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On the Expansion of Fibonacci and Lucas Polynomials

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Abstract

Recently, Belbachir and Bencherif have expanded Fibonacci and Lucas polynomials using bases of Fibonacci- and Lucas-like polynomials. Here, we provide simplified proofs for the expansion formulæthat in essence a computer can do. Furthermore, for 2 of the 5 instances, we find q-analogues.

1 Introduction

In [2], Belbachir and Bencherif studied the Fibonacci and Lucas polynomials:

$$U_0 = 0,\ U_1 = 1,\ U_n = xU_{n-1} + yU_{n-2},$$
$$V_0 = 2,\ V_1 = x,\ V_n = xV_{n-1} + yV_{n-2}.$$

We prefer the modified polynomials

$$u_0 = 0,\ u_1 = 1,\ u_n = u_{n-1} + zu_{n-2},$$
$$v_0 = 2,\ v_1 = 1,\ v_n = v_{n-1} + zv_{n-2},$$

so that

$$U_n(x,y) = x^{n-1}u_n\Big(\frac{y}{x^2}\Big), \quad V_n(x,y) = x^n v_n\Big(\frac{y}{x^2}\Big).$$

Then, with

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+4z}}{2},$$

$$u_n = \frac{1}{\sqrt{1+4z}}(\lambda_1^n - \lambda_2^n), \qquad v_n = \lambda_1^n + \lambda_2^n.$$

Substituting $z = t/(1-t)^2$, these formulæ become particularly nice:

$$u_n = \frac{1-(-t)^n}{(1+t)(1-t)^{n-1}}, \qquad v_n = \frac{1+(-t)^n}{(1-t)^n}.$$

The main result of [2] are the following 5 formulæ:

$$2u_{2n+1} = \sum_{k=0}^{n} a_{n,k} v_{2n-k}, \qquad a_{n,k} = 2\sum_{j=0}^{n} (-1)^{j+k}\binom{j}{k} - (-1)^{n+k}\binom{n}{k}. \tag{1}$$

$$u_{2n} = \sum_{k=1}^{n} b_{n,k} u_{2n-k}, \qquad b_{n,k} = (-1)^{k+1}\binom{n}{k}. \tag{2}$$

$$v_{2n-1} = \sum_{k=1}^{n} c_{n,k} u_{2n-k}, \qquad c_{n,k} = 2(-1)^{k+1}\binom{n}{k} - [k=1]. \tag{3}$$

$$2v_{2n-1} = \sum_{k=1}^{n} d_{n,k} v_{2n-1-k}, \qquad d_{n,k} = (-1)^{k+1}\frac{2n-k}{n}\binom{n}{k}. \tag{4}$$

$$2u_{2n} = \sum_{k=1}^{n} e_{n,k} v_{2n-1-k}, \tag{5}$$

$$e_{n,k} = (-1)^{k+1}\frac{2n-k}{2n}\binom{n}{k} + \sum_{j=0}^{n-1}(-1)^{j+k-1}\binom{j}{k-1} - \frac{1}{2}(-1)^{n+k}\binom{n-1}{k-1}.$$

But the proofs of all these, using the simple forms for u_n and v_n, can be done by a computer! To give the reader an idea, let us do the last one, which seems to be the most complicated:

$$\begin{aligned}
\sum_{k=1}^{n} e_{n,k} v_{2n-1-k} &= \sum_{k=1}^{n} (-1)^{k+1}\frac{2n-k}{2n}\binom{n}{k} v_{2n-1-k} \\
&\quad + \sum_{j=0}^{n-1}\sum_{k=1}^{j+1}(-1)^{j+k-1}\binom{j}{k-1} v_{2n-1-k} - \sum_{k=1}^{n}\frac{1}{2}(-1)^{n+k}\binom{n-1}{k-1} v_{2n-1-k} \\
&= \frac{1-t^{2n-1}}{(1-t)^{2n-1}} + \frac{1+t^{2n-1}}{(1-t)^{2n-2}(1+t)} - \frac{(-1)^n t^{n-1}}{(1-t)^{2n-2}} + \frac{(-1)^n t^{n-1}}{(1-t)^{2n-2}} \\
&= \frac{2(1-t^{2n})}{(1-t)^{2n-1}(1+t)} = 2u_{2n}.
\end{aligned}$$

The other proofs are similar/easier:

$$\begin{aligned}
\sum_{k=0}^{n} a_{n,k} v_{2n-k} &= \frac{2[(-t)^n(1+t) + 1 + t^{2n+1}]}{(1-t)^{2n}(1+t)} - \frac{2(-t)^n}{(1-t)^{2n}} \\
&= \frac{2[1+t^{2n+1}]}{(1-t)^{2n}(1+t)} = 2u_{2n+1}.
\end{aligned}$$

$$\sum_{k=1}^{n} c_{n,k} u_{2n-k} = \frac{2(1-t^{2n})}{(1-t)^{2n-1}(1+t)} - \frac{1+t^{2n-1}}{(1+t)(1-t)^{2n-2}}$$
$$= \frac{1-t^{2n-1}}{(1-t)^{2n-1}} = v_{2n-1}.$$

Remark 1. The polynomials $u_n(x)$ and $v_n(x)$ are essentially Chebyshev polynomials. The authors of [2] have also published a companion paper [3]; according to a remark in [2], the results in [2] are more general than the ones in [3].

2 q-analogues

Now we are interested in q-analogues. For this, we replace u_n by

$$\mathrm{Fib}_n = \sum_{0 \le k \le \frac{n-1}{2}} q^{\binom{k+1}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q z^k$$

and v_n by

$$\mathrm{Luc}_n = \sum_{0 \le k \le \frac{n}{2}} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{[n]_q}{[n-k]_q} z^k,$$

as suggested by Cigler [4]. We use standard q-notation here:

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q [2]_q \ldots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!},$$

compare [1]; the notions of the Introduction are the special instance $q = 1$.

Theorem 2.

$$\mathrm{Luc}_{2n-1} = \sum_{k=1}^{n} d_{n,k} \mathrm{Luc}_{2n-1-k},$$

with

$$d_{n,k} = (-1)^{k-1} \frac{q^{\binom{k}{2}}}{1+q^{n-1}} \left(\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \right).$$

Proof. We must prove that

$$\sum_{0 \le k \le n-1} q^{\binom{k}{2}} \begin{bmatrix} 2n-1-k \\ k \end{bmatrix}_q \frac{[2n-1]_q}{[2n-1-k]_q} z^k$$
$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{q^{\binom{j}{2}}}{1+q^{n-1}} \left(\begin{bmatrix} n-1 \\ j \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \right)$$
$$\times \sum_{0 \le k \le \frac{2n-j-1}{2}} q^{\binom{k}{2}} \begin{bmatrix} 2n-j-1-k \\ k \end{bmatrix}_q \frac{[2n-j-1]_q}{[2n-j-1-k]_q} z^k.$$

Comparing coefficients, we have to prove that

$$q^{\binom{k}{2}} \begin{bmatrix} 2n-1-k \\ k \end{bmatrix}_q \frac{[2n-1]_q}{[2n-1-k]_q}$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{q^{\binom{j}{2}}}{1+q^{n-1}} \left(\begin{bmatrix} n-1 \\ j \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \right) q^{\binom{k}{2}} \begin{bmatrix} 2n-j-1-k \\ k \end{bmatrix}_q \frac{[2n-j-1]_q}{[2n-j-1-k]_q}.$$

Simplifying, we must prove that

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \left(\begin{bmatrix} n-1 \\ j \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \right) \begin{bmatrix} 2n-j-2-k \\ k-1 \end{bmatrix}_q [2n-j-1]_q = 0.$$

Another form of this is

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \left(1 - q^{2n-1} - q^{n-j} + q^{n-1}\right) \left(1 - q^{2n-j-1}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} 2n-j-2-k \\ k-1 \end{bmatrix}_q = 0.$$

Notice that

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{-aj} = 0$$

for $0 \le a \le n-1$. This follows from *Rothe's* formula [1, p. 490]

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j = (1-x)(1-xq)\dots(1-q^{n-1}).$$

We write the desired identity as

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \left(A + Bq^{-j} + Cq^{-2j}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q \left(D_0 q^{-0} + \dots + D_{k-1} q^{-j(k-1)}\right) = 0.$$

Therefore, for $k \le n-2$, the identity holds. For $k = n-1$,

$$\sum_{j=0}^{1} (-1)^j q^{\binom{j}{2}} \left(1 - q^{2n-1} - q^{n-j} + q^{n-1}\right) \left(1 - q^{2n-j-1}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ n-2 \end{bmatrix}_q = 0$$

can be shown by inspection, and for $k = n$, the identity holds, since the sum is empty. □

Theorem 3.

$$\mathrm{Fib}_{2n} = \sum_{k=1}^{n} b_{n,k} \mathrm{Fib}_{2n-k}$$

with

$$b_{n,k} = (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Proof. We must prove that

$$\sum_{0\le k\le n-1} q^{\binom{k+1}{2}} \begin{bmatrix} 2n-k-1 \\ k \end{bmatrix}_q z^k$$

$$= \sum_{j=1}^{n} (-1)^{j-1} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \sum_{0\le k\le \frac{2n-j-1}{2}} q^{\binom{k+1}{2}} \begin{bmatrix} 2n-j-k-1 \\ k \end{bmatrix}_q z^k.$$

Comparing coefficients, this means

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} 2n-j-k-1 \\ k \end{bmatrix}_q = 0,$$

which follows by a similar but simpler argument than before. □

3 Conclusion

We found 2 q-analogues; for the remaining 3 instances we were not successful and leave this as a challenge for anybody who is interested.

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