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Some Postage Stamp 2-Bases

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Abstract

A set of k integers is a 2-basis if every positive integer up to n can be expressed as the sum of no more than 2 values from the set; an extremal 2-basis is one for which n is as large as possible. A new algorithm extends the lower bound of Mossige for symmetric bases. An assumed modulo structure is combined with local search. These 2-bases match all known extremal values for k from 11 to 20. Bases out to $k = 82$ are given.

1 Introduction

The global postage-stamp problem (GPSP) consists of determining, for given positive integers h and k, a set of k positive integers

$$A_k = \{1 = a_1 < \cdots < a_k\}$$

such that

1. sums of h (or fewer) of these a_j can realize the numbers $1, 2, \ldots, n$,
2. the value of n is as large as possible.

This extremal total is denoted $n_h(k)$. The h-range for a particular set A is denoted $n_h(A)$. Informally, h denotes the maximum number of stamps on an envelope, while k is the number of stamp denominations.

Mossige [1] presented efficient search algorithms for determining $n_h(k)$, which Shallit [4] has shown to be NP-hard. The extremal GPSP for $k = 3$ has been solved for any h and Mossige has constructed very good h-bases for $k = 4$ which are believed to be extremal [2, 3].

This note considers only the $h = 2$ case. Most of the known extremal GPSP 2-bases are symmetric, i.e., $a_i + a_{k-i} = a_k$ for $i = 1, \ldots, k$. For some values of k there are several extremal 2-bases solutions. For $k = 11$, there are 4 solutions, two of them being symmetric. The instance $k = 10$ is the only known case with no symmetric extremal solution.

Mossige [1] gave extremal symmetric 2-bases for k from 15 to 30 (Table 4). To date, these out to $k = 20$ have been shown to be globally extremal (see A001212 of Sloane's table [5]). Our search algorithm uses some properties of these bases.

2 Construction

We consider only the $h = 2$ case and will generate symmetric 2-bases, i.e. $a_i + a_{k-i} = a_k$ for $i = 1, \ldots, k$. Let s denote a repeated central difference following an initial preamble, e.g., row 4 of Table 4 [1], $k = 18$, has the first half difference sequence $1, 2, 1, 1, 3, 6, 6, 6, 6, 6$, and we say $s = 6$. The complete difference sequence is k elements long with the second half the reverse of the first half.

We refer to s as the stride of the bases. The initial s elements a_i we call the preamble; the central basis elements a_i corresponding to the differences of s we call the amble. Mossige [1] has the same 6 element preamble $(1, 2, 1, 1, 3, 6)$ for the first 7 values of k; the amble just gets longer as k increases.

From Table 4 [1] we observe the following for these globally extremal bases:

Property 1: The first $s - 1$ elements a_j have distinct nonzero residues modulo s.

Property 2: Element a_s repeats a nonzero residue modulo s.

Property 3: The first s elements match any following sequence of difference values s.

Preamble construction:

1. Assume a positive integer value for s.
2. Enumerate all possible s element 2-range preambles with properties 1 and 2.
3. Report those sets with property 3 with maximum a_s.

It is easy to show that this construction is an algorithm. Property 1 is combined with admissibility to prune the combinations. Table 1 lists the experiential number of preamble cases that were examined. This number grows by a factor of about 4 for each increment in s. We call a resulting preamble a *PA* which has two parameters s and a_s.

Table 1: Number of candidate preamble cases for various stride lengths s.

Stride s	Cases	Stride s	Cases
11	192	18	1,044,846
12	634	19	3,822,468
13	1,654	20	17,365,943
14	6,277	21	69,076,273
15	18,757	22	334,698,203
16	73,775	23	1,438,317,540
17	246,196	24	7,367,635,861

From a PA we can construct a basis for $k \geq 2s$. The PA and its reversal have $2s$ elements; thus the length of the amble is $k - 2s$. The largest element a_k is equal to the sum of all the differences. For a symmetric basis, $n_2(A) = 2a_k$, and

$$n_2(A) = 2ks + 4a_s - 4s^2 \text{ for } k \geq 2s.$$

Note that this 2-range grows as $4a_s$. Our construction yields all known extremal GPSP 2-bases for $k > 11$ [5]. Representative most efficient PAs are listed in Table 2.

Table 2: Most efficient PAs for $s = 11, \ldots, 24$

s	$\{a_s\}$
11	$\{1, 3, 4, 7, 8, 9, 16, 17, 21, 24, 35\}$
13	$\{1, 2, 5, 7, 10, 11, 19, 21, 22, 25, 29, 30, 43\}$
15	$\{1, 2, 5, 6, 8, 9, 13, 19, 22, 27, 29, 33, 40, 41, 56\}$
16	$\{1, 2, 5, 8, 10, 12, 19, 22, 23, 25, 30, 31, 36, 43, 45, 61\}$
19	$\{1, 2, 3, 6, 9, 11, 12, 15, 16, 27, 32, 37, 45, 48, 52, 55, 61, 62, 80\}$
20	$\{1, 2, 4, 5, 11, 13, 14, 19, 29, 35, 37, 43, 46, 47, 50, 52, 56, 58, 68, 88\}$
21	$\{1, 2, 3, 6, 10, 14, 17, 19, 26, 29, 36, 41, 49, 51, 54, 55, 58, 60, 67, 74, 95\}$
22	$\{1, 3, 5, 7, 8, 12, 14, 18, 26, 32, 33, 42, 43, 50, 60, 63, 68, 79, 81, 83, 97, 105\}$
24	$\{1, 2, 3, 5, 9, 12, 15, 17, 23, 28, 32, 35, 37, 44, 45, 66, 79, 82, 86, 91, 94, 102, 112, 118\}$

3 Observations

Some values of s yield poor bases, e.g. $s = 7$ is bested by either $s = 6$ or $s = 8$. Other values have singular points e.g. $s = 10$ has 2 symmetric bases [1] which tie $s = 9$ and $s = 11$ at $k = 30$. In Table 3 we report a best s for k from 30 to about 82. The corresponding PAs are in Table 2.

The bases constructed from Table 2, for the ranges in Table 3, are all above the $2k^2/7$ lower bound construction of Mossige [1]. Our bases suggest extremal $n_2(k)$ of about $5k^2/16$.

Table 3: Best $n_2(A)$ for the symmetric bases

s	a_s	k range	$n_2(A)$
11	35	30-40	$22k - 344$
13	43	40-43	$26k - 504$
15	56	43-50	$30k - 676$
16	61	50-58	$32k - 780$
19	80	58-62	$38k - 1124$
20	88	62-66	$40k - 1248$
21	95	66-68	$42k - 1380$
22	105	68-79	$44k - 1516$
24	118	79-82	$48k - 1832$

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