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Some Congruences for the Partial Bell Polynomials

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Abstract

Let $B_{n,k}$ and $A_n = \sum_{j=1}^{n} B_{n,j}$ with $A_0 = 1$ be, respectively, the $(n,k)^{\text{th}}$ partial and the n^{th} complete Bell polynomials with indeterminate arguments $x_1, x_2, \ldots$. Congruences for A_n and $B_{n,k}$ with respect to a prime number have been studied by several authors. In the present paper, we propose some results involving congruences for $B_{n,k}$ when the arguments are integers. We give a relation between Bell polynomials and we apply it to several congruences. The obtained congruences are connected to binomial coefficients.

1 Introduction

Let $x_1, x_2, \ldots$ denote indeterminates. Recall that the partial Bell polynomials $B_{n,k}(x_1, x_2, \ldots)$ are given by

$$B_{n,k}(x_1, x_2, \ldots) = \sum \frac{n!}{k_1! k_2! \cdots} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{k_{n-k+1}}, \qquad (1)$$

where the summation takes place over all integers $k_1, k_2, \ldots \geq 0$ such that

$$k_1 + 2k_2 + \cdots + (n-k+1)\, k_{n-k+1} = n \quad \text{and} \quad k_1 + k_2 + \cdots + k_{n-k+1} = k.$$

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For references, see Bell [1], Comtet [4] and Riordan [7].

Congruences for Bell polynomials have been studied by several authors. Bell [1] and Carlitz [3] give some congruences for complete Bell polynomials. In this paper, we propose some congruences for partial Bell polynomials when the arguments are integers. Indeed, we give a relation between Bell polynomials, given by Theorem 1 below, and we use it in the first part of the paper, and with connection of the results of Carlitz [3] in the second part, to deduce some congruences for partial Bell polynomials. Some applications to Stirling numbers of the first and second kind and to the binomial coefficients are given.

2 Main results

The next theorem gives an interesting relation between Bell polynomials. We use it to establish some congruences for partial Bell polynomials.

Theorem 1. *Let $\{x_n\}$ be a real sequence. Then for n, r, k integers with $n, r, k \geq 1$, we have*

$$x_1^k \sum_{j=1}^{n} B_{n,j}\left(y_1, y_2, \ldots\right)\left(k - nr\right)^{j-1} = x_1^{nr} \frac{B_{n+k,k}\left(x_1, x_2, x_3, \ldots\right)}{k\binom{n+k}{k}} \tag{2}$$

with $y_n = \dfrac{B_{(r+1)n,nr}\left(x_1, x_2, x_3, \ldots\right)}{nr\binom{(r+1)n}{nr}}$.

For $k = nr + s$, Identity (2) becomes

Remark 2. Let $\{x_n\}$ be a real sequence. Then for n, r, s integers with $n, r \geq 1$, we get

$$x_1^s A_n\left(sy_1, sy_2, \ldots\right) = \frac{s}{nr+s} \frac{B_{(r+1)n+s,nr+s}\left(x_1, x_2, x_3, \ldots\right)}{\binom{(r+1)n+s}{nr+s}}, \quad s \geq -nr+1. \tag{3}$$

For $s \geq 0$, we obtain Proposition 8 in [5], (see also [6]).

Theorem 3. *Let k, s be a nonnegative integers and p be a prime number. Then for any sequence $\{x_j\}$ of integers we have*

$$\left(k + s + 1\right) B_{sp,k+s+1}\left(x_1, x_2, \ldots\right) \equiv 0 \pmod{p}.$$

Application 4. If we denote by $s\left(n, k\right)$ and $S\left(n, k\right)$ for Stirling numbers of first and second kind respectively, then from the well-known identities

$$B_{n,k}\left(0!, -1!, 2!, \ldots\right) = s\left(n, k\right) \quad \text{and} \quad B_{n,k}\left(1, 1, 1, \ldots\right) = S\left(n, k\right)$$

when $x_n = 1$ or $x_n = \left(-1\right)^{n-1}\left(n-1\right)!$ in Theorem 3 we obtain

$$\left(k + s + 1\right) S\left(sp, k + s + 1\right) \equiv \left(k + s + 1\right) s\left(sp, k + s + 1\right) \equiv 0 \pmod{p}.$$

Theorem 5. *Let n, k, s be integers with $n \geq k \geq 1$, $s \geq 1$ and p be a prime number. Then for any sequence $\{x_j\}$ of integers with x_1 not a multiple of p we have*

$$\begin{aligned} \frac{B_{n+sp,k+sp}(x_1,x_2,x_3,\ldots)}{(k+sp)\binom{n+sp}{k+sp}} &\equiv x_1^s \frac{B_{n,k}(x_1,x_2,x_3,\ldots)}{k\binom{n}{k}} \pmod p \quad \text{if } p > n-k+1 \\ x_1^n \frac{B_{n+sp,sp}(x_1,x_2,x_3,\ldots)}{s\binom{n+sp}{sp}} &\equiv x_1^s \frac{B_{(p+1)n,np}(x_1,x_2,x_3,\ldots)}{n\binom{(p+1)n}{np}} \pmod{p^2} \quad \text{if } p > n+1. \end{aligned} \tag{4}$$

Application 6. If we consider the cases $k = 1$ and $k = 2$ in Theorem 5 we obtain

$$B_{n+sp,1+sp}(x_1,x_2,x_3,\ldots) \equiv x_1^s x_n \pmod p \text{ for } p > n,$$

$$B_{n+sp,2+sp}(x_1,x_2,x_3,\ldots) \equiv \frac{x_1^s}{2}\sum_{j=1}^{n-1}\binom{n}{j}x_j x_{n-j} \pmod p \text{ for } p > n-1.$$

Then, when $x_n = 1$ or $x_n = (-1)^{n-1}(n-1)!$ we obtain

$$S(n+sp,1+sp) \equiv 1 \pmod p \text{ for } p > n,$$

$$s(n+sp,1+sp) \equiv (-1)^{n-1}(n-1)! \pmod p \text{ for } p > n,$$

$$S(n+sp,2+sp) \equiv 2^{n-1}-1 \pmod p \text{ for } p > n-1 \text{ and}$$

$$s(n+sp,2+sp) \equiv (-1)^{n-1}\tfrac{n(n+1)^2}{2} \pmod p \text{ for } p > n-1,$$

Theorem 7. *Let n, k, s, p be integers with $n \geq k \geq 1$, $s \geq 1$, $p \geq 1$. Then for any sequence $\{x_j\}$ of integers with x_1 not a multiple of p we have*

$$\begin{aligned} \frac{B_{(s+1)n,sn}(x_1,2x_2,3x_3,\ldots)}{\binom{(s+1)n}{sn}} &\equiv s x_1^{n(s-1)} \frac{B_{2n,n}(x_1,2x_2,3x_3,\ldots)}{\binom{2n}{n}} \pmod{n^2}, \\ \frac{B_{n+sp,k+sp}(x_1,2x_2,3x_3,\ldots)}{(k+sp)\binom{n+sp}{k+sp}} &\equiv x_1^s \frac{B_{n,k}(x_1,2x_2,3x_3,\ldots)}{k\binom{n}{k}} \pmod p, \\ x_1^n \frac{B_{n+sp,sp}(x_1,2x_2,3x_3,\ldots)}{s\binom{n+sp}{sp}} &\equiv x_1^s \frac{B_{(p+1)n,np}(x_1,2x_2,3x_3,\ldots)}{n\binom{(p+1)n}{np}} \pmod{n^2}. \end{aligned} \tag{5}$$

Application 8. Belbachir et al. [2] have proved that

$$B_{n,k}(1!,2!,\ldots,(q+1)!,0,\ldots) = \frac{n!}{k!}\binom{k}{n-k}_q, \tag{6}$$

then, for $s \geq 1$ and $p \nmid j$, the two last congruences of (5) and Identity (6) prove that

$$\binom{k+sp}{j}_q \equiv \binom{k}{j}_q \pmod p \quad \text{and} \quad j\binom{sp}{j}_q \equiv s\binom{jp}{j}_q \pmod{p^2}.$$

Corollary 9. *Let n, k, s be integers with $n \geq k \geq 1$ and p be a prime number. Then for any sequence $\{x_j\}$ of integers with x_1 not a multiple of p we have*

$$\frac{B_{(p+1)n,np}\left(x_1, x_2, \ldots\right)}{n\binom{(p+1)n}{np}} \equiv x_1^{n-1} \frac{B_{n+p,p}\left(x_1, x_2, x_3, \ldots\right)}{\binom{n+p}{p}} \pmod{p^2} \quad \text{if } p > n+1,$$

$$\frac{B_{(p+1)n,np}\left(x_1, 2x_2, 3x_3, \ldots\right)}{n\binom{(p+1)n}{np}} \equiv x_1^{n-1} \frac{B_{n+p,p}\left(x_1, 2x_2, 3x_3, \ldots\right)}{\binom{n+p}{p}} \pmod{p^2}.$$

Application 10. As in Application 8, we have

$$\binom{jp}{j}_q \equiv j \binom{p}{j}_q \pmod{p^2}.$$

Theorem 11. *Let $k \geq 2$, $j \geq 1$ be integers and p be an odd prime number. Then for any sequence of integers $\{x_j\}$ we have*

$$\begin{aligned} \frac{B_{p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{p^j+k}{k}} &\equiv x_1^{k-1} x_{p^j+1} \pmod{p} \quad \text{if } p \nmid kx_1, \\ \frac{B_{(r+1)p^j,p^j r}\left(x_1, 2x_2, 3x_3, \ldots\right)}{p^j r\binom{(r+1)p^j}{p^j r}} &\equiv x_1^{r-1}\left(x_{p^j+1} - x_{p^{j-1}+1}\right) \pmod{p} \quad \text{if } p \nmid x_1. \end{aligned} \tag{7}$$

Application 12. As in Application (8), let $j = 1$ in the second congruence of Theorem 11. Then

$$\frac{(p-1)!}{r}\binom{pr}{p}_q \equiv -1 \pmod{p}.$$

Theorem 13. *Let $k \geq 2$, $j \geq 1$ be integers and p be an odd prime number. Then for any sequence of integers $\{x_j\}$ we have*

$$\frac{B_{2p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{2p^j+k}{k}} \equiv x_1^{k-2}\left((k-1)\, x_{p^j+1}^2 + x_1 x_{2p^j+1}\right) \pmod{p} \quad \text{if } p \nmid kx_1$$

$$\frac{B_{2(r+1)p^j,2p^j r}\left(x_1, 2x_2, 3x_3, \ldots\right)}{2p^j r\binom{2(r+1)p^j}{2p^j r}} \equiv x_1^{2r-2}\left(x_1 x_{2p^j+1} - x_{p^j+1}^2\right) \pmod{p} \quad \text{if } p \nmid x_1.$$

Remark 14. Similarly to the last proofs, one can exploit the results of Carlitz [3] with connection to Theorem 1 to obtain more congruences for partial Bell polynomials.

3 Proof of the main results

Proof of Theorem 1. Let $\{x_n\}$ be a sequence of real numbers with $x_1 := 1$ and let $\{f_n(x)\}$ be a sequence of polynomials defined by

$$f_n(x) = \sum_{j=1}^{n} B_{n,j}\left(\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4} \ldots\right)(x)_j\,,$$

with $f_0(x)=1$, $(x)_j := x(x-1)\cdots(x-j+1)$ for $j\geq 1$ and $(x)_0 := 1$.

We have $nf_{n-1}(1) = n\sum_{j=1}^{n-1} B_{n-1,j}\left(\frac{x_2}{2},\frac{x_3}{3},\ldots\right)(1)_j = x_n$ and $D_{x=0}f_1(0)=1\neq 0$.

It is well known that $\{f_n(x)\}$ presents a sequence of binomial type, see [4]. Then, from Proposition 1 in [5] we have

$$y_n = \frac{1}{nr\binom{(r+1)n}{nr}} B_{(r+1)n,nr}(1,x_2,x_3,\ldots) = \frac{f_n(nr)}{nr} = D_{x=0}f_n(x;r), \tag{8}$$

where $\{f_n(x;a)\}$ is a sequence of binomial type defined by

$$f_n(x;a) := \frac{x}{an+x} f_n(an+x) \tag{9}$$

with a is a real number, see [5]. From Proposition 1 in [5] we have also

$$\frac{B_{n+k,k}(1,x_2,x_3,\ldots)}{k\binom{n+k}{k}} = \frac{f_n(k)}{k} = \frac{f_n(k-nr;r)}{k-nr}, \tag{10}$$

but from [8] we can write $f_n(k-nr;r)$ as

$$f_n(k-nr;r) = \sum_{j=1}^{n} B_{n,j}\left(D_{x=0}f_1(x;r), D_{x=0}f_2(x;r),\ldots\right)(k-nr)^j. \tag{11}$$

Then, by substitution (11) in (10) and by using (8) we obtain

$$\frac{B_{n+k,k}(1,x_2,x_3,\ldots)}{k\binom{n+k}{k}} = \sum_{j=1}^{n} B_{n,j}(y_1,y_2,\ldots)(k-nr)^{j-1}. \tag{12}$$

We can verify that Identity (2) is true for $x_1=0$, and, for $x_1\neq 0$ it can be derived from (12) by replacing x_n by $\frac{x_n}{x_1}$ and by using the well known identities

$$\begin{aligned} B_{n,k}(xa_1,xa_2,xa_3,\ldots) &= x^k B_{n,k}(a_1,a_2,a_3,\ldots) \quad \text{and} \\ B_{n,k}(xa_1,x^2a_2,x^3a_3,\ldots) &= x^n B_{n,k}(a_1,a_2,a_3,\ldots), \end{aligned} \tag{13}$$

where $\{a_n\}$ is any real sequence. □

Proof of Theorem 3. We prove that $kB_{sp,k}\equiv 0 \pmod p$, $k\geq s+1$. From the identities

$$\binom{sp}{j}\equiv 0 \pmod p,\ \text{for } p\nmid j \text{ and } \binom{sp}{pj}\equiv\binom{s}{j} \pmod p, \tag{14}$$

and from the recurrence relation given by

$$kB_{n,k} = \sum_j \binom{n}{j} x_j B_{n-j,k-1}$$

with $B_{n,k} := B_{n,k}(x_1, x_2, \ldots)$ and $x_j = 0$ for $j \leq 0$, we obtain

$(k+1) B_{sp,k+1} = \sum_j \binom{sp}{j} x_j B_{sp-j,k} \equiv \sum_{j=1}^{s} \binom{s}{j} x_{jp} B_{(s-j)p,k} \pmod p.$

Then, for $s = 0$, we get $kB_{0,k} \equiv 0 \pmod p$, $k \geq 0$.

For $s = 1$, we get $(k+1) B_{p,k+1} \equiv x_p B_{0,k} \equiv 0 \pmod p$, $k \geq 1$.

For $s = 2$, the last congruences imply that

$(k+1) B_{2p,k+1} \equiv 2x_p B_{p,k} + x_{2p} B_{0,k} = 0 \pmod p$, $k \geq 2$ and $p \nmid k$.

The induction on s proves that $kB_{sp,k} \equiv 0 \pmod p$ when $k \geq s+1$. □

Proof of Theorem 5. From [4] we have

$$B_{n,k}(x_1, x_2, \ldots) = \frac{n!}{(n-k)!} \sum_{j=0}^{k} B_{n-k,k-j}\left(\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots\right) \frac{x_1^j}{j!}, \quad n \geq k \geq 1. \tag{15}$$

Then, for $i \in \{1, \ldots, n\}$, the last identity and Identities (13) imply

$$t_i = ((n+1)!)^i y_i = \frac{((n+1)!)^i}{ir\binom{(r+1)i}{ir}} B_{(r+1)i,ir}(x_1, x_2, \ldots, x_{i-j+1}) =$$
$$\sum_{j=1}^{i} \frac{(ir-1)!}{(ir-j)!} x_1^{ir-j} B_{i,j}\left(\frac{(n+1)!}{2} x_2, \frac{((n+1)!)^2}{3} x_3, \ldots, \frac{((n+1)!)^{i-j}}{i-j+1} x_{i-j+1}\right),$$

from which we deduce that $t_1, \ldots, t_n$ are integers, and then $B_{n,1}(t_1, t_2, \ldots), \ldots, B_{n,n}(t_1, t_2, \ldots)$ are also integers. Therefore, by using the second identity of (13), Identity (2) becomes

$$x_1^k \sum_{j=1}^{n} B_{n,j}(t_1, t_2, \ldots)(k - nr)^{j-1} = x_1^{nr} \frac{((n+1)!)^n B_{n+k,k}(x_1, x_2, x_3, \ldots)}{k\binom{n+k}{k}}.$$

Hence, when we replace k by $\alpha + sp$ in the last identity we obtain

$$x_1^{\alpha+sp} \sum_{j=1}^{n} B_{n,j}(t_1, t_2, \ldots)(\alpha + sp - nr)^{j-1} = x_1^{nr} \frac{((n+1)!)^n B_{n+\alpha+sp,\alpha+sp}(x_1, x_2, x_3, \ldots)}{(\alpha+sp)\binom{n+\alpha+sp}{\alpha+sp}},$$

and when we reduce modulo p in the last identity we obtain

$$x_1^{nr} \frac{((n+1)!)^n B_{n+\alpha+sp,\alpha+sp}(x_1, x_2, x_3, \ldots)}{(\alpha+sp)\binom{n+\alpha+sp}{\alpha+sp}} \equiv x_1^{\alpha+s} \sum_{j=1}^{n} B_{n,j}(t_1, t_2, \ldots)(\alpha - nr)^{j-1} \pmod p.$$

But from (2) we have

$$x_1^{\alpha}\sum_{j=1}^{n}B_{n,j}(t_1,t_2,\ldots)(\alpha-nr)^{j-1}=((n+1)!)^n x_1^{\alpha}\sum_{j=1}^{n}B_{n,j}(y_1,y_2,\ldots)(\alpha-nr)^{j-1}$$
$$=((n+1)!)^n x_1^{nr}\frac{B_{n+\alpha,\alpha}(x_1,x_2,x_3,\ldots)}{\alpha\binom{n+\alpha}{\alpha}},$$

from which the last congruence becomes

$$\frac{((n+1)!)^n B_{n+\alpha+sp,\alpha+sp}(x_1,x_2,x_3,\ldots)}{(\alpha+sp)\binom{n+\alpha+sp}{\alpha+sp}}\equiv x_1^s((n+1)!)^n\frac{B_{n+\alpha,\alpha}(x_1,x_2,x_3,\ldots)}{\alpha\binom{n+\alpha}{\alpha}}\pmod p.$$

Now, when we replace n by $n-\alpha$, the last congruence becomes

$$\frac{((n-\alpha+1)!)^n B_{n+sp,\alpha+sp}(x_1,x_2,x_3,\ldots)}{(\alpha+sp)\binom{n+sp}{\alpha+sp}}\equiv x_1^s((n-\alpha+1)!)^n\frac{B_{n,\alpha}(x_1,x_2,x_3,\ldots)}{\alpha\binom{n}{\alpha}}\pmod p.$$

Then, if $p>n-\alpha+1$ we obtain

$$\frac{B_{n+sp,\alpha+sp}(x_1,x_2,\ldots)}{(\alpha+sp)\binom{n+sp}{\alpha+sp}}\equiv x_1^s\frac{B_{n,\alpha}(x_1,x_2,x_3,\ldots)}{\alpha\binom{n}{\alpha}}\pmod p.$$

For the second part of theorem, when we replace k by sr in (2) we get

$$x_1^{sr}\sum_{j=1}^{n}B_{n,j}(t_1,t_2,\ldots)r^{j-1}(s-n)^{j-1}=x_1^{nr}\frac{((n+1)!)^n B_{n+sr,sr}(x_1,x_2,x_3,\ldots)}{sr\binom{n+sr}{sr}},$$

and, because $B_{n,j}(t_1,t_2,\ldots)$ $(1\leq j\leq n)$ are integers, the last identity proves that

$$x_1^{nr}\frac{((n+1)!)^{nr} B_{n+sr,sr}(x_1,x_2,x_3,\ldots)}{sr\binom{n+sr}{sr}}\equiv x_1^{sr}z_n$$
$$\equiv x_1^{sr}\frac{((n+1)!)^{nr} B_{(r+1)n,nr}(x_1,x_2,x_3,\ldots)}{nr\binom{(r+1)n}{nr}}\pmod r.$$

Let $r=p>n+1$ be a prime number. Now, because the expressions

$$\frac{((n+1)!)^{np} B_{n+sp,sp}(x_1,x_2,x_3,\ldots)}{sp\binom{n+sp}{sp}}\text{ and }\frac{((n+1)!)^{np} B_{(p+1)n,np}(x_1,x_2,x_3,\ldots)}{np\binom{(p+1)n}{np}}$$

are integers, we obtain

$$x_1^{n}\frac{B_{n+sp,sp}(x_1,x_2,x_3,\ldots)}{s\binom{n+sp}{sp}}\equiv x_1^{s}\frac{B_{(p+1)n,np}(x_1,x_2,x_3,\ldots)}{n\binom{(p+1)n}{np}}\pmod{p^2}.$$

□

Proof of Theorem 7. From Identity (15) we get

$$\frac{B_{n+k,k}(x_1,2x_2,3x_3,\ldots)}{k\binom{n+k}{k}}=\sum_{j=1}^{k}\frac{(k-1)!}{(k-j)!}B_{n,j}(x_2,x_3,x_4,\ldots)x_1^{k-j},\quad n,k\geq 1 \tag{16}$$

and this implies that the numbers

$$z_n = \frac{B_{(r+1)n,nr}(x_1, 2x_2, 3x_3, \ldots)}{nr\binom{(r+1)n}{nr}}, \ n \geq 1 \tag{17}$$

are integers, and then, the numbers $B_{n,j}(z_1, z_2, \ldots)$ $(1 \leq j \leq n)$ are also integers. From Identity (3), when we replace r by 1 and s by $n(s-1)$ we obtain

$$B_{(s+1)n,sn}(x_1, 2x_2, 3x_3, \ldots) = x_1^{n(s-1)} ns \binom{(s+1)n}{sn} \sum_{j=1}^{n} B_{n,j}(\overline{z}_1, \overline{z}_2, \ldots)((s-1)n)^{j-1}, \tag{18}$$

with $\overline{z}_n := \frac{1}{n\binom{2n}{n}} B_{2n,n}(x_1, 2x_2, \ldots)$.

Furthermore, from (18), we have

$$\begin{aligned}
\frac{B_{(s+1)n,sn}(x_1, 2x_2, 3x_3, \ldots)}{\binom{(s+1)n}{sn}} &= nx_1^{n(s-1)} s \sum_{j=1}^{n} B_{n,j}(z_1, z_2, \ldots)((s-1)n)^{j-1} \\
&\equiv n\left\{x_1^{n(s-1)} s z_n\right\} \\
&\equiv n\left\{x_1^{n(s-1)} s \frac{1}{n\binom{2n}{n}} B_{2n,n}(x_1, 2x_2, \ldots)\right\} \\
&\equiv x_1^{n(s-1)} s \frac{1}{\binom{2n}{n}} B_{2n,n}(x_1, 2x_2, \ldots) \pmod{n^2}, \text{ i.e.,}
\end{aligned}$$

$$\frac{B_{(s+1)n,sn}(x_1, 2x_2, 3x_3, \ldots)}{\binom{(s+1)n}{sn}} \equiv sx_1^{n(s-1)} \frac{B_{2n,n}(x_1, 2x_2, 3x_3, \ldots)}{\binom{2n}{n}} \pmod{n^2}.$$

For the second part of (5), when we replace k by $\alpha + sp$ in (2), we obtain

$$x_1^{\alpha+sp} \sum_{j=1}^{n} B_{n,j}(z_1, z_2, \ldots)(\alpha + sp - nr)^{j-1} = x_1^{nr} \frac{B_{n+\alpha+sp,\alpha+sp}(x_1, 2x_2, 3x_3, \ldots)}{(\alpha+sp)\binom{n+\alpha+sp}{\alpha+sp}}$$

with z_n is given by (17). Because the numbers $B_{n,j}(z_1, z_2, \ldots)$, $1 \leq j \leq n$, are integers, then when we reduce modulo p in the last identity we get

$$x_1^{nr} \frac{B_{n+\alpha+sp,\alpha+sp}(x_1, 2x_2, 3x_3, \ldots)}{(\alpha+sp)\binom{n+\alpha+sp}{\alpha+sp}} \equiv x_1^{\alpha+s} \sum_{j=1}^{n} B_{n,j}(z_1, z_2, \ldots)(\alpha - nr)^{j-1} \pmod{p}$$

and by (2) the last congruence becomes

$$\frac{B_{n+\alpha+sp,\alpha+sp}(x_1, 2x_2, 3x_3, \ldots)}{(\alpha+sp)\binom{n+\alpha+sp}{\alpha+sp}} \equiv x_1^{s} \frac{B_{n+\alpha,\alpha}(x_1, 2x_2, 3x_3, \ldots)}{\alpha\binom{n+\alpha}{\alpha}} \pmod{p}.$$

To terminate, it suffices to replace n by $n - \alpha$ in the last congruence.

For the third part of (5), when we replace k by kr in (2), we obtain

$$x_1^{kr} \sum_{j=1}^{n} B_{n,j}(z_1, z_2, \ldots)(kr - nr)^{j-1} = x_1^{nr} \frac{B_{n+kr,kr}(x_1, 2x_2, 3x_3, \ldots)}{rk\binom{n+kr}{kr}}$$

and because the numbers $B_{n,j}(z_1, z_2, \ldots)$, $1 \leq j \leq n$, are integers, then when we reduce modulo r in the last identity we get

$$x_1^{nr} \frac{B_{n+kr,kr}(x_1, 2x_2, 3x_3, \ldots)}{rk\binom{n+kr}{kr}} \equiv x_1^{kr} z_n \equiv \frac{x_1^{kr} B_{(r+1)n,nr}(x_1, 2x_2, 3x_3, \ldots)}{nr\binom{(r+1)n}{nr}} \pmod{p}.$$

Now, because

$$\frac{B_{n+kr,kr}(x_1, 2x_2, 3x_3, \ldots)}{rk\binom{n+kr}{kr}} \text{ and } \frac{x_1^{kr} B_{(r+1)n,nr}(x_1, 2x_2, 3x_3, \ldots)}{nr\binom{(r+1)n}{nr}}$$

are integers and $x_1^p \equiv x_1 \pmod{p}$ for any prime number p, then when we put $r = p$, the last congruence becomes

$$x_1^n \frac{B_{n+kp,kp}(x_1, 2x_2, 3x_3, \ldots)}{kp\binom{n+kp}{kp}} \equiv x_1^k \frac{B_{(p+1)n,np}(x_1, 2x_2, 3x_3, \ldots)}{np\binom{(p+1)n}{np}} \pmod{p}.$$

To complete this proof, it suffices to multiply the two sides of the last congruence by p. □

Proof of Corollary 9. From the first congruence of (4) when we replace s by $s-1$, n by $n+p$ and k by p we get

$$\frac{B_{n+sp,sp}(x_1, x_2, x_3, \ldots)}{s\binom{n+sp}{sp}} \equiv x_1^{s-1} \frac{B_{n+p,p}(x_1, x_2, x_3, \ldots)}{\binom{n+p}{p}} \pmod{p^2}, \ p > n+1, \ s \geq 1,$$

and by combining the last congruence and the second congruence of (4) we obtain

$$\frac{B_{(p+1)n,np}(x_1, x_2, x_3, \ldots)}{n\binom{(p+1)n}{np}} \equiv x_1^{n-1} \frac{B_{n+p,p}(x_1, x_2, x_3, \ldots)}{\binom{n+p}{p}} \pmod{p^2}, \ p > n+1.$$

Similarly, we use the second and the third congruences of (5) to get the second part of the corollary. □

Proof of Theorem 11. Identity (2) can be written as

$$\begin{gathered} x_1^{pr}(k-p) \frac{B_{p+k,k}(x_1, 2x_2, 3x_3, \ldots)}{k\binom{p+k}{k}} = x_1^k A_p((k-p)z_1, (k-p)z_2, \ldots), \\ \text{with } z_n = \frac{B_{(r+1)n,nr}(x_1, 2x_2, 3x_3, \ldots)}{nr\binom{(r+1)n}{nr}}, \ k \geq 1. \end{gathered} \tag{19}$$

Bell [1] showed, for any indeterminates $x_1, x_2, \ldots$, that

$$A_p(x_1, x_2, x_3, \ldots) \equiv x_1^p + x_p \pmod{p}. \tag{20}$$

Therefore, from (20) and (19), we obtain

$$x_1^{pr}(k-p) \frac{B_{p+k,k}(x_1, 2x_2, 3x_3, \ldots)}{k\binom{p+k}{k}} \equiv x_1^k \left\{(k-p)^p z_1^p + (k-p) z_p\right\} \pmod{p},$$

and Identity (16) shows that $\dfrac{B_{p+k,k}(x_1, 2x_2, 3x_3, \ldots)}{k\binom{p+k}{k}}$ and the terms of the sequence $\{z_n; n \geq 1\}$ are integers. Now, because $z_1 = x_1^{r-1}x_2$, then, when k is not a multiple of p, the last congruence and Fermat little Theorem prove that

$$x_1^r \frac{B_{p+k,k}(x_1, 2x_2, 3x_3, \ldots)}{k\binom{p+k}{k}} \equiv x_1^{k-1}\left\{x_1^r x_2 + x_1 y_p\right\} \pmod{p}.$$

For $k = 1$ in the last congruence we have

$$y_p \equiv x_1^{r-1}x_{p+1} - x_1^{r-1}x_2 \pmod{p}.$$

The proof for $j = 1$ results from the two last congruences.
Assume now that the congruences given by (7) are true for the index j.
Carlitz [1] showed, for any indeterminates $x_1, x_2, \ldots,$ that

$$A_{p^j} \equiv x_1^{p^j} + x_p^{p^{j-1}} + x_{p^2}^{p^{j-2}} + \cdots + x_{p^j} \pmod{p}.$$

For $x_1, x_2, \ldots$ integers we obtain

$$A_{p^j} \equiv x_1 + x_p + x_{p^2} + \cdots + x_{p^j} \pmod{p}.$$

Then, when we use Identity (19) and the fact that the sequence $\{z_n; n \geq 1\}$ is a sequence of integers, we obtain when $p \nmid kx_1$

$$\begin{aligned} x_1^r \frac{B_{p^{j+1}+k,k}(x_1, 2x_2, 3x_3, \ldots)}{k\binom{p^{j+1}+k}{k}} &\equiv x_1^k \left(z_1 + z_p + z_{p^2} + \cdots + z_{p^{j+1}}\right) \\ &\equiv x_1^k \left(z_1 + z_p + z_{p^2} + \cdots + z_{p^j}\right) + x_1^k z_{p^{j+1}} \\ &\equiv x_1^r \frac{B_{p^j+k,k}(x_1, 2x_2, 3x_3, \ldots)}{k\binom{p^j+k}{k}} + x_1^k z_{p^{j+1}} \\ &\equiv x_1^{k-1}x_{p^j+1} + x_1^k z_{p^{j+1}} \pmod{p}. \end{aligned}$$

For $k = 1$ in the last congruence we have

$$x_1^r x_{p^{j+1}+1} \equiv x_{p^j+1} + x_1 z_{p^{j+1}} \pmod{p}.$$

From the two last congruences we deduce that

$$\begin{aligned} &\frac{B_{p^{j+1}+k,k}(x_1, 2x_2, 3x_3, \ldots)}{k\binom{p^j+k}{k}} \equiv x_1^{k-1}x_{p^{j+1}+1} \pmod{p} \quad \text{if } p \nmid kx_1, \\ &\frac{B_{(r+1)p^{j+1},p^{j+1}r}(x_1, 2x_2, 3x_3, \ldots)}{p^{j+1}r\binom{(r+1)p^{j+1}}{p^{j+1}r}} \equiv x_1^{r-1}\left(x_{p^{j+1}+1} - x_{p^j+1}\right) \pmod{p} \;\; \text{if } p \nmid x_1, \end{aligned}$$

which completes the proof. □

Proof of Theorem 13. Carlitz [1] showed, for any indeterminates $x_1, x_2, \ldots,$ that

$$A_{2p^j} \equiv A_{p^j}^2 + x_{2p^j} \pmod p.$$

Then, for $x_1, x_2, \ldots$ integers we get

$$A_{2p^j} \equiv \left(x_1^{p^j} + x_p^{p^{j-1}} + \cdots + x_{p^j}\right)^2 + x_{2p^j} \equiv \left(x_1 + x_p + \cdots + x_{p^j}\right)^2 + x_{2p^j} \pmod p,$$

and, when we use Identity (19), we obtain

$$x_1^{2p^j r}\left(k - 2p^j r\right)\frac{B_{2p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{2p^j+k}{k}} = x_1^k A_{2p^j}\left(\left(k - 2p^j r\right)z_1, \left(k - 2p^j r\right)z_2, \ldots\right),$$

and because $\{z_n; n \geq 1\}$ is a sequence of integers, the last identity gives

$$x_1^{2p^j r}\left(k - 2p^j r\right)\frac{B_{2p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{2p^j+k}{k}} \equiv$$
$$x_1^k\left(\left(k - 2p^j r\right)^2\left(z_1 + z_p + z_{p^2} + \cdots + z_{p^j}\right)^2 + \left(k - 2p^j r\right)z_{2p^j}\right) \pmod p.$$

From the proof of Theorem 11, the last congruence gives when $p \nmid kx_1$

$$\begin{aligned} x_1^k\left(x_1^{2r}\frac{B_{2p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{2p^j+k}{k}}\right) &\equiv kx_1^{2k}\left(z_1 + z_p + z_{p^2} + \cdots + z_{p^j}\right)^2 + x_1^{2k}z_{2p^j} \\ &\equiv k\left(x_1^r\frac{B_{p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{p^j+k}{k}}\right)^2 + x_1^{2k}z_{2p^j} \\ &\equiv k\left(x_1^{r+k-1}x_{p^j+1}\right)^2 + x_1^{2k}z_{2p^j} \pmod p, \text{ i.e.,} \end{aligned}$$

$$x_1^{2r}\frac{B_{2p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{2p^j+k}{k}} \equiv x_1^k\left(kx_1^{2r-2}x_{p^j+1}^2 + z_{2p^j}\right) \pmod p.$$

For $k = 1$ in the last congruence we get $x_1^{2r-1}x_{2p^j+1} \equiv x_1^{2r-2}x_{p^j+1}^2 + z_{2p^j}$, i.e.,

$$z_{2p^j} \equiv x_1^{2r-2}\left(x_1 x_{2p^j+1} - x_{p^j+1}^2\right) \pmod p.$$

Then

$$x_1^2\frac{B_{2p^j+k,k}\left(x_1, 2x_2, 3x_3, \ldots\right)}{k\binom{2p^j+k}{k}} \equiv x_1^k\left((k-1)\,x_{p^j+1}^2 + x_1 x_{2p^j+1}\right) \pmod p.$$

□

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