Journal of Integer Sequences, Vol. 12 (2009),
Article 09.8.5

Ramanujan Type Trigonometric Formulas: The General Form for the Argument $\frac{2\pi}{7}$

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Abstract

In this paper, we present many general identities connected with the classical Ramanujan equality. Moreover, we give Binet formulas for an accelerator sequence for Catalan's constant.

1 Introduction

The main objective of this paper is to obtain some general trigonometric formulas related to the known Ramanujan equality [1, 2, 3, 5, 7]:

$$\sqrt[3]{\cos\alpha} + \sqrt[3]{\cos 2\alpha} + \sqrt[3]{\cos 4\alpha} = \sqrt[3]{\tfrac{1}{2}\big(5 - 3\sqrt[3]{7}\big)}, \tag{1}$$

where $\alpha = 2\pi/7$. Other formulas of this type, referring to the ninth and eleventh primitive roots of unity, etc., will be published in separate papers. The present paper, in a way, is a continuation of previous papers [11, 12, 13] and I will take advantage of some results from those papers. The quasi-Fibonacci sequences of the seventh order discussed in the above mentioned papers are applied here for describing some attractive formulas involving radicals.

The paper is divided into five parts:

- Section 2 – where some striking equalities related to equality (1) are presented. Furthermore, Binet formulas for two new sequences $\{\mathcal{S}_n\}$ and $\{\mathcal{S}_n^*\}$ are derived, which, at the same time, resolves the problem of an algebraic description of the zeros of polynomials $x^3 - \sqrt[3]{7}x - 1$ and $x^3 - \sqrt[3]{49}x - 1$ (see Remark 1).
- Section 3 – where the fundamental formula (10) for a sum of the cube roots of the three roots of a cubic polynomial is given.
- Section 4 – where many basic sequences of integers, reals and complex numbers, introduced and discussed earlier by the authors in papers [11, 12, 13], are presented. In addition, some new relations between the elements of the sequences are discussed. Furthermore, a new description of Binet's formula is introduced for an accelerator sequence for Catalan's constant, which, naturally, makes it possible to extend this formula for all integers (see Remark 7).
- Section 5 – where applications of the formula (10) to many special kinds of polynomial of degree three are given. This section contains many Ramanujan type trigonometric formulas. Moreover in Remark 10 the nontrivial theoretical discussion of some numerical case is presented.

2 Delicious

Now we are going to prove the following three interesting identities:

$$\begin{aligned}
&\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\,\big(2\,\cos\alpha\big)^k + \sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\,\big(2\,\cos 2\alpha\big)^k + \sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}\,\big(2\,\cos 4\alpha\big)^k = \\
&\qquad = \sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\,\big(2\,\cos 2\alpha\big)^{k+1} + \sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\,\big(2\,\cos 4\alpha\big)^{k+1} + \\
&\qquad\qquad + \sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}\,\big(2\,\cos\alpha\big)^{k+1} = \sqrt[3]{7}\,\psi_k, \qquad (2)
\end{aligned}$$

where $\psi_0 = -1$, $\psi_1 = 0$, $\psi_2 = -3$ and

$$\psi_{k+3} + \psi_{k+2} - 2\,\psi_{k+1} - \psi_k = 0, \qquad k \in \mathbb{Z};$$

$$\begin{aligned}
&\sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}\,\big(2\,\cos\alpha\big)^k + \sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}\,\big(2\,\cos 2\alpha\big)^k + \sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}\,\big(2\,\cos 4\alpha\big)^k = \\
&\qquad = \sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}\,\big(2\,\cos\alpha\big)^{k+1} + \sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}\,\big(2\,\cos 2\alpha\big)^{k+1} + \\
&\qquad\qquad + \sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}\,\big(2\,\cos 4\alpha\big)^{k+1} = \sqrt[3]{49}\,\varphi_k, \qquad (3)
\end{aligned}$$

where $\varphi_0 = 0$, $\varphi_1 = -1$, $\varphi_2 = 1$ and

$$\varphi_{k+3} + \varphi_{k+2} - 2\,\varphi_{k+1} - \varphi_k = 0, \qquad k \in \mathbb{Z};$$

$$\begin{aligned}
&\sqrt[3]{\sec 2\alpha}\,\big(2\,\cos\alpha\big)^k + \sqrt[3]{\sec 4\alpha}\,\big(2\,\cos 2\alpha\big)^k + \sqrt[3]{\sec\alpha}\,\big(2\,\cos 4\alpha\big)^k = \\
&\qquad = \delta_k\,\sqrt[3]{8 - 6\,\sqrt[3]{7}} + \sigma_k\,\sqrt[3]{6\,\big(1+\sqrt[3]{7}\big)^2} + \xi_k\,\sqrt[3]{2\,\big(5 - 3\,\sqrt[3]{7}\big)^2} = \\
&\qquad = \sqrt[3]{f_{3k+1} + 6 - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{3k+1,8} + \sqrt{\mathcal{T}_{3k+1,8}}} + \sqrt[3]{\mathcal{S}_{3k+1,8} - \sqrt{\mathcal{T}_{3k+1,8}}}\,\right)},
\end{aligned} \tag{4}$$

where

$$\begin{array}{lll}
\delta_0 = 1, & \delta_1 = 0, & \delta_2 = 0, \\
\sigma_0 = 0, & \sigma_1 = -1, & \sigma_2 = 0, \\
\xi_0 = 0, & \xi_1 = 0, & \xi_2 = 1,
\end{array}$$

$$\mathbb{X}_{k+3} + \mathbb{X}_{k+2} - 2\,\mathbb{X}_{k+1} - \mathbb{X}_k = 0, \qquad k = 0, 1, 2, \ldots,$$

for every $\mathbb{X} \in \{\delta, \sigma, \xi\}$, whereas the sequences f_{3k+1}, $\mathcal{S}_{3k+1,8}$ and $\mathcal{T}_{3k+1,8}$ are defined by formulas (21), (99) and (100) (in Section 4, other sequences occurring in the definition of sequences $\mathcal{S}_{3k+1,8}$ and $\mathcal{T}_{3k+1,8}$ are defined as well). The first twelve values of numbers ψ_n and φ_n are presented in Table 1.

Moreover an interesting numerical link to the formula (2) are the considerations from Remark 10.

Proof of formulas (2)–(4).

For $k = 0, 1, 2$, the formulas (2)–(4) follow from (34), (37), (40), (41) and (44), and (or) from the following equalities (in both cases, equality (5) below for $\mathbb{X} = 0$ is needed):

$$\begin{aligned}
&\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\,\big(2\,\cos\alpha\big)^2 + \sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\,\big(2\,\cos 2\alpha\big)^2 + \sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}\,\big(2\,\cos 4\alpha\big)^2 = \\
&\quad = \sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\,\big(2 + 2\,\cos 2\alpha\big) + \sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\,\big(2 + 2\,\cos 4\alpha\big) + \sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}\,\big(2 + 2\,\cos\alpha\big) = \\
&\quad = 2\,\sqrt[3]{7}\,\psi_0 + 2\left(\sqrt[3]{\cos\alpha\,\cos^2 2\alpha} + \sqrt[3]{\cos 2\alpha\,\cos^2 4\alpha} + \sqrt[3]{\cos 4\alpha\,\cos^2\alpha}\,\right) = \\
&\quad = -2\,\sqrt[3]{7} + \sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}} + \sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}} + \sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}} = -2\,\sqrt[3]{7} + \sqrt[3]{7}\,\psi_0 = -3\,\sqrt[3]{7},
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}\,\big(2\,\cos\alpha\big)^2 + \sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}\,\big(2\,\cos 2\alpha\big)^2 + \sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}\,\big(2\,\cos 4\alpha\big)^2 = \\
&\quad = \sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}\,\big(2 + 2\,\cos 2\alpha\big) + \sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}\,\big(2 + 2\,\cos 4\alpha\big) + \sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}\,\big(2 + 2\,\cos\alpha\big) =
\end{aligned}$$

$$= 2\sqrt[3]{49}\,\varphi_0 + \left(\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\right)^2 + \left(\sqrt[3]{\frac{\cos 4\alpha}{\cos \alpha}}\right)^2 + \left(\sqrt[3]{\frac{\cos \alpha}{\cos 2\alpha}}\right)^2 =$$

$$=\left(\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}} + \sqrt[3]{\frac{\cos 4\alpha}{\cos \alpha}} + \sqrt[3]{\frac{\cos \alpha}{\cos 2\alpha}}\right)^2 - 2\left(\sqrt[3]{\frac{\cos 2\alpha}{\cos \alpha}} + \sqrt[3]{\frac{\cos \alpha}{\cos 4\alpha}} + \sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}\right)=$$

$$= \left(\sqrt[3]{7}\,\psi_0\right)^2 - 2\sqrt[3]{49}\,\varphi_0 = \sqrt[3]{49}.$$

Now let us set

$$\mathfrak{B}_n := \sum_{k=0}^{2} x_k \left(\cos\left(2^k\alpha\right)\right)^n,$$

where $x_k \in \mathbb{R}$, $k = 1, 2, 3$, are given. Then, from Newton's formula we obtain

$$\mathfrak{B}_{n+3} + \mathfrak{B}_{n+2} - 2\,\mathfrak{B}_{n+1} - \mathfrak{B}_n = 0$$

since [5, 11, 12]:

$$\prod_{k=0}^{2}\Big(\mathbb{X} - 2\,\cos\left(2^k\alpha\right)\Big) = \mathbb{X}^3 + \mathbb{X}^2 - 2\,\mathbb{X} - 1. \tag{5}$$

Hence, on account of the definitions of sequences φ_k and ψ_k, $k \in \mathbb{N}$, by applied induction arguments the formulas (2) and (3) follow. Similarly, by applying (33), (36) and (39) we deduce the first part of (4). The second part of (4) from (98) follows.

Remark 1. Let $a, b, c \in \mathbb{C}$ and $a + b + c = 0$. Put

$$s_k := a^k + b^k + c^k, \qquad k \in \mathbb{N}.$$

Then the following relations hold [5, 6]:

$$\begin{aligned} 2\,s_4 &= s_2^2; & 6\,s_5 &= 5\,s_2\,s_3;\\ 6\,s_7 &= 7\,s_3\,s_4; & 10\,s_7 &= 7\,s_2\,s_5;\\ 25\,s_3\,s_7 &= 21 s_5^2; & 50\,s_7^2 &= 49\,s_4\,s_5^2 \end{aligned}$$

and the respective Newton formula has the form

$$s_{n+3} = a\,b\,c\,s_n + \frac{1}{2}\,s_2\,s_{n+1}, \qquad n \in \mathbb{N}.$$

Hence, and from (3) for $k = 0$, we get

$$\mathcal{S}_{n+3} = \sqrt[3]{7}\,\mathcal{S}_{n+1} + \mathcal{S}_n,$$

where $\mathcal{S}_0 = 3$, $\mathcal{S}_1 = 0$, $\mathcal{S}_2 = 2\sqrt[3]{7}$,

$$\mathcal{S}_n := \left(\frac{\cos\alpha}{\cos 4\alpha}\right)^{n/3} + \left(\frac{\cos 2\alpha}{\cos\alpha}\right)^{n/3} + \left(\frac{\cos 4\alpha}{\cos 2\alpha}\right)^{n/3}$$

which implies the following formula

$$\mathcal{S}_n = \widehat{a}_n + \widehat{b}_n \sqrt[3]{7} + \widehat{c}_n \sqrt[3]{49},$$

where

$$\begin{aligned}
\widehat{a}_{n+3} &= \widehat{a}_n + 7\,\widehat{c}_n,\\
\widehat{b}_{n+3} &= \widehat{b}_n + \widehat{a}_{n+1},\\
\widehat{c}_{n+3} &= \widehat{c}_n + \widehat{b}_{n+1},\\
\widehat{a}_0 &= 3, \quad \widehat{a}_1 = 0, \quad \widehat{a}_2 = 0,\\
\widehat{b}_0 &= 0, \quad \widehat{b}_1 = 0, \quad \widehat{b}_2 = 1,\\
\widehat{c}_0 &= 0, \quad \widehat{c}_1 = 0, \quad \widehat{c}_2 = 0.
\end{aligned}$$

On the other hand, from (2) for $k=1$, we obtain

$$\mathcal{S}^*_{n+3} = \frac{1}{2}\,\mathcal{S}^*_2\,\mathcal{S}^*_{n+1} + \mathcal{S}^*_n, \qquad n \in \mathbb{N}_0,$$

where $\mathcal{S}^*_0 = 3$, $\mathcal{S}^*_1 = 0$,

$$\mathcal{S}^*_n := \left(2\,\cos\alpha\,\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\right)^n + \left(2\,\cos 2\alpha\,\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}\right)^n + \left(2\,\cos 4\alpha\,\sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}\right)^n, \quad n \in \mathbb{N},$$

and by (5) for $\mathbb{X}=0$:

$$\begin{aligned}
\mathcal{S}^*_2 := {}& \left(\frac{\cos\alpha}{\cos 2\alpha}\right)^{2/3}\left(2+2\,\cos 2\alpha\right) + \left(\frac{\cos 2\alpha}{\cos 4\alpha}\right)^{2/3}\left(2+2\,\cos 4\alpha\right) + \left(\frac{\cos 4\alpha}{\cos\alpha}\right)^{2/3}\left(2+2\,\cos\alpha\right) = \\
&= 2\left(\left(\frac{\cos\alpha}{\cos 2\alpha}\right)^{1/3} + \left(\frac{\cos 2\alpha}{\cos 4\alpha}\right)^{1/3} + \left(\frac{\cos 4\alpha}{\cos\alpha}\right)^{1/3}\right)^2 - \\
&- 4\left(\left(\frac{\cos\alpha}{\cos 4\alpha}\right)^{1/3} + \left(\frac{\cos 4\alpha}{\cos 2\alpha}\right)^{1/3} + \left(\frac{\cos 2\alpha}{\cos\alpha}\right)^{1/3}\right) + \\
&+ \left(2\,\cos\alpha\right)^{1/3}\left(4\,\cos\alpha\,\cos 2\alpha\right)^{1/3} + \left(2\,\cos 2\alpha\right)^{1/3}\left(4\,\cos 2\alpha\,\cos 4\alpha\right)^{1/3} + \\
&+ \left(2\,\cos 4\alpha\right)^{1/3}\left(4\,\cos\alpha\,\cos 4\alpha\right)^{1/3} = 2\,(\sqrt[3]{7}\,\psi_0)^2 - 4\,\mathcal{S}_1 + \mathcal{S}_1 = 2\,\sqrt[3]{49}.
\end{aligned}$$

So, we have

$$\mathcal{S}^*_{n+3} = \sqrt[3]{49}\,\mathcal{S}^*_{n+1} + \mathcal{S}^*_n, \qquad n \in \mathbb{N}_0,$$

which implies

$$\mathcal{S}^*_n = a^*_n + b^*_n \sqrt[3]{7} + c^*_n \sqrt[3]{49},$$

where

$$\begin{aligned}
a^*_{n+3} &= a^*_n + 7\,b^*_{n+1},\\
b^*_{n+3} &= b^*_n + 7\,c^*_{n+1},\\
c^*_{n+3} &= c^*_n + a^*_{n+1},
\end{aligned}$$

$$
\begin{aligned}
a_0^* &= 3, \quad a_1^* = 0, \quad a_2^* = 0,\\
b_0^* &= 0, \quad b_1^* = 0, \quad b_2^* = 0,\\
c_0^* &= 0, \quad c_1^* = 0, \quad c_2^* = 2.
\end{aligned}
$$

We note that the elements $\mathcal{S}_{3n} = \widehat{a}_n$ and $\mathcal{S}_{3n}^* = a_n^*$, $n = 0, 1, \ldots$, are all integers.

3 Some theoretical deliberations

Let us assume that ξ_1, ξ_2, ξ_3 are complex roots of the following polynomial with complex coefficients

$$f(z) := z^3 + p\,z^2 + q\,z + r.$$

The symbols $\sqrt[3]{\xi_1}$, $\sqrt[3]{\xi_2}$, $\sqrt[3]{\xi_3}$ will denote any of the third complex roots of the numbers ξ_1, ξ_2 and ξ_3, respectively (only in the case where ξ_1, ξ_2 and ξ_3 are real numbers, we will assume that $\sqrt[3]{\xi_1}$, $\sqrt[3]{\xi_2}$ and $\sqrt[3]{\xi_3}$ also denote the respective real roots).

Let us assume that

$$A := \left(\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3}\right)^3$$

and

$$B := \left(\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2} + \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_3} + \sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3}\right)^3.$$

Thus, the numbers

$$\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3} \qquad \text{and} \qquad \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2} + \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_3} + \sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3}$$

belong to the sets of the third complex roots of A and B, respectively, which, for conciseness of notation, will be denoted by the symbols $\sqrt[3]{A}$ and $\sqrt[3]{B}$, respectively. In other words, we have

$$\sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3} \in \sqrt[3]{A}$$

and

$$\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2} + \sqrt[3]{\xi_1}\,\sqrt[3]{\xi_3} + \sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3} \in \sqrt[3]{B}.$$

Hence, after two-sided raising of the numbers to the third power, we obtain the following formulas:

$$A = \xi_1 + \xi_2 + \xi_3 + 3\sum_{k \neq l}\left(\sqrt[3]{\xi_k}\right)^2\,\sqrt[3]{\xi_l} + 6\,\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3},$$

and

$$B = \xi_1\,\xi_2 + \xi_1\,\xi_3 + \xi_2\,\xi_3 + 3\sum_{k \neq l}\left(\sqrt[3]{\xi_k}\right)^2\sqrt[3]{\xi_l}\,\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3} + 6\left(\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3}\right)^2,$$

where

$$\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3} \in \sqrt[3]{\xi_1\,\xi_2\,\xi_3} = \sqrt[3]{-r} = -\sqrt[3]{r}.$$

Also here, for abbreviation, the product $(-1)\,\sqrt[3]{\xi_1}\,\sqrt[3]{\xi_2}\,\sqrt[3]{\xi_3}$ will be denoted by the symbol $\sqrt[3]{r}$.

Taking into account Viète's formulas for polynomial $f(z)$, the expressions for A and B can be attributed the following form (from now on, symbols $\sqrt[3]{A}$ and $\sqrt[3]{B}$ will mean the properly selected elements from sets $\sqrt[3]{A}$ and $\sqrt[3]{B}$, respectively):

$$A = -p + 3\,\sqrt[3]{A}\,\sqrt[3]{B} + 3\sqrt[3]{r} \tag{6}$$

and

$$B = q - 3\,\sqrt[3]{A}\,\sqrt[3]{B}\,\sqrt[3]{r} - 3\,\left(\sqrt[3]{r}\right)^2. \tag{7}$$

By multiplying the first of these equations by $\sqrt[3]{r}$ and adding the equations side-by-side, we obtain

$$B = q - (A+p)\,\sqrt[3]{r}. \tag{8}$$

At the same time, the equation (6) yields

$$3\,\sqrt[3]{A}\,\sqrt[3]{B} = A + p - 3\,\sqrt[3]{r},$$

i.e.,

$$27\,A\,B = \left(A + p - 3\,\sqrt[3]{r}\right)^3,$$

hence, with respect to (8) we obtain

$$\begin{aligned}27\,A\left(q - (A+p)\,\sqrt[3]{r}\right) = A^3 + p^3 - 27\,r\,+&\\ +\,3\left(A^2\,(p - 3\sqrt[3]{r}) + A\,(p^2 + 9\,(\sqrt[3]{r})^2) - 3\,p^2\,\sqrt[3]{r} + 9\,p\,(\sqrt[3]{r})^2\right) - 18\,A\,p\,\sqrt[3]{r},\end{aligned}$$

and having rearrange the summands (with respect to the powers of A), we obtain the basic equality

$$A^3 + 3\left(p + 6\,\sqrt[3]{r}\right)A^2 + 3\left(p^2 + 3\,p\,\sqrt[3]{r} + 9\,(\sqrt[3]{r})^2 - 9\,q\right)A + \left(p - 3\,\sqrt[3]{r}\right)^3 = 0. \tag{9}$$

By applying Cardano's formula to this polynomial, we get the following basic formula (the right side of the formula below means a properly selected third root of the number present in the formula):

$$\begin{aligned}\sqrt[3]{A} = \sqrt[3]{\xi_1} + \sqrt[3]{\xi_2} + \sqrt[3]{\xi_3} =&\\ = \sqrt[3]{-p - 6\,\sqrt[3]{r} - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S} + \sqrt{\mathcal{T}}} + \sqrt[3]{\mathcal{S} - \sqrt{\mathcal{T}}}\right)},\end{aligned} \tag{10}$$

where

$$\begin{aligned}\mathcal{S} &:= p\,q + 6\,q\,\sqrt[3]{r} + 6\,p\,\sqrt[3]{r^2} + 9\,r,\\ \mathcal{T} &:= p^2\,q^2 - 4\,q^3 - 4\,p^3\,r + 18\,p\,q\,r - 27\,r^2.\end{aligned}$$

In the case when $\mathcal{T} \geq 0$, $\mathcal{S} \in \mathbb{R}$, $r \in \mathbb{R}$, we assume that all the roots appearing here are real.

Remark 2. We note that, if in the formula (9) the following condition holds

$$\left(p+6\sqrt[3]{r}\right)^2 = p^2+3\,p\sqrt[3]{r}+9\,(\sqrt[3]{r})^2-9\,q,$$

i.e.,

$$p\sqrt[3]{r}+3\,(\sqrt[3]{r})^2+q=0, \tag{11}$$

then the equation (9) could be given in the form

$$\left(A+p+6\sqrt[3]{r}\right)^3 = \left(p+6\sqrt[3]{r}\right)^3-\left(p-3\sqrt[3]{r}\right)^3,$$

hence we get

$$A=-p-6\sqrt[3]{r}+\sqrt[3]{\left(p+6\sqrt[3]{r}\right)^3-\left(p-3\sqrt[3]{r}\right)^3}. \tag{12}$$

Remark 3. The analysis which enabled describing the value of A by means of coefficients of polynomial $f(z)$ comes from the papers [7, 5] (see also [4]).

4 Basic sequences

We will now provide definitions of a dozen basic sequences (not only integer sequences) used further in the paper. For more information concerning these sequences (including the trigonometric relationships defining their terms), see the papers [11, 12].

The sequences $\{A_n(\delta)\}_{n=0}^{\infty}$, $\{B_n(\delta)\}_{n=0}^{\infty}$ and $\{C_n(\delta)\}_{n=0}^{\infty}$ are the so-called quasi-Fibonacci numbers of the seventh order described in [11] by means of relations

$$(1+\delta\,(\xi^k+\xi^{6k}))^n = A_n(\delta)+B_n(\delta)\,(\xi^k+\xi^{6k})+C_n(\delta)\,(\xi^{2k}+\xi^{5k}) \tag{13}$$

for $k=1,2,3$, where $\xi\in\mathbb{C}$ is a primitive root of unity of the seventh order ($\xi^7=1$ and $\xi\neq 1$), $\delta\in\mathbb{C}$, $\delta\neq 0$. These sequences satisfy the following recurrence relations

$$\begin{cases} A_0(\delta)=1, \quad B_0(\delta)=C_0(\delta)=0, \\ A_{n+1}(\delta)=A_n(\delta)+2\,\delta\,B_n(\delta)-\delta\,C_n(\delta), \\ B_{n+1}(\delta)=\delta\,A_n(\delta)+B_n(\delta), \\ C_{n+1}(\delta)=\delta\,B_n(\delta)+(1-\delta)\,C_n(\delta), \end{cases} \tag{14}$$

for every $n\in\mathbb{N}$.

Two auxiliary sequences $\{\mathcal{A}_n(\delta)\}_{n=0}^{\infty}$ and $\{\mathcal{B}_n(\delta)\}_{n=0}^{\infty}$ connected with these ones are defined by the following relations:

$$\mathcal{A}_n(\delta) := 3\,A_n(\delta)-B_n(\delta)-C_n(\delta) \tag{15}$$

and

$$\mathcal{B}_n(\delta) := \frac{1}{2}\left(\left(\mathcal{A}_n(\delta)\right)^2-\mathcal{A}_{2n}(\delta)\right). \tag{16}$$

Furthermore, to simplify notation, we will write

$$\mathcal{A}_n=\mathcal{A}_n(1),\ \mathcal{B}_n=\mathcal{B}_n(1),\ A_n=A_n(1),\ B_n=B_n(1) \text{ and } C_n=C_n(1), \tag{17}$$

for every $n \in \mathbb{N}$.

We note that the elements of the sequences $\{\mathcal{A}_n\}_{n=0}^{\infty}$, $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ respectively, satisfy the following recurrence relation (see [12, eq. (3.20)]):

$$\mathbb{X}_{n+3} - 2\,\mathbb{X}_{n+2} - \mathbb{X}_{n+1} + \mathbb{X}_n \equiv 0.$$

Simultaneously, the elements of sequence $\{\mathcal{B}_n\}_{n=0}^{\infty}$ by [12, eqs. (3.18), (3.13)] satisfy the following relation

$$\mathcal{B}_{n+3} + \mathcal{B}_{n+2} - 2\,\mathcal{B}_{n+1} - \mathcal{B}_n \equiv 0. \tag{18}$$

Remark 4. The sequence $\{\mathcal{B}_n\}_{n=0}^{\infty}$ is an accelerator sequence for Catalan's constant (see [10, sequence A094648] and papers [11, 12]).

The elements of the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are defined by the following recurrence relations:

$$a_0 = b_0 = c_0 = \sqrt{7}$$

and

$$\begin{cases} a_{n+1} = 2\,a_n + b_n, \\ b_{n+1} = a_n + 2\,b_n - c_n, \\ c_{n+1} = c_n - b_n, \end{cases} \tag{19}$$

for $n = 0, 1, 2, \ldots$.

Moreover, we will use the following sequences

$$\begin{cases} \overline{\alpha}_n := c_{n+1}, \\ \overline{\beta}_n := -a_n - b_n, \\ \overline{\gamma}_n := a_n, \end{cases} \tag{20}$$

Next, sequences $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$, $\{h_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$ are defined in the following way

$$f_0 = g_0 = h_0 = -1,$$

and

$$\begin{cases} f_{n+1} = f_n + g_n, & n \geq 0, \\ g_{n+1} = f_n + h_n, & n \geq 0, \\ h_n = \mathcal{B}_{n+1}, & n \geq 1, \\ 2\,H_n = f_n^2 + f_{2n} - g_n^2 + h_n^2 - h_{2n+1} - 2\,h_{2n}, & n \geq 0, \end{cases} \tag{21}$$

(the numbers $\mathcal{B}_n$ are defined by the formula (17) above).

And at last, the elements of sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ are defined by

$$\begin{cases} u_{n+1} = x_n, \\ v_{n+1} = -y_n - z_n = x_n - \sqrt{7}\,z_{n-1}, \\ w_{n+1} = y_n - x_n, \\ x_{n+1} = u_n - w_n, \\ y_{n+1} = w_n - v_n, \\ z_{n+1} = 2\,z_{n-1} - v_n, \end{cases} \tag{22}$$

for $n = 0, 1, 2, \ldots$, where $u_0 = v_0 = w_0 = -1$, $x_0 = y_0 = z_0 = \sqrt{7}$ and $z_1 = 7$.

Remark 5. All the recurrence sequences above have a third order; the selective identities, Binet formulas and generating functions, and some different identities for these numbers, are presented in papers [11, 12]. For example, the following equivalent recurrence relations hold

$$(19) \iff \mathbb{X}_{n+3} - 5\,\mathbb{X}_{n+2} + 6\,\mathbb{X}_{n+1} - \mathbb{X}_n = 0,$$

for $n = 0, 1, 2, \ldots$, and $\mathbb{X} \in \{a, b, c\}$, and $a_0 = b_0 = c_0 = \sqrt{7}$, $a_1 = 3\sqrt{7}$, $b_1 = 2\sqrt{7}$, $c_1 = 0$, $a_2 = 8\sqrt{7}$, $b_2 = \sqrt{7^3}$, $c_1 = -2\sqrt{7}$. We also have

$$\begin{aligned} a_n &= 2^{2n+1}\left[\sin\alpha\left(\cos 4\alpha\right)^{2n} + \sin 2\alpha\left(\cos\alpha\right)^{2n} + \sin 4\alpha\left(\cos 2\alpha\right)^{2n}\right], \\ b_n &= 2^{2n+1}\left[\sin 2\alpha\left(\cos 4\alpha\right)^{2n} + \sin 4\alpha\left(\cos\alpha\right)^{2n} + \sin\alpha\left(\cos 2\alpha\right)^{2n}\right], \\ c_n &= 2^{2n+1}\left[\sin 4\alpha\left(\cos 4\alpha\right)^{2n} + \sin\alpha\left(\cos\alpha\right)^{2n} + \sin 2\alpha\left(\cos 2\alpha\right)^{2n}\right], \end{aligned}$$

etc.

Remark 6. Now we present some new identities for the above sequences (identities (23)–(26), (28)–(31)), which will be used in subsection 5.2. These identities significantly complete those obtained in paper [12]. By [12, Lemma 3.14 (a)], equality (5) and [12, eq. (3.21)], the following identity holds

$$(-1)^n\,\mathcal{A}_n = H_{n+1}. \tag{23}$$

Hence, by [12, eq. (3.23)], we obtain

$$\begin{aligned} &2\,\cos\alpha\left(2\,\cos 2\alpha\right)^{-n} + 2\,\cos 2\alpha\left(2\,\cos 4\alpha\right)^{-n} + 2\,\cos 4\alpha\left(2\,\cos\alpha\right)^{-n} = \\ &\qquad = (-1)^n\left(\mathcal{A}_n + \mathcal{A}_{n-1} - \mathcal{A}_{n-2}\right) = (-1)^n\left(\mathcal{A}_{n+1} - \mathcal{A}_n\right), \end{aligned} \tag{24}$$

and next, by [12, eq. (3.22)], we obtain

$$\begin{aligned} &2\,\cos\alpha\left(2\,\cos 4\alpha\right)^{-n} + 2\,\cos 2\alpha\left(2\,\cos\alpha\right)^{-n} + 2\,\cos 4\alpha\left(2\,\cos 2\alpha\right)^{-n} = \\ &\qquad = (-1)^n\left(\mathcal{A}_{n+1} - \mathcal{A}_{n-1} + \mathcal{A}_{n-2} - 7\,A_n\right) = (-1)^n\left(\mathcal{A}_{n-1} - \mathcal{A}_{n+1}\right), \end{aligned} \tag{25}$$

since, by [12, Remark 3.11], we have the identity

$$\mathcal{A}_{n+3} = 2\,\mathcal{A}_{n+2} + \mathcal{A}_{n+1} - \mathcal{A}_n$$

and by [12, Remark 3.8], we have

$$4\,\mathcal{A}_n - \mathcal{A}_{n-2} = 7\,A_n.$$

Moreover, the following formula can be easily generated

$$\mathcal{B}_n = g_{n+1} + h_{n+1}. \tag{26}$$

By [12, eq. (3.18)], we have

$$h_{n-1} = \mathcal{B}_n, \tag{27}$$

which by (26) implies

$$g_{n+1} = \mathcal{B}_n - \mathcal{B}_{n+2}, \tag{28}$$

and next, by [12, eq. (3.12)] and by (18), we get

$$f_n = g_{n+1} - h_n = \mathcal{B}_n - \mathcal{B}_{n+2} - \mathcal{B}_{n+1} = -\mathcal{B}_{n-1} - \mathcal{B}_n. \tag{29}$$

Moreover, we obtain

$$f_n + g_n = -\mathcal{B}_n - \mathcal{B}_{n+1} \tag{30}$$

and

$$f_n + h_n = \mathcal{B}_n - \mathcal{B}_{n+2}. \tag{31}$$

Remark 7. From (27) we get (see [12, eq. (3.11)]):

$$\mathcal{B}_n = 2^n \left(\cos^n \alpha + \cos^n 2\alpha + \cos^n 4\alpha\right),$$

from which the next form can be deduced

$$\mathcal{B}_n = \left(\frac{\sin 2\alpha}{\sin \alpha}\right)^n + \left(\frac{\sin 4\alpha}{\sin 2\alpha}\right)^n + \left(\frac{\sin \alpha}{\sin 4\alpha}\right)^n. \tag{32}$$

This forms of Binet's formula for $\mathcal{B}_n$ are more attractive than the Sloane's ones (see [10, sequence A094648]). Moreover, the formula (32) makes it possible to extend the definition $\mathcal{B}_n$ for negative integers n. So $\{\mathcal{B}_n\}_{n=-\infty}^{\infty}$ is a two-sided sequence of integers which is defined for all integers either by recurrence formula (18), or equivalently by Binet's formula (32).

5 Applications of the formula (10)

5.1 Some special formulas

By [12, eq. (3.30)]

for $n = 1$:

$$\sqrt[3]{\sec \alpha} + \sqrt[3]{\sec 2\alpha} + \sqrt[3]{\sec 4\alpha} = \sqrt[3]{8 - 6\sqrt[3]{7}}\,; \tag{33}$$

for $n = 2$:

$$\sqrt[3]{\frac{\cos \alpha}{\cos 2\alpha}} + \sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}} + \sqrt[3]{\frac{\cos 4\alpha}{\cos \alpha}} = -\sqrt[3]{7}\,; \tag{34}$$

for $n = 3$:

$$\cos 2\alpha \sqrt[3]{2\cos \alpha} + \cos 4\alpha \sqrt[3]{2\cos 2\alpha} + \cos \alpha \sqrt[3]{2\cos 4\alpha} = \frac{1}{2}\sqrt[3]{5 + 3\sqrt[3]{7} - 3\sqrt[3]{49}}\,; \tag{35}$$

for $n = 4$:

$$\cos 2\alpha \sqrt[3]{\sec 4\alpha} + \cos 4\alpha \sqrt[3]{\sec \alpha} + \cos \alpha \sqrt[3]{\sec 2\alpha} = -\sqrt[3]{\frac{3}{4}\left(1 + \sqrt[3]{7}\right)^2}\,; \tag{36}$$

for $n=5$:

$$\cos 2\alpha\,\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}+\cos 4\alpha\,\sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}+\cos\alpha\,\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}=0\,; \tag{37}$$

for $n=6$:

$$\cos^2(2\alpha)\,\sqrt[3]{2\,\cos\alpha}+\cos^2(4\alpha)\,\sqrt[3]{2\,\cos 2\alpha}+\cos^2(\alpha)\,\sqrt[3]{2\,\cos 4\alpha}= \\ =\frac{1}{4}\,\sqrt[3]{12\,\sqrt[3]{7}-\left(4+3\,\sqrt[3]{7}\right)^2}\,; \tag{38}$$

for $n=7$:

$$\cos^2(2\alpha)\,\sqrt[3]{\sec 4\alpha}+\cos^2(4\alpha)\,\sqrt[3]{\sec\alpha}+\cos^2(\alpha)\,\sqrt[3]{\sec 2\alpha}=\frac{1}{4}\,\sqrt[3]{2\left(5-3\,\sqrt[3]{7}\right)^2}\,; \tag{39}$$

for $n=8$:

$$\cos^3(2\alpha)\,\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}+\cos^3(4\alpha)\,\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}+\cos^3(\alpha)\,\sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}= \\ =\frac{1}{2}\left(\cos^2(2\alpha)\,\sqrt[3]{\frac{\cos 2\alpha}{\cos 4\alpha}}+\cos^2(4\alpha)\,\sqrt[3]{\frac{\cos 4\alpha}{\cos\alpha}}+\cos^2(\alpha)\,\sqrt[3]{\frac{\cos\alpha}{\cos 2\alpha}}\right)=-\frac{3}{8}\,\sqrt[3]{7}\,, \tag{40}$$

which also can be generated from (34).

Remark 8. The formula (39) also follows from (1) and (5) for $\mathbb{X}=0$:

$$2\,\cos^2(2\alpha)\,\sqrt[3]{\sec 4\alpha}+2\,\cos^2(4\alpha)\,\sqrt[3]{\sec\alpha}+2\,\cos^2(\alpha)\,\sqrt[3]{\sec 2\alpha}= \\ =\sum_{k=0}^{2}\sqrt[3]{\sec(2^k\alpha)}+\sum_{k=0}^{2}\left(\sqrt[3]{\cos(2^k\alpha)}\right)^2= \\ =\sum_{k=0}^{2}\sqrt[3]{\sec(2^k\alpha)}+\left(\sum_{k=0}^{2}\sqrt[3]{\cos(2^k\alpha)}\right)^2-2\sum_{k=0}^{2}\sqrt[3]{\cos(2^k\alpha)\,\cos(2^{k+1}\alpha)}= \\ =\left(\sum_{k=0}^{2}\sqrt[3]{\cos(2^k\alpha)}\right)^2.$$

By [12, eq. (3.31)]

for $n=2$:

$$\sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}+\sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}+\sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}=0\,; \tag{41}$$

for $n=3$:

$$\cos\alpha\,\sqrt[3]{\cos 2\alpha}+\cos 2\alpha\,\sqrt[3]{\cos 4\alpha}+\cos 4\alpha\,\sqrt[3]{\cos\alpha}=-\frac{1}{2}\,\sqrt[3]{1+\frac{3}{2}\,\sqrt[3]{49}}\,; \tag{42}$$

for $n=4$:

$$\cos\alpha\,\sqrt[3]{\sec 4\alpha}+\cos 4\alpha\,\sqrt[3]{\sec 2\alpha}+\cos 2\alpha\,\sqrt[3]{\sec\alpha}=\sqrt[3]{\frac{9}{4}\left(2-\sqrt[3]{7}\right)}\,; \tag{43}$$

for $n=5$:

$$\cos^2(2\alpha)\,\sqrt[3]{\frac{\cos 4\alpha}{\cos 2\alpha}}+\cos^2(4\alpha)\,\sqrt[3]{\frac{\cos\alpha}{\cos 4\alpha}}+\cos^2(\alpha)\,\sqrt[3]{\frac{\cos 2\alpha}{\cos\alpha}}=-\frac{1}{4}\,\sqrt[3]{49}; \tag{44}$$

for $n=6$:

$$\cos^2(2\alpha)\,\sqrt[3]{\cos 4\alpha}+\cos^2(\alpha)\,\sqrt[3]{\cos 2\alpha}+\cos^2(4\alpha)\,\sqrt[3]{\cos\alpha}=$$
$$=2^{-7/3}\,\sqrt[3]{47+3\,\sqrt[3]{7}-12\,\sqrt[3]{49}}\,; \tag{45}$$

for $n=7$:

$$\cos^2(\alpha)\,\sqrt[3]{\sec 4\alpha}+\cos^2(4\alpha)\,\sqrt[3]{\sec 2\alpha}+\cos^2(2\alpha)\,\sqrt[3]{\sec\alpha}=$$
$$=-2^{5/3}\,\sqrt[3]{73+36\,\sqrt[3]{7}+3\,\sqrt[3]{49}}\,; \tag{46}$$

Remark 9. We note that

$$\begin{aligned}
&2\,\cos^2(2\alpha)\,\sqrt[3]{2\,\cos\alpha}+2\,\cos^2(4\alpha)\,\sqrt[3]{2\,\cos 2\alpha}+2\,\cos^2(\alpha)\,\sqrt[3]{2\,\cos 4\alpha}=\\
&=\left(\sqrt[3]{2\,\cos\alpha}+\sqrt[3]{2\,\cos 2\alpha}+\sqrt[3]{2\,\cos 4\alpha}\right)+\\
&+\cos 4\alpha\,\sqrt[3]{2\,\cos\alpha}+\cos\alpha\,\sqrt[3]{2\,\cos 2\alpha}+\cos 2\alpha\,\sqrt[3]{2\,\cos 4\alpha}=\\
&\overset{(1),(42)}{=}\sqrt[3]{5-3\,\sqrt[3]{7}}-\frac{1}{2}\,\sqrt[3]{2+3\,\sqrt[3]{49}}\overset{(38)}{=}\frac{1}{2}\,\sqrt[3]{12\,\sqrt[3]{7}-\left(4+3\,\sqrt[3]{7}\right)^2},
\end{aligned}$$

which implies the identity

$$\sqrt[3]{16+12\,\sqrt[3]{7}+9\,\sqrt[3]{49}}=2\,\sqrt[3]{3\,\sqrt[3]{7}-5}+\sqrt[3]{2+3\,\sqrt[3]{49}}.$$

By [12, eq. (3.32)]

for $n=2$:

$$\sqrt[3]{\frac{\cos\alpha}{\cos^2(2\alpha)}}+\sqrt[3]{\frac{\cos 2\alpha}{\cos^2(4\alpha)}}+\sqrt[3]{\frac{\cos 4\alpha}{\cos^2(\alpha)}}=\sqrt[3]{2\left(11-3\,\sqrt[3]{49}\right)}\,; \tag{47}$$

for $n=3$:

$$\frac{\sqrt[3]{\cos\alpha}}{\cos 2\alpha}+\frac{\sqrt[3]{\cos 2\alpha}}{\cos 4\alpha}+\frac{\sqrt[3]{\cos 4\alpha}}{\cos\alpha}=-\sqrt[3]{36\left(1+\sqrt[3]{7}\right)}\,; \tag{48}$$

By [12, eq. (3.33)]

for $n = 2$:

$$\sqrt[3]{\frac{\cos\alpha}{\cos^2(4\alpha)}} + \sqrt[3]{\frac{\cos 2\alpha}{\cos^2(\alpha)}} + \sqrt[3]{\frac{\cos 4\alpha}{\cos^2(2\alpha)}} = \sqrt[3]{6\left(-1+\sqrt[3]{7}-\sqrt[3]{49}\right)} = -2\sqrt[3]{\frac{6}{1+\sqrt[3]{7}}}\,; \qquad (49)$$

for $n = 3$:

$$\frac{\sqrt[3]{\cos\alpha}}{\cos 4\alpha} + \frac{\sqrt[3]{\cos 2\alpha}}{\cos\alpha} + \frac{\sqrt[3]{\cos 4\alpha}}{\cos 2\alpha} = \sqrt[3]{4\left(26-6\sqrt[3]{7}-3\sqrt[3]{49}\right)}\,; \qquad (50)$$

By [12, eq. (3.34)]

for $n = 2$ we get (33);

for $n = 3$:

$$\left(2\,\cos\alpha\right)^{-2/3} + \left(2\,\cos 2\alpha\right)^{-2/3} + \left(2\,\cos 4\alpha\right)^{-2/3} = \sqrt[3]{12+6\sqrt[3]{7}+3\sqrt[3]{49}}\,; \qquad (51)$$

By [12, eq. (4.30)]

for $n = 1$:

$$\sqrt[18]{7}\left(\sqrt[3]{\cot\alpha} + \sqrt[3]{\cot 2\alpha} + \sqrt[3]{\cot 4\alpha}\right) =$$

$$= \sqrt[3]{\sqrt[3]{49}-6+3\sqrt[3]{3\,(1-\sqrt[3]{7}+\sqrt[3]{49})}-3\sqrt[3]{5+3\sqrt[3]{7}-3\sqrt[3]{49}}}\,; \qquad (52)$$

for $n = 2$:

$$\sqrt[9]{\frac{7}{8}}\left(\sqrt[3]{\cot\alpha\,\csc\alpha} + \sqrt[3]{\cot 2\alpha\,\csc 2\alpha} + \sqrt[3]{\cot 4\alpha\,\csc 4\alpha}\right) =$$

$$= \sqrt[3]{6-2\sqrt[3]{7}-3\sqrt[3]{3\,(1+\sqrt[3]{7})^2}-3\sqrt[3]{-26+6\sqrt[3]{7}+3\sqrt[3]{49}}}\,, \qquad (53)$$

i.e,

$$\sum_{k=0}^{2}\sqrt[3]{\cot 2^k\alpha\,\csc 2^k\alpha} = \sqrt[3]{\frac{6}{\sqrt[3]{7}}\left(2-\frac{2}{3}\sqrt[3]{7}-\sqrt[3]{3(1+\sqrt[3]{7})^2}-\sqrt[3]{3(1+\sqrt[3]{7})^2-27}\right)}\,. \qquad (54)$$

Moreover from (117) below we have for $n = 1$:

$$\sqrt[3]{\tan\alpha} + \sqrt[3]{\tan 2\alpha} + \sqrt[3]{\tan 4\alpha} =$$

$$= \sqrt[18]{7}\sqrt[3]{3\sqrt[3]{3\,(1-\sqrt[3]{7}+\sqrt[3]{49})}-3\sqrt[3]{5+3\sqrt[3]{7}-3\sqrt[3]{49}}-6-\sqrt[3]{7}}\,; \qquad (55)$$

5.2 Some general formulas

1. By [12, eq. (2.1)] we obtain (for $n = 0, 1, 2, \ldots$):

$$\big(2\,\cos\alpha\big)^{n/3} + \big(2\,\cos 2\alpha\big)^{n/3} + \big(2\,\cos 4\alpha\big)^{n/3} = \\ = \sqrt[3]{\mathcal{B}_n + 6 - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,1} + \sqrt{\mathcal{T}_{n,1}}} + \sqrt[3]{\mathcal{S}_{n,1} - \sqrt{\mathcal{T}_{n,1}}}\,\right)}, \tag{56}$$

where

$$\mathcal{S}_{n,1} := (-1)^{n-1}\,\mathcal{A}_n\,\big(\mathcal{B}_n + 6\big) - 6\,\mathcal{B}_n - 9, \tag{57}$$

$$\mathcal{T}_{n,1} := \mathcal{A}_n^2\,\mathcal{B}_n^2 + 4\,(-1)^{n-1}\,\mathcal{A}_n^3 - 4\,\mathcal{B}_n^3 + 18\,(-1)^n\,\mathcal{A}_n\,\mathcal{B}_n - 27. \tag{58}$$

We note that

$$\mathcal{B}_n = \big(2\,\cos\alpha\big)^n + \big(2\,\cos 2\alpha\big)^n + \big(2\,\cos 4\alpha\big)^n, \tag{59}$$

so we get the following interesting identity

$$\frac{1}{3}\left(6 + \mathcal{B}_{3n} - \mathcal{B}_n^3\right) = \sqrt[3]{\tfrac{1}{2}\big(\mathcal{S}_{3n,1} + \sqrt{\mathcal{T}_{3n,1}}\,\big)} + \sqrt[3]{\tfrac{1}{2}\big(\mathcal{S}_{3n,1} - \sqrt{\mathcal{T}_{3n,1}}\,\big)}. \tag{60}$$

Moreover, if $n \in \mathbb{N}$ then by [12, eq. (3.34)] and by (23) and (26) we obtain

$$\big(2\,\cos\alpha\big)^{-n/3} + \big(2\,\cos 2\alpha\big)^{-n/3} + \big(2\,\cos 4\alpha\big)^{-n/3} = \\ = \sqrt[3]{(-1)^n\,\mathcal{A}_n + 6 - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,1} + \sqrt{\mathcal{T}_{n,1}}} + \sqrt[3]{\mathcal{S}_{n,1} - \sqrt{\mathcal{T}_{n,1}}}\,\right)}. \tag{61}$$

Hence, for example for $n = 2$ the formula (33) follows.

2. By [12, eq. (3.27)] we obtain

$$\sqrt[3]{2\,\sin\alpha\,\big(2\,\cos 4\alpha\big)^n} + \sqrt[3]{2\,\sin 2\alpha\,\big(2\,\cos\alpha\big)^n} + \sqrt[3]{2\,\sin 4\alpha\,\big(2\,\cos 2\alpha\big)^n} = \\ = \sqrt[3]{p_{n,2} - 6\,\sqrt[6]{7} - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,2} + \sqrt{\mathcal{T}_{n,2}}} + \sqrt[3]{\mathcal{S}_{n,2} - \sqrt{\mathcal{T}_{n,2}}}\,\right)}, \tag{62}$$

where

$$\mathcal{S}_{n,2} = 7\,(-1)^n\,B_n\,\big(6\,\sqrt[6]{7} - p_{n,2}\big) - 6\,\sqrt[3]{7}\,p_{n,2} + 9\,\sqrt{7}, \tag{63}$$

$$\mathcal{T}_{n,2} = 49\,B_n^2\,\big(p_{n,2}^2 - 28\,(-1)^n\,B_n\big) + \\ + 2\,\sqrt{7}\,p_{n,2}\,\big(2\,p_{n,2}^2 - 63\,(-1)^n B_n\big) - 189, \tag{64}$$

$$p_{n,2} = \begin{cases} a_{n/2} & \text{if } n \text{ is even,} \\ \overline{\alpha}_{(n-1)/2} & \text{if } n \text{ is odd,} \end{cases} \tag{65}$$

$$q_{n,2} = 7\,(-1)^n\,B_n, \tag{66}$$

$$r_{n,2} \equiv \sqrt{7}. \tag{67}$$

3. By [12, eq. (3.28)] we obtain

$$\sqrt[3]{2\,\sin\alpha\,\big(2\,\cos 2\alpha\big)^n}+\sqrt[3]{2\,\sin 2\alpha\,\big(2\,\cos 4\alpha\big)^n}+\sqrt[3]{2\,\sin 4\alpha\,\big(2\,\cos\alpha\big)^n}=$$
$$=\sqrt[3]{p_{n,3}-6\,\sqrt[6]{7}-\frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,3}+\sqrt{\mathcal{T}_{n,3}}}+\sqrt[3]{\mathcal{S}_{n,3}-\sqrt{\mathcal{T}_{n,3}}}\,\right)},\quad (68)$$

where

$$\mathcal{S}_{n,3}=7\,(-1)^n\,\big(B_n-C_n\big)\,\big(p_{n,3}-6\,\sqrt[6]{7}\big)-6\,\sqrt[3]{7}\,p_{n,3}+9\,\sqrt{7},\quad (69)$$
$$\mathcal{T}_{n,3}=49\,\big(B_n-C_n\big)^2\,\big(p_{n,3}^2+28\,(-1)^n\,\big(B_n-C_n\big)\big)+$$
$$+\,2\,\sqrt{7}\,p_{n,3}\,\big(2\,p_{n,3}^2+63\,(-1)^n\big(B_n-C_n\big)\big)-189,\quad (70)$$
$$p_{n,3}=\begin{cases} b_{n/2} & \text{if } n \text{ is even,}\\ \overline{\beta}_{(n-1)/2} & \text{if } n \text{ is odd,}\end{cases}\quad (71)$$
$$q_{n,3}=7\,(-1)^{n-1}\,\big(B_n-C_n\big),\quad (72)$$
$$r_{n,3}\equiv\sqrt{7}.\quad (73)$$

4. By [12, eq. (3.29)] we obtain

$$\sqrt[3]{2\,\sin\alpha\,\big(2\,\cos\alpha\big)^n}+\sqrt[3]{2\,\sin 2\alpha\,\big(2\,\cos 2\alpha\big)^n}+\sqrt[3]{2\,\sin 4\alpha\,\big(2\,\cos 4\alpha\big)^n}=$$
$$=\sqrt[3]{p_{n,4}-6\,\sqrt[6]{7}-\frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,4}+\sqrt{\mathcal{T}_{n,4}}}+\sqrt[3]{\mathcal{S}_{n,4}-\sqrt{\mathcal{T}_{n,4}}}\,\right)},\quad (74)$$

where

$$\mathcal{S}_{n,4}=7\,(-1)^n\,C_n\,\big(p_{n,4}-6\,\sqrt[6]{7}\big)-6\,\sqrt[3]{7}p_{n,4}+9\,\sqrt{7},\quad (75)$$
$$\mathcal{T}_{n,4}=\big(7\,p_{n,4}\,C_n+9\,(-1)^n\,\sqrt{7}\big)^2\,+4\,\big(7^3\,(-1)^n\,C_n^3+\sqrt{7}\,p_{n,4}^3\big)-756,\quad (76)$$
$$p_{n,4}=\begin{cases} c_{n/2} & \text{if } n \text{ is even,}\\ \overline{\gamma}_{(n-1)/2} & \text{if } n \text{ is odd,}\end{cases}\quad (77)$$
$$q_{n,4}=7\,(-1)^{n-1}\,C_n,\quad (78)$$
$$r_{n,4}\equiv\sqrt{7}.\quad (79)$$

5. By [12, eq. (3.31)] we obtain

$$\sqrt[3]{2\,\cos\alpha\,\big(2\,\cos 4\alpha\big)^n}+\sqrt[3]{2\,\cos 2\alpha\,\big(2\,\cos\alpha\big)^n}+\sqrt[3]{2\,\cos 4\alpha\,\big(2\,\cos 2\alpha\big)^n}=$$
$$=\sqrt[3]{g_n+6-\frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,5}+\sqrt{\mathcal{T}_{n,5}}}+\sqrt[3]{\mathcal{S}_{n,5}-\sqrt{\mathcal{T}_{n,5}}}\,\right)},\quad (80)$$

where

$$\mathcal{S}_{n,5}=-g_n\,q_{n,5}-6\,\big(g_n+q_{n,5}\big)-9,\quad (81)$$
$$\mathcal{T}_{n,5}=\big(g_n\,q_{n,5}+9\big)^2-4\,\big(g_n^3+q_{n,5}^3\big)-108,\quad (82)$$

$$p_{n,5} = -g_n, \tag{83}$$
$$q_{n,5} = (-1)^n \big(\mathcal{A}_n + \mathcal{A}_{n+1} - 7\, A_n\big), \tag{84}$$
$$r_{n,5} \equiv -1. \tag{85}$$

6. By [12, eq. (3.32)] and (24) we obtain

$$\sqrt[3]{\cos\alpha\,\big(\sec 2\alpha\big)^n} + \sqrt[3]{\cos 2\alpha\,\big(\sec 4\alpha\big)^n} + \sqrt[3]{\cos 4\alpha\,\big(\sec\alpha\big)^n} = \\ = \sqrt[3]{2^{n-1}\left(-p_{n,6} + 6 - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,6} + \sqrt{\mathcal{T}_{n,6}}} + \sqrt[3]{\mathcal{S}_{n,6} - \sqrt{\mathcal{T}_{n,6}}}\ \right)\right)}, \tag{86}$$

where

$$\mathcal{S}_{n,6} = (-1)^n \big(\mathcal{A}_{n+1} - \mathcal{A}_n\big)\big(\mathcal{B}_{n+2} - \mathcal{B}_n - 6\big) + 6\big(\mathcal{B}_{n+2} - \mathcal{B}_n\big) - 9, \tag{87}$$
$$\mathcal{T}_{n,6} = \big(\mathcal{B}_{n+2} - \mathcal{B}_n\big)^2 \Big(\big(\mathcal{A}_n - \mathcal{A}_{n+1}\big)^2 + 4\big(\mathcal{B}_{n+2} - \mathcal{B}_n\big)\Big) + \\ + 2(-1)^n\big(\mathcal{A}_n - \mathcal{A}_{n+1}\big)\Big(2\big(\mathcal{A}_n - \mathcal{A}_{n+1}\big)^2 + 9\big(\mathcal{B}_{n+2} - \mathcal{B}_n\big)\Big) - 27, \tag{88}$$
$$p_{n,6} = (-1)^n \big(\mathcal{A}_n - \mathcal{A}_{n+1}\big), \tag{89}$$
$$q_{n,6} = f_n + h_n = \mathcal{B}_n - \mathcal{B}_{n+2}, \tag{90}$$
$$r_{n,6} \equiv -1. \tag{91}$$

7. By [12, eq. (3.33)] and (25) we obtain

$$\sqrt[3]{\cos\alpha\,\big(\sec 4\alpha\big)^n} + \sqrt[3]{\cos 2\alpha\,\big(\sec\alpha\big)^n} + \sqrt[3]{\cos 4\alpha\,\big(\sec 2\alpha\big)^n} = \\ = \sqrt[3]{2^{n-1}\left(-p_{n,7} + 6 - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,7} + \sqrt{\mathcal{T}_{n,7}}} + \sqrt[3]{\mathcal{S}_{n,7} - \sqrt{\mathcal{T}_{n,7}}}\ \right)\right)}, \tag{92}$$

where

$$\mathcal{S}_{n,7} = (-1)^n \big(\mathcal{A}_{n+1} - \mathcal{A}_{n-1}\big)\big(6 - \mathcal{B}_{n+1} - \mathcal{B}_n\big) + 6\big(\mathcal{B}_{n+1} + \mathcal{B}_n\big) - 9, \tag{93}$$
$$\mathcal{T}_{n,7} = \big(\mathcal{B}_{n+1} + \mathcal{B}_n\big)^2 \Big(\big(\mathcal{A}_{n+1} - \mathcal{A}_{n-1}\big)^2 + 4\big(\mathcal{B}_{n+1} + \mathcal{B}_n\big)\Big) + \\ + 2(-1)^n\big(\mathcal{A}_{n+1} - \mathcal{A}_{n-1}\big)\Big(2\big(\mathcal{A}_{n+1} - \mathcal{A}_{n-1}\big)^2 + 9\big(\mathcal{B}_{n+1} + \mathcal{B}_n\big)\Big) - 27, \tag{94}$$
$$p_{n,7} = (-1)^n \big(\mathcal{A}_{n+1} - \mathcal{A}_{n-1}\big), \tag{95}$$
$$q_{n,7} = f_n + g_n = -\mathcal{B}_n - \mathcal{B}_{n+1}, \tag{96}$$
$$r_{n,7} \equiv -1. \tag{97}$$

8. By [12, eq. (3.30)], we obtain

$$\sqrt[3]{2\ \cos\alpha\,\big(2\ \cos 2\alpha\big)^n} + \sqrt[3]{2\ \cos 2\alpha\,\big(2\ \cos 4\alpha\big)^n} + \sqrt[3]{2\ \cos 4\alpha\,\big(2\ \cos\alpha\big)^n} = \\ = \sqrt[3]{f_n + 6 - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,8} + \sqrt{\mathcal{T}_{n,8}}} + \sqrt[3]{\mathcal{S}_{n,8} - \sqrt{\mathcal{T}_{n,8}}}\ \right)}, \tag{98}$$

where

$$\mathcal{S}_{n,8} = -f_n\, q_{n,8} - 6\,\big(f_n + q_{n,8}\big) - 9, \tag{99}$$

$$\mathcal{T}_{n,8} = \big(f_n\, q_{n,8} + 9\big)^2 - 4\,\big(f_n^3 + q_{n,8}^3\big) - 108, \tag{100}$$

$$p_{n,8} = -f_n, \tag{101}$$

$$q_{n,8} = (-1)^n\,\big(7\,A_n - 3\,\mathcal{A}_n\big) = (-1)^n\big(\mathcal{A}_n - \mathcal{A}_{n-2}\big), \tag{102}$$

$$r_{n,8} \equiv -1. \tag{103}$$

Remark 10. As results from direct observation of the value of expression

$$\sqrt[3]{\frac{1}{2}\left(\mathcal{S}_{n,8} - \sqrt{\mathcal{T}_{n,8}}\right)}$$

for $n = 0, 1, \ldots, 2000$, for the indicated index values, the following equalities hold

$$\sqrt[3]{\frac{1}{2}\left(\mathcal{S}_{n,8} - \sqrt{\mathcal{T}_{n,8}}\right)} = \begin{cases} (-1)^{k-1}\,\widehat{x}_k, & \text{for} \;\; n = 3\,k - 1 \geq 5, \\ \sqrt[3]{7}\,(-1)^k\,\widehat{y}_k, & \text{for} \;\; n = 3\,k, \\ \sqrt[3]{49}\,(-1)^{k+1}\,\widehat{z}_k, & \text{for} \;\; n = 3\,k + 1, \end{cases}$$

for $k = 1, 2, \ldots$, where

$$\begin{array}{lll} \widehat{x}_1 = 2, & \widehat{x}_2 = 5, & \widehat{x}_3 = 16, \\ \widehat{y}_1 = 1, & \widehat{y}_2 = 4, & \widehat{y}_3 = 12, \\ \widehat{z}_1 = 1, & \widehat{z}_2 = 3, & \widehat{z}_3 = 9, \end{array}$$

and the elements of any of the sequences: $\{\widehat{x}_k\}_{k=1}^{\infty}$, $\{\widehat{y}_k\}_{k=1}^{\infty}$ and $\{\widehat{z}_k\}_{k=1}^{\infty}$, satisfy the following recurrent relation

$$\mathbb{X}_{n+3} - 4\,\mathbb{X}_{n+2} + 3\,\mathbb{X}_{n+1} + \mathbb{X}_n = 0.$$

Also, the following interesting relationships occur (see (14)):

$$\begin{aligned} \widehat{x}_k &= A_k(-1) + 2\,C_k(-1) - C_{k-2}(-1), \qquad k \geq 2, \\ \widehat{y}_k &= A_k(-1) + C_k(-1), \\ \widehat{z}_k &= C_k(-1) - B_k(-1). \end{aligned}$$

Hence, by [11, eqs. (3.17), (3.18), (3.19)], we obtain, inter alia, the following Binet formulas:

$$\begin{aligned} \widehat{x}_k = {} & \big(2 - 2\,\cos\alpha + 4\,\cos 2\alpha\big)\,\big(1 - 2\,\cos\alpha\big)^{k-2} + \\ & + \big(2 - 2\,\cos 2\alpha + 4\,\cos 4\alpha\big)\,\big(1 - 2\,\cos 2\alpha\big)^{k-2} + \\ & + \big(2 - 2\,\cos 4\alpha + 4\,\cos\alpha\big)\,\big(1 - 2\,\cos 4\alpha\big)^{k-2}, \end{aligned}$$

$$\begin{aligned} \widehat{y}_k = {} & \frac{2}{7}\,\big(1 + \cos 2\alpha - 2\,\cos 4\alpha\big)\,\big(1 - 2\,\cos\alpha\big)^{k} + \\ & + \frac{2}{7}\,\big(1 + \cos 4\alpha - 2\,\cos\alpha\big)\,\big(1 - 2\,\cos 2\alpha\big)^{k} + \\ & + \frac{2}{7}\,\big(1 + \cos\alpha - 2\,\cos 2\alpha\big)\,\big(1 - 2\,\cos 4\alpha\big)^{k}, \end{aligned}$$

$$\widehat{z}_k = \frac{2}{7}\big(\cos 2\alpha - \cos\alpha\big)\big(1 - 2\,\cos\alpha\big)^k + \\ + \frac{2}{7}\big(\cos 4\alpha - \cos 2\alpha\big)\big(1 - 2\,\cos 2\alpha\big)^k + \\ + \frac{2}{7}\big(\cos\alpha - \cos 4\alpha\big)\big(1 - 2\,\cos 4\alpha\big)^k.$$

Next, as results from direct observation of the value of expression

$$\sqrt[3]{\frac{1}{2}\Big(\mathcal{S}_{n,8} + \sqrt{\mathcal{T}_{n,8}}\Big)}$$

for $n = 0, 1, \ldots, 2000$, for the indicated index values, the following equations hold

$$\sqrt[3]{\frac{1}{2}\Big(\mathcal{S}_{n,8} + \sqrt{\mathcal{T}_{n,8}}\Big)} = \begin{cases} \widetilde{x}_k, & \text{for} \;\; n = 3\,k + 2, \\ \sqrt[3]{7}\,\widetilde{y}_k, & \text{for} \;\; n = 3\,k + 1, \\ \sqrt[3]{49}\,\widetilde{z}_k, & \text{for} \;\; n = 3\,k, \end{cases}$$

for $k = 1, 2, \ldots$, where $x_0 = 2$ and

$$\begin{array}{lll} \widetilde{x}_1 = 8, & \widetilde{x}_2 = 29, & \widetilde{x}_3 = 120, \\ \widetilde{y}_1 = 1, & \widetilde{y}_2 = 2, & \widetilde{y}_3 = 10, \\ \widetilde{z}_1 = 1, & \widetilde{z}_2 = 3, & \widetilde{z}_3 = 13, \end{array}$$

and the elements of any of the sequences $\{\widetilde{x}_k\}_{k=2}^{\infty}$, $\{\widetilde{y}_k\}_{k=1}^{\infty}$ and $\{\widetilde{z}_k\}_{k=1}^{\infty}$ satisfy the following recurrent relation:

$$\mathbb{X}_{n+3} - 3\,\mathbb{X}_{n+2} - 4\,\mathbb{X}_{n+1} - \mathbb{X}_n = 0.$$

After substitution $x \mapsto (x-1)$ in the respective characteristic polynomial of this relation, we obtain polynomial $x^3 - 7\,x - 7$, for which by [12, eq. (4.14)], for $n = 1$, we obtain

$$\mathbb{X}^3 - 7\,\mathbb{X} - 7 = \prod_{k=0}^{2}\Big(\mathbb{X} + \frac{\sqrt{7}}{2}\,\csc(2^k\alpha)\Big).$$

Hence, the following Binet formulas hold

$$p_n = a_p\,\big(1 - \frac{\sqrt{7}}{2}\,\csc\alpha\big)^n + b_p\,\big(1 - \frac{\sqrt{7}}{2}\,\csc 2\alpha\big)^n + c_p\,\big(1 - \frac{\sqrt{7}}{2}\,\csc 4\alpha\big)^n,$$

for every $p \in \{\widetilde{x}, \widetilde{y}, \widetilde{z}\}$, and where

$$\begin{array}{lll} a_{\tilde{x}} \approx -1.246979604, & b_{\tilde{x}} \approx 0.4450418679, & c_{\tilde{x}} \approx 1.801937736, \\ a_{\tilde{y}} \approx -1.064961507, & b_{\tilde{y}} \approx 0.9189943261, & c_{\tilde{y}} \approx 0.1459671806, \\ a_{\tilde{z}} \approx -0.4355596199, & b_{\tilde{z}} \approx 0.2417173531, & c_{\tilde{z}} \approx 0.1938422668. \end{array}$$

Additionally, we note that

$$\begin{aligned} \widetilde{x}_0 &= a_{\tilde{x}} + b_{\tilde{x}} + c_{\tilde{x}} = 1, \\ \widetilde{y}_0 &= a_{\tilde{y}} + b_{\tilde{y}} + c_{\tilde{y}} = 0, \\ \widetilde{z}_0 &= a_{\tilde{z}} + b_{\tilde{z}} + c_{\tilde{z}} = 0. \end{aligned}$$

By juxtaposing the obtained values of expressions

$$\sqrt[3]{\frac{1}{2}\left(\mathcal{S}_{n,8}\pm\sqrt{\mathcal{T}_{n,8}}\right)}$$

we can now invest formula (98) to the new interesting form (for cases $n=3k,3k+1,3k+2$ respectively, and only within the indicated range of values $n=1,2,\ldots,2000$):

$$\sqrt[3]{2\,\cos\alpha}\left(2\,\cos 2\alpha\right)^k+\sqrt[3]{2\,\cos 2\alpha}\left(2\,\cos 4\alpha\right)^k+\sqrt[3]{2\,\cos 4\alpha}\left(2\,\cos\alpha\right)^k= \\ =\sqrt[3]{f_{3k}+6+3\,(-1)^{k-1}\widehat{y}_k\,\sqrt[3]{7}-3\,\widetilde{z}_k\,\sqrt[3]{49}}\,,$$

$$\sqrt[3]{\sec 4\alpha}\left(2\,\cos 2\alpha\right)^k+\sqrt[3]{\sec\alpha}\left(2\,\cos 4\alpha\right)^k+\sqrt[3]{\sec 2\alpha}\left(2\,\cos\alpha\right)^k= \\ =\sqrt[3]{2\left(f_{3k+1}+6-3\,\widetilde{y}_k\,\sqrt[3]{7}+3\,(-1)^k\,\widehat{z}_k\,\sqrt[3]{49}\right)}\,,$$

and the formula which is equivalent to relation (2) and which generates the identity

$$7\,\psi_k^3=f_{3k+2}+6-3\left(\widetilde{x}_k+(-1)^k\,\widehat{x}_{k+1}\right). \tag{104}$$

9. By [12, eq. (4.15)] we obtain

$$\left(2\,\sin\alpha\right)^{n/3}+\left(2\,\sin 2\alpha\right)^{n/3}+\left(2\,\sin 4\alpha\right)^{n/3}= \\ =\sqrt[3]{z_{n-1}+6\,(-1)^n\,7^{n/6}-\frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_{n,9}+\sqrt{\mathcal{T}_{n,9}}}+\sqrt[3]{\mathcal{S}_{n,9}-\sqrt{\mathcal{T}_{n,9}}}\,\right)}, \tag{105}$$

where

$$\mathcal{S}_{n,9}=\left(\tfrac{1}{2}\,z_{n-1}+3\,(-1)^n\,7^{n/6}\right)\left(z_{2n-1}-z_{n-1}^2\right)- \\ -6\cdot 7^{n/3}\,z_{n-1}-9\,(-1)^n\,7^{n/2}, \tag{106}$$

$$\mathcal{T}_{n,9}=(-1)^n\,7^{n/2}\,z_{n-1}\left(5\,z_{n-1}^2-9\,z_{2n-1}\right)+ \\ +\tfrac{1}{4}\left(2\,z_{2n-1}-z_{n-1}^2\right)\left(z_{n-1}^2-z_{2n-1}\right)^2-27\cdot 7^n, \tag{107}$$

$$p_{n,9}=-z_{n-1},\quad q_{n,9}=\tfrac{1}{2}\left(z_{n-1}^2-z_{2n-1}\right),\quad r_{n,9}=(-1)^{n-1}\,7^{n/2}. \tag{108}$$

10. By [12, eq. (2.2)] we have (for $\delta\in\mathbb{R}$):

$$\left(1+2\,\delta\,\cos\alpha\right)^{n/3}+\left(1+2\,\delta\,\cos 2\alpha\right)^{n/3}+\left(1+2\,\delta\,\cos 4\alpha\right)^{n/3}= \\ =\left[\mathcal{A}_n(\delta)+6\,\widehat{\delta}^{\,n/3}-\frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_n(\delta)+\sqrt{\mathcal{T}_n(\delta)}}+\sqrt[3]{\mathcal{S}_n(\delta)-\sqrt{\mathcal{T}_n(\delta)}}\,\right)\right]^{1/3}, \tag{109}$$

where

$$\widehat{\delta}=\delta^3-2\,\delta^2-\delta+1, \tag{110}$$

$$\mathcal{S}_n(\delta)=-\mathcal{A}_n(\delta)\,\mathcal{B}_n(\delta)-6\,\mathcal{B}_n(\delta)\,\widehat{\delta}^{\,n/3}-6\,\mathcal{A}_n(\delta)\,\widehat{\delta}^{\,2n/3}-9\,\widehat{\delta}^{\,n}, \tag{111}$$

$$\mathcal{T}_n(\delta)=\mathcal{A}_n^2(\delta)\,\mathcal{B}_n^2(\delta)-4\,\mathcal{A}_n^3(\delta)\,\widehat{\delta}^{\,n}-4\,\mathcal{B}_n^3(\delta)+18\,\mathcal{A}_n(\delta)\,\mathcal{B}_n(\delta)\,\widehat{\delta}^{\,n}-27\,\widehat{\delta}^{\,2n} \tag{112}$$

$$=\left(\mathcal{A}_n(\delta)\,\mathcal{B}_n(\delta)+9\,\widehat{\delta}^{\,n}\right)^2-4\left(\mathcal{A}_n^3(\delta)\,\widehat{\delta}^{\,n}+\mathcal{B}_n^3(\delta)\right)-108\,\widehat{\delta}^{\,2n}. \tag{113}$$

11. By [12, eq. (6.14)] we have

$$7^{n/6}\left((\cot\alpha)^{n/3}+(\cot 2\alpha)^{n/3}+(\cot 4\alpha)^{n/3}\right)= \\ =\left[3^n\,\mathcal{A}_n(\tfrac{2}{3})+6\,(-1)^n\,7^{n/3}+\frac{3}{\sqrt[3]{2}}\,7^{n/3}\left(\sqrt[3]{\mathcal{S}_n'+\sqrt{\mathcal{T}_n'}}+\sqrt[3]{\mathcal{S}_n'-\sqrt{\mathcal{T}_n'}}\,\right)\right]^{1/3}, \tag{114}$$

where

$$\mathcal{S}_n'=\left(3^n\,\mathcal{A}_n(\tfrac{2}{3})+6\,(-1)^n\,7^{n/3}\right)\Omega_n\left(\tfrac{2\,i}{\sqrt{7}}\right)+6\left(\tfrac{3}{\sqrt[3]{7}}\right)^n\mathcal{A}_n(\tfrac{2}{3})+9\,(-1)^n, \tag{115}$$

$$\mathcal{T}_n'=\left((-3)^n\,\mathcal{A}_n(\tfrac{2}{3})\,\Omega_n\left(\tfrac{2\,i}{\sqrt{7}}\right)+9\right)^2-4\left(7^n\,\Omega_n^3\left(\tfrac{2\,i}{\sqrt{7}}\right)+\left(-\tfrac{27}{7}\right)^n\mathcal{A}_n^3(\tfrac{2}{3})\right)-108. \tag{116}$$

The numbers $\Omega_n(\delta)$ are defined for $n\in\mathbb{N}$ and $\delta\in\mathbb{C}$, in the following way

$$\Omega_n(\delta):=\left(1+2\,i\,\delta\,\sin\alpha\right)^n+\left(1+2\,i\,\delta\,\sin 2\alpha\right)^n+\left(1+2\,i\,\delta\,\sin 4\alpha\right)^n,$$

(see [12, Section 6] for more details).

Remark 11. Moreover, we have

$$\left(\mathbb{X}-(\tan\alpha)^n\right)\left(\mathbb{X}-(\tan 2\alpha)^n\right)\left(\mathbb{X}-(\tan 4\alpha)^n\right)= \\ =\mathbb{X}^3-(-\sqrt{7})^n\,\Omega_n\left(\tfrac{2\,i}{\sqrt{7}}\right)\mathbb{X}^2+(-3)^n\,\mathcal{A}_n(\tfrac{2}{3})\,\mathbb{X}-(-\sqrt{7})^n. \tag{117}$$

This "distribution" easily results from [12, eq. (6.14)]. Now we will present a direct proof of the relation (117), because the formula (6.14) in [12] is presented without a proof. For this purpose, let us suppose that $\xi=\exp(i\,2\,\pi/7)$. Then we have

$$(\tan\alpha)^n+(\tan 2\alpha)^n+(\tan 4\alpha)^n= \\ =\left(-i\,\frac{\xi-\xi^6}{\xi+\xi^6}\right)^n+\left(-i\,\frac{\xi^2-\xi^5}{\xi^2+\xi^5}\right)^n+\left(-i\,\frac{\xi^4-\xi^3}{\xi^4+\xi^3}\right)^n= \\ =\left(\frac{-i}{(\xi+\xi^6)(\xi^2+\xi^5)(\xi^4+\xi^3)}\right)^n\left[\left((\xi-\xi^6)(\xi^2+\xi^5)(\xi^4+\xi^3)\right)^n+\right. \\ \left.+\left((\xi^2-\xi^5)(\xi+\xi^6)(\xi^4+\xi^3)\right)^n+\left((\xi^4-\xi^3)(\xi+\xi^6)(\xi^2+\xi^5)\right)^n\right]=$$

$$\overset{[12,\ \text{eq. (1.4)}]}{=}(-i)^n\left[\left(2\,(\xi^2-\xi^5)-(\xi-\xi^6)-(\xi^2-\xi^5)-(\xi^4-\xi^3)\right)^n+\right. \\ +\left(2\,(\xi^4-\xi^3)-(\xi-\xi^6)-(\xi^2-\xi^5)-(\xi^4-\xi^3)\right)^n+ \\ \left.+\left(2\,(\xi-\xi^6)-(\xi-\xi^6)-(\xi^2-\xi^5)-(\xi^4-\xi^3)\right)^n\right]= \\ \overset{[12,\ \text{eq. (1.1)}]}{=}\left[\left(-2\,i\,(\xi^2-\xi^5)-\sqrt{7}\right)^n+\left(-2\,i\,(\xi^4-\xi^3)-\sqrt{7}\right)^n+\right. \\ \left.+\left(-2\,i\,(\xi-\xi^6)-\sqrt{7}\right)^n\right]=(-\sqrt{7})^n\,\Omega_n\left(\tfrac{2\,i}{\sqrt{7}}\right),$$

and

$$\big(\tan\alpha\,\tan 2\alpha\big)^n+\big(\tan\alpha\,\tan 4\alpha\big)^n+\big(\tan 2\alpha\,\tan 4\alpha\big)^n=$$

$$=\Big(\Big(-i\,\frac{\xi-\xi^6}{\xi+\xi^6}\Big)\Big(-i\,\frac{\xi^2-\xi^5}{\xi^2+\xi^5}\Big)\Big)^n+\Big(\Big(-i\,\frac{\xi-\xi^6}{\xi+\xi^6}\Big)\Big(-i\,\frac{\xi^4-\xi^3}{\xi^4+\xi^3}\Big)\Big)^n+$$

$$+\Big(\Big(-i\,\frac{\xi^2-\xi^5}{\xi^2+\xi^5}\Big)\Big(-i\,\frac{\xi^4-\xi^3}{\xi^4+\xi^3}\Big)\Big)^n=$$

$$=\Big((\xi+\xi^6)(\xi^2+\xi^5)(\xi^4+\xi^3)\Big)^{-n}\Big[\Big((\xi^6-\xi)(\xi^2-\xi^5)(\xi^4+\xi^3)\Big)^n+$$

$$+\Big((\xi^6-\xi)(\xi^4-\xi^3)(\xi^2+\xi^5)\Big)^n+\Big((\xi^5-\xi^2)(\xi^4-\xi^3)(\xi+\xi^6)\Big)^n\Big]=$$

$$=\Big(-3-2\,(\xi+\xi^6)\Big)^n+\Big(-3-2\,(\xi^4+\xi^3)\Big)^n+\Big(-3-2\,(\xi^2+\xi^5)\Big)^n=(-3)^n\,\mathcal{A}_n(\tfrac{2}{3}).$$

Final remark. I was only after I received the referee report on my paper that I learnt about two important publications in this field [8, 9]. Certainly, both papers supplement and enrich the contents of Section 3. As a spontaneous reaction to [8] and the report on the present paper, two more papers sprang up [14] and [15].

6 Acknowledgments

The author wish to express their gratitude to the Reviewer for several helpful comments concerning the first version of my paper.

Table 1:

n	0	1	2	3	4	5	6	7	8	9	10	11
ψ_n	−1	0	−3	2	−8	9	−23	33	−70	113	−220	376
φ_n	0	−1	1	−3	4	−9	14	−28	47	−89	155	−286

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2000 *Mathematics Subject Classification*: Primary 11B37; Secondary 11B83, 11Y55, 33B10.
Keywords: Ramanujan equalities, trigonometric recurrences.

(Concerned with sequences A094648 and A006053.)

Received July 6 2009; Revised version received December 3 2009. Published in *Journal of Integer Sequences*, December 3 2009.

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