$$

The purpose of this section is to show that the error term can be improved to

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \nu_{\mathbb{K}}((i, n)) \right) = \frac{x^2 L(2, \chi)}{2} + O\left( d^{1/2} x (\log x)^{2/3} \right). \tag{18}$$

The identity $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s, \chi)$ implies that $\nu_{\mathbb{K}} \star \mu = \chi$ and thus it is easy to verify that $\nu_{\mathbb{K}}$ satisfies hypotheses (6) and (7) of the class $\mathcal{M}_2(\alpha)$ with $\alpha = 0$. We also have $\left| (\nu_{\mathbb{K}} \star \mu) \left( p^l \right) \right| = \left| \chi \left( p^l \right) \right| \leq 1$ of hypothesis (8). In fact, the only condition which fails is that $p \longmapsto (\nu_{\mathbb{K}} \star \mu) (p) = \chi(p)$ is not ultimately monotone since $\chi(p) = 1,\ 0,\ -1$ depending on whether $p$ completely splits, is ramified or is inert in $\mathbb{K}$. The following result is a first step into the direction of (18).

**Lemma 11.** *For every real number $x \geq 1$ sufficiently large and every real number $T$ such that $1 \leq T \leq x$, we have*

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \nu_{\mathbb{K}}((i, n)) \right) = \frac{x^2 L(2, \chi)}{2} - x \sum_{n \leq T} \frac{\chi(n)}{n} \psi \left( \frac{x}{n} \right) + O\left( x^2 T^{-2} d^{1/2} \log d + T \right).$$

*In particular, we have*

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \nu_{\mathbb{K}}((i, n)) \right) = \frac{x^2 L(2, \chi)}{2} - x \sum_{n \leq x^{1/2}} \frac{\chi(n)}{n} \psi \left( \frac{x}{n} \right) + O\left( x d^{1/2} \log d \right).$$

*Proof.* Using Dirichlet's hyperbola principle, we get

$$\sum_{n \leq x} \left( \sum_{i=1}^{n} \nu_{\mathbb{K}}((i, n)) \right) = \sum_{n \leq x} (\chi \star \mathrm{Id}) (n)$$

$$= \sum_{n \leq T} \chi(n) \sum_{k \leq x/n} k + \sum_{n \leq x/T} n \sum_{k \leq x/n} \chi(k) - \sum_{n \leq T} \chi(n) \sum_{n \leq x/T} n$$

$$= \frac{1}{2} \sum_{n \leq T} \chi(n) \left[ \frac{x}{n} \right] \left( \left[ \frac{x}{n} \right] + 1 \right) + O\left\{ \sum_{n \leq x/T} n \left| \sum_{k \leq x/n} \chi(k) \right| \right\}$$

$$+ O\left\{ \left( \sum_{n \leq x/T} n \right) \left| \sum_{n \leq T} \chi(n) \right| \right\}$$

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and the use of the Pólya–Vinogradov inequality and the estimate

$$\left[\frac{x}{n}\right]\left(\left[\frac{x}{n}\right]+1\right) = \frac{x^2}{n^2} - \frac{2x}{n}\psi\left(\frac{x}{n}\right) + O(1)$$

give

$$\sum_{n\leq x}\left(\sum_{i=1}^{n}\nu_{\mathbb{K}}((i,n))\right) = \frac{x^2}{2}\sum_{n\leq T}\frac{\chi(n)}{n^2} - x\sum_{n\leq T}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) + O\left(x^2 T^{-2}d^{1/2}\log d + T\right).$$

We conclude the proof by noticing that

$$\sum_{n\leq T}\frac{\chi(n)}{n^2} = L(2,\chi) - \sum_{n>T}\frac{\chi(n)}{n^2}$$

and we get by Abel summation and the Pólya–Vinogradov inequality the estimate

$$\left|\sum_{n>T}\frac{\chi(n)}{n^2}\right| \leq \frac{4d^{1/2}\log d}{T^2}$$

giving the asserted result. $\qquad\square$

*Remark* 12. The choice of $T = x^{1/2}$ is obviously not the best possible since $T = x^{2/3}$ provides an error-term of the form $O\left(x^{2/3}d^{1/2}\log d\right)$, but it will be sufficient for our purpose.

Using Lemma 11, we can see that (18) follows at once from the estimate

$$\sum_{n\leq x^{1/2}}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) \ll d^{1/2}(\log x)^{2/3} \tag{19}$$

where $\chi$ is any primitive real Dirichlet character of modulus $d \geq 2$. For $x$ large we set $w(x) := \exp\left(c(\log x)^{2/3}\right)$ where $c > 0$ is an absolute constant. Since

$$\sum_{n\leq w(x)}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) \ll \sum_{n\leq w(x)}\frac{1}{n} \ll (\log x)^{2/3}$$

it is sufficient to prove

$$\sum_{w(x)<n\leq x^{1/2}}\frac{\chi(n)}{n}\psi\left(\frac{x}{n}\right) \ll d^{1/2}(\log x)^{2/3} \tag{20}$$

so that we set $N \geq 1$ to be a large integer satisfying $w(x) < N \leq x^{1/2}$ and consider sums of the type

$$S_N(\chi) := \sum_{N<n\leq N_1}\chi(n)\psi\left(\frac{x}{n}\right) \tag{21}$$

with $N_1 \geq 1$ integer such that $N < N_1 \leq 2N$.

The proof of (20) uses ideas developed by Walfisz in [15] and exploited by Pétermann [9] and Pétermann & Wu [10]. The first step to estimate (21) is to use an approximation of the function $\psi$ by trigonometric polynomials as it was shown by Vaaler [14].

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**Lemma 13.** *For all real number $x \geq 1$ and all integer $H \geq 1$ we have*

$$\psi(x) = -\sum_{0<|h|\leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(hx)}{2\pi i h} + \mathcal{R}_H(x)$$

*where $\Phi(t) := \pi t\, (1 - |t|) \cot(\pi t) + |t|$ for $0 < |t| < 1$ and*

$$|\mathcal{R}_H(x)| \leq \frac{1}{2H+2} \sum_{|h|\leq H} \left(1 - \frac{|h|}{H+1}\right) e(hx).$$

*Moreover, we have $0 < \Phi(t) < 1$ for $0 < |t| < 1$.*

Note that

$$\sum_{|h|\leq H} \left(1 - \frac{|h|}{H+1}\right) e(hx) = \frac{1}{H+1} \left|\sum_{h=0}^{H} e(hx)\right|^2$$

so that the sum in the error-term is a nonnegative real number. Using this useful tool by multiplying by $\chi(n)$ and summing over $(N, N_1]$ we get

$$S_N(\chi) = -\frac{1}{2\pi i} \sum_{0<|h|\leq H} \Phi\left(\frac{h}{H+1}\right) \frac{1}{h} \sum_{N<n\leq N_1} \chi(n) e\left(\frac{hx}{n}\right) + \sum_{N<n\leq N_1} \chi(n) \mathcal{R}_H\left(\frac{x}{n}\right)$$

with

$$\left|\sum_{N<n\leq N_1} \chi(n) \mathcal{R}_H\left(\frac{x}{n}\right)\right| \leq \sum_{N<n\leq N_1} \left|\mathcal{R}_H\left(\frac{x}{n}\right)\right|$$

$$\leq \frac{1}{2H+2} \sum_{|h|\leq H} \left(1 - \frac{|h|}{H+1}\right) \sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)$$

$$= \frac{N}{2H+2} + \frac{1}{2H+2} \sum_{0<|h|\leq H} \left(1 - \frac{|h|}{H+1}\right) \sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)$$

$$\leq \frac{N}{2H+2} + \frac{1}{H+1} \sum_{1\leq h\leq H} \left|\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)\right|$$

so that

$$S_N(\chi) = -\frac{1}{2\pi i} \sum_{0<|h|\leq H} \Phi\left(\frac{h}{H+1}\right) \frac{1}{h} \sum_{N<n\leq N_1} \chi(n) e\left(\frac{hx}{n}\right)$$

$$+ O\left\{NH^{-1} + H^{-1} \sum_{1\leq h\leq H} \left|\sum_{N<n\leq N_1} e\left(\frac{hx}{n}\right)\right|\right\}.$$

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Since $\chi$ is primitive, we can expand it as a linear combination of additive characters using Gauss sums, which gives

$$S_N(\chi) = -\frac{1}{2\pi i \tau(\overline{\chi})} \sum_{a \bmod d} \overline{\chi}(a) \sum_{0 < |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{1}{h} \sum_{N < n \leq N_1} e\left(\frac{hx}{n} + \frac{an}{d}\right)$$

$$+ O\left\{ NH^{-1} + H^{-1} \sum_{1 \leq h \leq H} \left| \sum_{N < n \leq N_1} e\left(\frac{hx}{n}\right) \right| \right\}$$

where $\tau(\overline{\chi})$ is the Gauss sum associated to $\overline{\chi}$. Since $\chi$ is primitive, we have $|\tau(\overline{\chi})| = d^{1/2}$ so that

$$S_N(\chi) \ll NH^{-1} + d^{-1/2} \sum_{a \bmod d} \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{N < n \leq N_1} e\left(\frac{hx}{n} + \frac{an}{d}\right) \right| + H^{-1} \sum_{1 \leq h \leq H} \left| \sum_{N < n \leq N_1} e\left(\frac{hx}{n}\right) \right|.$$

The second step is given by the following lemma which lies at the heart of Walfisz's method.

**Lemma 14.** *Suppose that $e^{200} \leq N < N_1 \leq 2N$ and $T \geq N^2$ and let $\alpha \in \mathbb{R}$. Then there exists $c_0 > 0$ such that uniformly in $\alpha$ we have*

$$\sum_{N < n \leq N_1} e\left(\frac{T}{n} + \alpha n\right) \ll N \exp\left\{ -c_0 \frac{(\log N)^3}{(\log TN^{-1})^2} \right\}.$$

*Proof.* Set $G_\alpha(y) := T/y + \alpha y$ for $N \leq y \leq 2N$. The case $\alpha = 0$ is Lemma 2.5 of [10]. The proof of this result uses a general theorem obtained by Karatsuba (see [6] Theorem 1, [9] Lemma C or [10] Lemma 2.4) which requires conditions on derivatives of orders $\geq 2$ of $G_\alpha$. Since $G_\alpha^{(j)}(y) = G_0^{(j)}(y)$ for all integer $j \geq 2$, we can see that the linear phase $e(\alpha n)$ does not affect Karatsuba's result and thus we can closely follow the proof of Lemma 2.5 of [10] giving the asserted estimate. It should be mentioned that the condition $T \geq N^2$ ensures that Karatsuba's theorem is used with derivatives of $G_\alpha$ having orders $\geq 2$. $\square$

Applying Lemma 14 we obtain with $e^{200} \leq N < N_1 \leq 2N$ and $N \leq x^{1/2}$

$$S_N(\chi) \ll NH^{-1} + Nd^{1/2} \exp\left\{ -c_0 \frac{(\log N)^3}{(\log HxN^{-1})^2} \right\} \log H$$

and choosing $H = [\exp\{(\log N)^3/(\log x)^2\}]$ gives for $e^{c(\log x)^{2/3}} \leq N \leq x^{1/2}$

$$S_N(\chi) \ll \frac{Nd^{1/2}(\log N)^3}{(\log x)^2} \exp\left\{ -c_1(\log N)^3/(\log x)^2 \right\}$$

with some absolute constants $c, c_1 > 0$ depending only on $c_0$ and where we have used the bounds $N \geq e^{c(\log x)^{2/3}}$ and $H \leq x^{1/8}$. An application of Abel summation yields

$$\sum_{N < n \leq N_1} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) \ll \frac{d^{1/2}(\log N)^3}{(\log x)^2} \exp\left\{ -c_1(\log N)^3/(\log x)^2 \right\}$$

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for $e^{c(\log x)^{2/3}} \leq N \leq x^{1/2}$ and a similar argument to that used in the proof of Lemma 2.3 of [10] finally gives

$$\sum_{w(x)<n\leq x^{1/2}} \frac{\chi(n)}{n} \psi\left(\frac{x}{n}\right) \ll d^{1/2}(\log x)^{2/3}$$

which completes the proof of (20) and (19). The following result has thus been proved.

**Theorem 15.** *Let $\mathbb{K}/\mathbb{Q}$ be a quadratic field of discriminant $D$ and let $\chi$ be the quadratic Dirichlet character associated to $\mathbb{K}$. For every real number $x \geq \exp\left((\log|D|)^{3/2}\right)$ sufficiently large, we have*

$$\sum_{n\leq x} \left(\sum_{i=1}^{n} \nu_{\mathbb{K}}((i,n))\right) = \frac{x^2 L(2,\chi)}{2} + O\left(|D|^{1/2} x (\log x)^{2/3}\right).$$

# 5   Acknowledgments

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(Concerned with sequence A018804.)

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