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Introduction to the "Prisoners and Guards" Game

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Abstract

We study the half-dependent problem for the king graph K_n. We give proofs to establish the values $h(K_n)$ for $n \in \{1, 2, 3, 4, 5, 6\}$ and an upper bound for $h(K_n)$ in general. These proofs are independent of computer assisted results. Also, we introduce a two-player game whose winning strategy is tightly related with the values $h(K_n)$. This strategy is analyzed here for $n = 3$ and some facts are given for the case $n = 4$. Although the rules of the game are very simple, the winning strategy is highly complex even for $n = 4$.

1 Introduction

Suppose that you are competing in a two-player game in which you and your opponent attempt to pack as many "prisoners" as possible on the squares of an $n \times n$ checkerboard; each prisoner has to be "protected" by an appropriate number of guards. Initially, the board is covered entirely with guards. The players – designated as "red" and "blue", with red going first – take turns adjusting the board configuration using one of the following rules in each turn:

I. Replace one guard with a prisoner of the player's color.

II. Replace one prisoner of either color with a guard and replace two other guards with prisoners of the player's color.

That is, each player takes a turn increasing the total number of prisoners by one. We require that, at every stage of the game, *the number of guards adjacent to a given prisoner is not less than the number of prisoners lying adjacent to that prisoner*. The squares *adjacent* to a given square are those squares, situated directly above, below, to the left, to the right, or diagonal to the square in question. An arrangement of prisoners and guards that satisfies this requirement and has exactly one occupant per square is called a *valid board*. The game ends when neither player can further adjust the board using rules I and II while maintaining a valid board. The player whose color represents more prisoners is the winner. This is the game of Prisoners and Guards – a game that can be played and analyzed without extensive knowledge of mathematics. We invite the reader to play the game online by running the Java Applet available at http://csc.colstate.edu/woolbright/.

The guards in this game are related to the half domination set in the king's graph as introduced in a paper by Dunbar, Hoffman, Laskar, and Markus [3]. Similar domination problems have been studied by Bode, Harborth, and Harporth [2], Dutton, Lee, and Brigham [4], Watkins, Ricci, and McVeigh [9] and many others. Two concepts in the domination literature very closely related to ours are those of *unfriendly partition* [1] and *global offensive alliance* [8]. At the end of Section 4 we show how some of our estimates relate to a general result in [8].

The Prisoners and Guards game originated as a puzzle created by the third author with a focus on minimizing the size of the dominating set (the guards).

In the two-player game, one fundamental question that naturally arises is "How do we decide when the game is over?" The short answer is that the game is over when the board configuration has reached a maximal state. A valid board to which no adjustments can be made to increase the total number of prisoners is called a *maximal* board. One can also define a *maximum* configuration as being an arrangement of prisoners and guards that has the greatest number of prisoners of all valid boards. Clearly, every maximum configuration is a maximal configuration. For $n \in \{1, 2, 3\}$ all maximal boards are also maximum configurations. We will see examples of 4×4 boards that are maximal but do not contain maximum configurations. Let $P(n)$ denote the number of prisoners in a maximum configuration. Finding the exact values of the sequence $\{P(n)\}_{n=1}^{\infty}$ will help us determine when to end the game. It also proves an interesting avenue for exploration on its own.

Since the lone square on a 1×1 board has no adjacent squares, we can place a prisoner in it and be sure that there are at least as many guards as prisoners lying in all adjacent squares – none. Therefore, we have $P(1) = 1$. By exhaustively checking all sixteen 2×2 cases, we find eleven valid boards, each having zero, one, or two prisoners. Thus, $P(2) = 2$. We analyze the cases $n = 3$, 4, 5, and 6 in sections 2 and 3. Exact values for $P(n)$, $n \in \{7, 8, 9, 10, 11\}$, are $P(7) = 28$, $P(8) = 39$, $P(9) = 49$, $P(10) = 59$ and $P(11) = 73$, and for the corresponding maximum configuration one can consult a paper by Ionascu, Pritikin, and Wright [7], who employ among various methods binary linear programming techniques in the study of $P(n)$. However, in this paper we include proofs for the above mentioned cases which are independent of computer searches and short enough to be read with ease. In Section 4 we also obtain an upper bound on $P(n)$; this is about the best that we can say for $n \geq 12$. In Section 5 we show how this technique from Section 4 can be used in order to completely answer the best density 1/2-domination problem in grid type graphs.

2 The game analysis for $n = 3$ and $n = 4$

Playing Prisoners and Guards on a 1×1 board or on a 2×2 board is not all that interesting. When we increase the board size slightly and consider the game on a 3×3 board, strategy becomes more of a factor. We will see that the arrangement in Figure 1 is a maximum configuration, as we establish in Theorem 2. We use the diamond to represent prisoners and blank squares represent guards.

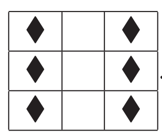

Figure 1: Maximum 3x3 Board

In fact, this is the only maximal arrangement (up to a rotation). Let us observe that a maximal board permits no adjustments using either Rule I or Rule II. First we consider arrangements that are maximal with respect to Rule I (i.e. one cannot simply add more prisoners in the existing configuration).

Perhaps it is not difficult to convince oneself that any valid board having zero, one, or two prisoners can be adjusted using Rule I; after factoring out rotations and reflections, there are ten unique cases to check. Therefore, a maximal board must have at least three prisoners. Figure 2 depicts all valid boards (up to rotations and reflections) that contain three, four, or five prisoners and are maximal with respect to Rule I.

Each one of these configurations can be adjusted to match the arrangement in Figure 1 by using Rule II (and one or more adjustments using Rules I and II in some of the cases). It follows that a maximal 3×3 board must contain at least six prisoners. We record this fact in the following lemma.

Lemma 1. *Any maximal 3×3 board has at least six prisoners.*

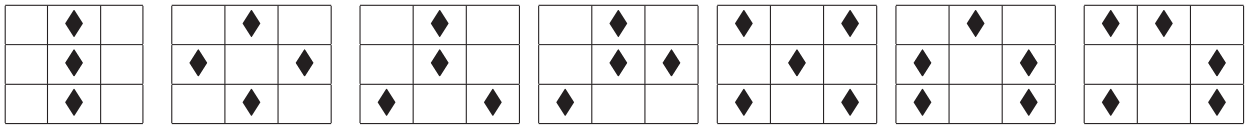

Figure 2: 3×3 Boards that are Maximal w.r.t. Rule I

With Figure 1, we see that a valid 3×3 configuration can have six prisoners. Does there exist a valid configuration with more than six prisoners? Suppose that a 3×3 board arrangement contains seven prisoners (and two guards). Since there are four non-corner edge squares, a prisoner must occupy at least one of them. Since this prisoner lies adjacent to at most two guards, the board cannot be valid. These observations, Lemma 1, and the fact that Figure 1 depicts a valid configuration with six prisoners lead us to the following conclusion.

Theorem 2. *A maximum 3×3 board contains six prisoners, i.e. $P(3) = 6$.*

As a matter of fact, the configuration shown in Figure 1 is the only maximal 3×3 board (up to a rotation of the board). From this we learn that the second player has a good chance to win by using Rule II all of the time. The first player may force a tie if she can lead the board configuration in such a manner that will require her opponent to use Rule I. In fact, this is manageable if she plays into the pattern in Figure 1, forcing the second player to use Rule I in the last step and so the final board will have an equal number of prisoners of each color.

We now turn our attention to 4×4 boards. This board size proves interesting because there exist many maximal arrangements that are not maximum configurations. We include some maximal arrangements with eight prisoners in Figure 3 that we found but there may be others.

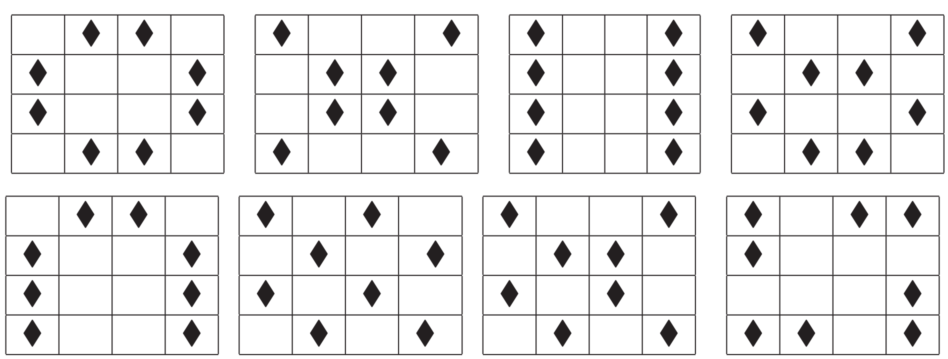

Figure 3: Some Maximal 4×4 Boards

If we factor out rotations and reflections of the board, there are three maximum arrangements as depicted in Figure 4. We obtained these via an exhaustive search of all 4×4 valid boards and verified that there are no other equivalence classes.

To prove something about maximum 4×4 board configurations, it helps to dissect the board and consider what can happen in the vicinity of the corner squares. Suppose that we

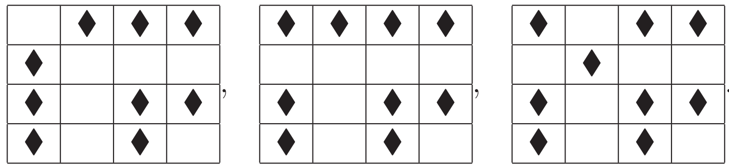

Figure 4: Maximum 4×4 Boards

have a 2×2 block C of squares situated in one corner of the board. If the corner square within C does not contain a guard, then it contains a prisoner. If the latter is the case, then C must contain at least two guards. Thus we have established the following fact.

Lemma 3. *If C is a 2×2 corner block within a valid board ($n \geq 2$), then it must contain at least one guard.*

With this in mind, we are equipped to consider maximum 4×4 boards by partitioning them into four 2×2 corner blocks and following through with the consequences. This will lead us to the conclusion of the next proposition.

Proposition 4. $P(4) = 9$*. That is, every maximum 4×4 valid board has nine prisoners.*

Proof. Since the configurations in Figure 4 are valid and each contains nine prisoners, it follows that $P(4) \geq 9$. It is enough to show that $P(4) \leq 9$. We therefore assume that there exists a valid board B with ten or more prisoners. By dropping prisoners if necessary, we can say there are exactly ten. We shall see that this leads to a contradiction. We partition B into four 2×2 blocks as indicated in Figure 5(a).

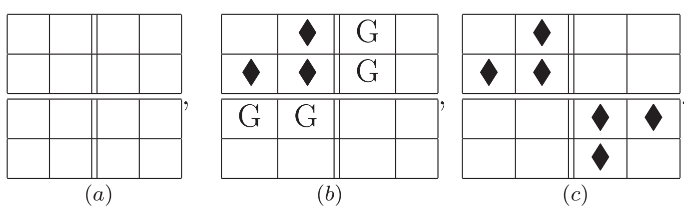

Figure 5: Block Partitions of a 4×4 Board

Since by assumption the board contains ten prisoners, it follows from Lemma 3 that at least two of these blocks must contain three prisoners each. Without loss of generality, assume that the upper left block is one of them. We see in the proof of Lemma 3 that the lone guard must lie in square b_{11}, as depicted in Figure 5(b).

For the board to be valid, the non-corner edge prisoners in b_{12} and b_{21} must each lie adjacent to three guards. This is only possible if guards lie in the squares b_{13}, b_{23}, b_{31}, and b_{32}, as Figure 5(b) indicates. As previously noted, at least two of the blocks must contain three prisoners each. The only way to achieve this will be for the lower right block to have three prisoners, with a guard in b_{44} as shown in Figure 5(c).

Now we see that the prisoners situated in squares b_{34} and b_{43} necessitate the presence of guards in squares b_{24} and b_{42}. By placing prisoners in all squares not yet committed, we will have a total of only eight prisoners on the board, contradicting our assumption that the board has ten prisoners. Thus, our assumption was invalid. □

3 Analysis of the 5×5 and 6×6 Cases

In our analysis of 5×5 and 6×6 board configurations, we will partition the boards into 3×3 blocks. The following lemma will help in the examination of these blocks.

Lemma 5. *If C is a 3×3 corner block within a valid board ($n > 3$), then it must contain at least three guards. Moreover, if C contains exactly three guards, then it must contain a prisoner diagonally opposite (within C) to the corner square.*

Proof. Assume that there exists a valid $n \times n$ ($n > 3$) board configuration with a 3×3 corner block C that contains only two guards. Without loss of generality, suppose that C is situated in the upper left corner of the board. Since four guards are required to protect a prisoner residing on an interior square, and we have two guards, a guard must occupy c_{22}. Since, by assumption, there remains only one more guard, there must lie a prisoner in c_{12} or c_{21}. However, three guards are required to cover a prisoner that is situated in a non-corner edge square. Hence, the board cannot be valid and we have a contradiction.

To establish the last part of our lemma, let us observe that since one guard must occupy c_{22}, if another one of the guards were located in c_{33} then we would have a prisoner in either c_{12} or c_{21} without a sufficient number of adjacent guards. □

Possible arrangements, up to a reflection about the main diagonal, appear in Figure 6. As we can see, all these configurations have a prisoner in c_{33}.

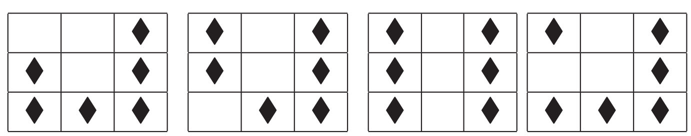

Figure 6: Possible 3×3 UL Corner Blocks With 3 Guards, ($n > 3$)

Now, we are ready to consider the 5×5 case. The only maximum configuration (up to rotations) is illustrated in Figure 7. Proposition 6 establishes that this is a maximum 5×5 board configuration.

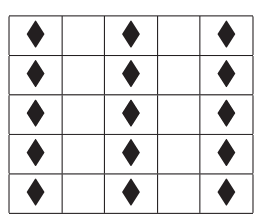

Figure 7: The Maximum 5×5 Board Configuration

Proposition 6. *A maximum* 5×5 *board configuration contains fifteen prisoners; that is,* $P(5) = 15$.

Proof. Assume that there exists a valid 5×5 board configuration with 16 or more prisoners. We will see that this leads to a contradiction.

Divide the 5×5 board into two opposite (overlapping) corner 3×3 blocks, A and C, that have a square in common and two 2×2 opposite corner blocks, B and D, as illustrated in Figure 8(a).

A	A	A	B	B
A	A	A	B	B
A	A	A/C	C	C
D	D	C	C	C
D	D	C	C	C

(*a*)

A	A		♦	
A	A		♦	♦
A	A	A/C		
D	D	C	C	C
D	D	C	C	C

(*b*)

A	A		♦	
A	A		♦	♦
		A/C		
♦	♦		C	C
	♦		C	C

(*c*)

Figure 8: Partitions of the 5×5 Board

According to Lemma 5, the two 3×3 blocks A and C collectively contain at most $2(6) - 1 = 11$ prisoners since the shared square (common to blocks A and C) must contain a prisoner if at least one of blocks A and C has six prisoners. Hence at least one of the 2×2 blocks must have three prisoners.

Recall that Lemma 3 establishes that each of blocks B and D contains at most three prisoners. Thus, for the board to contain a total of at least sixteen prisoners we must find either ten or eleven prisoners shared in blocks A and C.

Case 1. Blocks A **and** C **share 11 prisoners.** In this case, one of the 2×2 blocks holds three prisoners and the other holds at least two prisoners. Also the blocks A and C must have six prisoners each. Without loss of generality, let us suppose that the B block has three prisoners; then the one guard must lie in the (1,5) position. To cover the three prisoners in block B, guards must be placed in the (1,3), (2,3), (3,4), and (3,5) positions (refer to Figure 8(b)). For the board to be valid, block A (or its diagonal reflection) must match one of the corner blocks depicted in Figure 6. None of these allows guards in both the (1,3) and the (2,3) positions. Therefore, the board is not valid and we have a contradiction in this case.

Case 2. Blocks A **and** C **share 10 prisoners.** In this case, each of the 2×2 blocks B and D holds three prisoners. In order to maintain a valid board configuration, we are then forced to place guards in the (1,3), (1,5), (2,3), (3,1), (3,2), (3,4), (3,5), (4,3), and (5,3) positions as indicated in Figure 8(c). But then there remain only nine uncommitted squares in which to place the ten prisoners that blocks A and C are supposed to share. Thus, we also find a contradiction in this case. □

For $n = 6$ all maximum boards amount to rotations or small perturbations of the arrangement in Figure 3, the validity of which yields the lower bound $P(6) \geq 22$. We will show that, in fact, $P(6) = 22$ by an analysis of manageable size. Dunbar, Hoffman, Laskar, and

Markus assert (without proof) a fact about 1/2-domination in the king's graph dimension 6 which, if true, implies that $P(6) = 22$ [3]. This is indeed the case, as we shall establish next. We use a more specific version of Lemma 5 in order to obtain this fact.

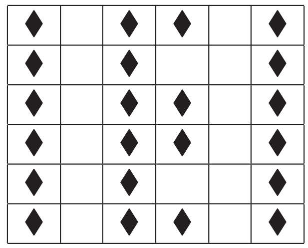

Figure 9: A Maximum 6×6 Board Configuration

Lemma 7. *If C is a 3×3 corner block holding six prisoners within a valid board ($n > 3$), where c_{11} is the corner square, then up to a diagonal symmetry the block has one of the four arrangements in Figure 6.*

Proof. By Lemma 5, the position c_{22} must have a guard as we have seen and also one of the positions c_{12} or c_{21} must have a guard or we will require four guards to cover it. By symmetry we can assume we have a guard at c_{12}. If we have a prisoner at c_{21} then this leaves three possibilities for the third guard. Adding in the case with guards in c_{21} and c_{21}, we see that there are only four configurations (up to rotation and/or diagonal reflection), as shown in Figure 6, of 3×3 corner blocks that contain six prisoners. □

Proposition 8. *A maximum 6×6 board contains twenty two prisoners. That is, $P(6) = 22$.*

Proof. We have observed that $P(6) \geq 22$. To verify that $P(6) \leq 22$ let us assume the existence of a valid arrangement C with 23 prisoners; we shall see that this leads to a contradiction. By Lemma 5, three of the four 3×3 corner blocks have six prisoners and one has five prisoners. Without loss of generality, we may assume that the block with five prisoners is the lower right one. By Lemma 7 and by symmetry, we can assume that the block in the upper left corner is one of those in Figure 6. Possibilities for the upper right 3×3 corner block can be generated from arrangements found in Figure 6 first by reflecting them about their vertical axis. Secondly, three more possibilities are generated by reflecting about counter-diagonal of what is obtained after the first reflection. We summarize the possibilities for the upper right block in Figure 10.

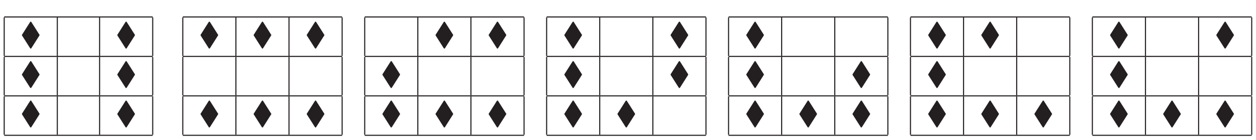

Figure 10: Possible Upper Right 3×3 Corner Blocks

Technically we need to analyze 28 possibilities but let us observe that by taking any of the four arrangements in Figure 6 as the upper left corner and any of the arrangements in

Figure 10 with the exception of the third one, in the upper right corner, will put a prisoner in the c_{14} position which will not be adequately covered by guards. If we choose the third configuration from Figure 10 in the upper right corner, then this will put a prisoner at c_{24} which will not have enough guards around it. This contradicts the existence of a configuration with 23 prisoners or more. □

We suspect that this block partition approach can be adapted to compute or bound $P(n)$ for larger sizes of n, although this approach could turn out to be quite lengthy. These proofs may very well be pursued as undergraduate research projects.

4 Upper bound for $P(n)$ and the deficiency function

As the board size grows larger, establishing the exact number of prisoners on a maximum board becomes increasingly difficult. The proof of Proposition 4 foreshadows the importance of finding useful upper bounds on $P(n)$. In this section, we construct a tool that will help in establishing these bounds – the deficiency matrix. We then use the deficiency matrix to determine a general upper bound for $P(n)$.

Suppose that we have fixed the board size at $n \times n$ ($n \geq 3$). With each configuration, we associate a binary matrix $X = (x_{ij})$ defined by

$$x_{ij} = \begin{cases} 1, & \text{if a prisoner lies in the } (i,j) \text{ position;} \\ 0, & \text{if a guard lies in the } (i,j) \text{ position.} \end{cases}$$

Many who work in combinatorics and graph theory, such as Hedetniemi, Hedetniemi, and Reynolds [6] have employed this idea. In any local measure of optimality we must be attentive to the number of prisoners lying in squares adjacent to a particular square; we let x^*_{ij} denote the number of prisoners lying in squares adjacent to the (i,j) square.

The deficiency matrix serves as an ad-hoc, local measure of the optimality of a given board configuration. Its construction arises from our observations and conjectures of maximum board configurations. We define the deficiency matrix $\delta = (\delta_{ij})$ by

$$\delta_{ij} = \text{expectation} - x^*_{ij}, \text{where}$$

$$\text{expectation} = \begin{cases} 1, & \text{if } (i,j) \text{ is a corner square with} & x_{ij} = 1; \\ 2, & \text{if } (i,j) \text{ is a corner square with} & x_{ij} = 0; \\ 2, & \text{if } (i,j) \text{ is an edge square with} & x_{ij} = 1; \\ 4, & \text{if } (i,j) \text{ is an edge square with} & x_{ij} = 0; \\ 4, & \text{if } (i,j) \text{ is an interior square with} & x_{ij} = 1; \\ 6, & \text{if } (i,j) \text{ is an interior square with} & x_{ij} = 0. \end{cases}$$

Figure 11 depicts a 4×4 non-maximal board configuration and its corresponding deficiency matrix. The positive entries in the deficiency matrix indicate areas of the board that are thought to be less than optimal; the 2's indicate that the "worst" deficiencies occur on the corresponding interior squares.

♦		♦	
	♦		♦
♦		♦	
	♦		♦

0	1	0	0
1	0	2	0
0	2	0	1
0	0	1	0

Figure 11: Non-maximal 4×4 Board and Its Deficiency Matrix

Since we use these values to obtain an upper bound on $P(n)$, it helps to first consider bounds on δ_{ij} for $1 \leq i, j \leq n$. Suppose that the (i, j) square contains a prisoner. If (i, j) is a corner square, then at least two of the three adjacent squares must contain guards; therefore $x^*_{ij} \leq 1$ and in checking our expectation value above we see that $\delta_{ij} \geq 0$. Likewise, if (i, j) is an edge square containing a prisoner or an interior square with a prisoner, we find that $\delta_{ij} \geq 0$.

Suppose that we find a guard in a corner square (i, j). Then three squares lie adjacent to this square, so we find at most three prisoners in the neighboring squares. Therefore, $x^*_{ij} \leq 3$ and so $\delta_{ij} \geq -1$. Via similar considerations, we find that if a guard occupies an edge square then $\delta_{ij} \geq -1$ and for an interior square we get $\delta_{ij} \geq -2$.

We define the *net deficiency of a board configuration* as the sum of all entries in the deficiency matrix,

$$\Delta = \sum_{i,j=1}^{n} \delta_{ij}.$$

For instance, the board configuration in Figure 11 has a net deficiency of 8. We expect maximum board configurations to correspond to minimum net deficiency values. We will relate Δ to the overall number of guards in a given board configuration. Let P_C and G_C denote the total number of prisoners and guards, respectively, found in the corner squares. Similarly, P_E and G_E refer to the prisoners and guards in edge squares, and P_I and G_I refer to prisoners and guards in interior squares. With this notation we have

$$\begin{aligned} \Delta &= \textstyle\sum_{\text{corners}} \delta_{ij} + \sum_{\text{edges}} \delta_{ij} + \sum_{\text{interior}} \delta_{ij} \\ &\geq (0 \cdot P_C - 1 \cdot G_C) + (0 \cdot P_E - 1 \cdot G_E) + (0 \cdot P_I - 2 \cdot G_I) \\ &= -G_C - G_E - 2 \cdot G_I. \end{aligned}$$

This establishes the next lemma.

Lemma 9. *The net deficiency of a given configuration satisfies the inequality* $\Delta \geq -G_C - G_E - 2 \cdot G_I$.

Now we are ready to think about bounding the size of $P(n)$. Observe that

$$4x_{ij} + x^*_{ij} = \begin{cases} 8 - \delta_{ij}, \text{if } x_{ij} = 1 \text{ and } (i,j) \text{ is an interior square;} \\ 6 - \delta_{ij}, \text{if } x_{ij} = 0 \text{ and } (i,j) \text{ is an interior square.} \end{cases}$$

$$3x_{ij} + x^*_{ij} = \begin{cases} 5 - \delta_{ij}, \text{ if } x_{ij} = 1 \text{ and } (i,j) \text{ is an edge square;} \\ 4 - \delta_{ij}, \text{if } x_{ij} = 0 \text{ and } (i,j) \text{ is an edge square.} \end{cases} \tag{1}$$

$$2x_{ij} + x^*_{ij} = \begin{cases} 3 - \delta_{ij}, \text{if } x_{ij} = 1 \text{ and } (i,j) \text{ is a corner square;} \\ 2 - \delta_{ij}, \text{if } x_{ij} = 0 \text{ and } (i,j) \text{ is a corner square.} \end{cases}$$

Theorem 10. *The number of prisoners in a valid configuration is given by*

$$P = \frac{3n^2}{5} - \frac{4n}{5} + \frac{1}{10}(3P_E + 6P_C - \Delta). \tag{2}$$

Proof. Summing the left hand sides of the equations in (1) over all squares of the board, we obtain

$$4 \cdot P_I + 3 \cdot P_E + 2 \cdot P_C + \sum_{1 \le i,j \le n} x^*_{ij} = 4 \cdot P_I + 3 \cdot P_E + 2 \cdot P_C + 8 \cdot P_I + 5 \cdot P_E + 3 \cdot P_C = 12 \cdot P_I + 8 \cdot P_E + 5 \cdot P_C.$$

We will equate this result with the sum of the right hand sides. In summing over the interior squares that contain prisoners, this contributes $8 - \delta_{ij} = 6 + 2 - \delta_{ij}$ for each such square, whereas the interior square that contain guards contribute only $6 - \delta_{ij}$ per square. There are $(n-2)^2$ interior squares, so altogether these sum to $6(n-2)^2 + 2 \cdot P_I - \sum_{\text{int. sqrs.}} \delta_{ij}$. Similarly summing the right hand sides over all edge squares we get $4\,[4(n-2)] + 1 \cdot P_E - \sum_{\text{edge sqrs.}} \delta_{ij}$. Summing over the corners yields $8 + 1 \cdot P_C - \sum_{\text{corner sqrs.}} \delta_{ij}$. Combining these right-hand sums and equating with the left-hand sum, we obtain the equation

$$12P_I + 8P_E + 5P_C = 6(n-2)^2 + 2P_I + 16(n-2) + P_E + 8 + P_C - \Delta$$

or

$$10P_I + 7P_E + 4P_C = 6n^2 - 8n - \Delta.$$

Then since $P = P_I + P_E + P_C$ we then obtain

$$10P = 6n^2 - 8n + 3P_E + 6P_C - \Delta,$$

which leads to (2). □

By combining the inequality in Lemma 9 with this theorem, we obtain a crude upper bound on $P(n)$.

Corollary 11. *In a maximum configuration of prisoners and guards on a $n \times n$ board the number of prisoners obeys the inequality*

$$P(n) \le \frac{2n^2 + n}{3}. \tag{3}$$

Proof. Using Lemma 9 and (2) we obtain

$$\begin{aligned} P(n) &\leq \frac{3n^2}{5} - \frac{4n}{5} + \frac{1}{10}(3P_E + 6P_C + 2G_I + G_E + G_C) \\ &= \frac{3n^2}{5} - \frac{4n}{5} + \frac{1}{10}\Big[2P_E + 5P_C + 2(n-2)^2 - 2P_I + 4(n-2) + 4\Big] \\ &= \frac{3n^2}{5} - \frac{4n}{5} + \frac{1}{10}\Big[2P_E + 5P_C + 2(n-2)^2 - 2P + 2P_E + 2P_C + 4n - 4\Big]. \end{aligned}$$

This implies

$$\begin{aligned} \left(1 + \frac{1}{5}\right) P(n) &\leq \frac{3n^2}{5} - \frac{4n}{5} + \frac{1}{10}(4P_E + 7P_C + 2n^2 - 4n + 4) \\ &\leq \frac{3n^2}{5} - \frac{4n}{5} + \frac{1}{10}\Big[(4)(4)(n-2) + (7)(4) + 2n^2 - 4n + 4\Big] \\ &= \frac{4n^2 + 2n}{5}. \end{aligned}$$

Therefore $P(n) \leq \left(\frac{5}{6}\right)\left(\frac{4n^2+2n}{5}\right) = \frac{2n^2+n}{3}$. □

From Proposition 14 in [3] one can heuristically obtain the upper bound of $P(n)$ as being about $\frac{2}{3}n^2$ as we have just seen, by neglecting the boundary vertices. We believe that $\Delta \leq O(n)$ in general. This fact is equivalent to $P(n) \leq 3n^2/5 + O(n)$. However we can tighten the upper bound (3) by getting a better estimate for Δ.

Lemma 12. *In a valid configuration the net deficiency satisfies*

$$\Delta \geq -1 \cdot G = -1(G_I + G_E + G_C).$$

Proof. Recall our previous observations about the possible range of values for δ_{ij}. If the (i,j) board position contains a prisoner then from the definition it follows that $\delta_{ij} \geq 0$. If the square is a corner or edge square containing a guard then $\delta_{ij} \geq -1$. For an interior square containing a guard, we have noted that $\delta_{ij} \geq -2$. Let us focus on this last case.

Suppose that a guard occupies the (i,j) interior position in a valid board configuration and that $\delta_{ij} = -2$. Then it must be the case that all adjacent squares contain prisoners, as depicted in Figure 12(a). The g's denote guards that are then forced into the arrangement in order for the configuration to be valid. We see that each of the prisoners in the squares diagonally adjacent to this position lies adjacent to five or six guards (depending on the occupants in the squares marked with asterisks). The possible deficiency values for neighboring squares appear in Figure 12(b). Summing these deficiency values, we find that the net contribution of the 3×3 block satisfies $2 \leq \Delta_{local} \leq 6$. Notice that, as the g's in Figure 12(a) suggest, it is not possible for two such 3×3 blocks around guards with deficiency -2 to overlap. Thus, we see that each guard on the board contributes a net deficiency value not less than -1.

Summing the δ_{ij}'s over all board positions, we have $\Delta = \sum_{prisoners} \delta_{ij} + \sum_{guards} \delta_{ij} \geq -1 \cdot G$. □

*	g	g	g	*
g	♦	♦	♦	g
g	♦		♦	g
g	♦	♦	♦	g
*	g	g	g	*

(a)

	1, 2	0	1, 2	
	0	-2	0	
	1, 2	0	1, 2	

(b)

Figure 12: Local Configuration Near a Guard with Deficiency -2

Using this bound on Δ in Theorem 10, we obtain a better upper bound for $P(n)$. The calculations parallel those used in the proof of Corollary 11.

Theorem 13. *For an $n \times n$ maximum arrangement of prisoners and guards, the number of prisoners, $P(n)$, satisfies the inequality*

$$P(n) \leq \frac{7n^2 + 4n}{11}. \tag{4}$$

Proof. By Lemma 12, $-\Delta \leq G$. Applying this upper bound in Theorem 10, we get

$$\begin{aligned} P &\leq \frac{3}{5}n^2 - \frac{4}{5}n + \frac{1}{10}\Big[3P_E + 6P_C + G\Big] = \frac{3}{5}n^2 - \frac{4}{5}n + \frac{1}{10}\Big[3P + 3P_C - 3P_I + G\Big] \\ &= \frac{3}{5}n^2 - \frac{4}{5}n + \frac{1}{10}\Big[2P + 3\big(P_C - P_I\big) + n^2\Big]. \end{aligned}$$

Subtracting $\frac{2}{10}P$ from both sides and combining the n^2 terms, we see that this implies

$$\begin{aligned} \frac{8}{10}P &\leq \frac{7}{10}n^2 - \frac{4}{5}n + \frac{3}{10}\big(P_C - P_I\big) \\ &\leq \frac{7}{10}n^2 - \frac{4}{5}n + \frac{3}{10}\big(4 - P + P_E + P_C\big) \\ &\leq \frac{7}{10}n^2 - \frac{4}{5}n + \frac{3}{10}\big(4 - P + 4n - 4\big). \end{aligned}$$

The claim now follows after a bit of arithmetic. □

Let us mention here that according to [8], the *global offensive number* γ_0 of a graph is the minimum cardinality of a global offensive alliance in that graph. An *offensive alliance in a graph* is a set of vertices, say O, with the property that a majority of the vertices in the neighborhood of every vertex in the boundary of O is in O. An offensive alliance O is said to be global if it affects all vertices not in O. It is easy to see that in the king's graph a minimum cardinality offensive alliance must be global. If we think of the vertices in such a global offensive alliance as guards and the rest of the vertices as prisoners we observe that the restriction on the prisoners is a little stronger than in our problem. This means that every minimum cardinality offensive alliance in the king's graph gives a valid configuration of prisoners versus guards in our domination problem. Hence $n^2 - \gamma_o \leq P(n)$. Since the number of edges in the king's graph is $m = 4n^2 - 6n + 2$ for an $n \times n$ board, and the maximum degree is 8, we get from Theorem 7 in [8] that

$$\gamma_o(K_n) \geq \lceil \frac{9n^2 - 12n + 4}{25} \rceil.$$

For every $n \geq 2$, this inequality follows from (4) as a corollary.

Corollary 14. *The global offensive alliance number γ_o for the king's graph associated with an $n \times n$ board satisfies*

$$\gamma_o(K_n) \geq \lceil \frac{4n^2 - 4n}{11} \rceil.$$

5 Grid type graphs and half-dependent best density arrangements

Thus far we have considered the Prisoners and Guards game using the king's graph to determine which board squares are adjacent. The "grid graph" provides another variation that graph theorists contemplate in domination problems; in this context we consider adjacent squares to be those that lie directly above/below or left/right of a particular square. This rule for adjacency resembles the movements of a rook on a chessboard, assuming the rook's move is limited to one square at a time.

If P_n denotes a path with n vertices and C_n is a cycle with n vertices, we will consider the naturally defined half-dependent problem in each of the three customarily grid type graphs: $G_n := P_n \times P_n$ (the usual grid graph), $GC_n := C_n \times P_n$ (cylindrical grid graph), and $GT_n := C_n \times C_n$ (toroidal grid graph). We are going to use the model of an $n \times n$ chess board in order to think about these graphs. The half-dependent problem in each one of these graphs is to determine the maximum cardinality of a set of prisoners such that each one has at least as many guards around (neighbors) as other prisoners. Let us denote by $P_{grid}(n)$, $P_{grid\ cylinder}(n)$, and $P_{grid\ torus}(n)$ respectively, the maximum cardinality of a (1/2)-dominating set of prisoners in each of the corresponding graphs as above.

As shown in [7] we can easily derive that the following limits exist and in fact

$$\lim_{n\to\infty} \frac{P_{grid}(n)}{n^2} = \lim_{n\to\infty} \frac{P_{grid\ cylinder}(n)}{n^2} = \lim_{n\to\infty} \frac{P_{grid\ torus}(n)}{n^2} := \rho_{1/2}(grid).$$

This is based on the fact that $\limsup_{n\to\infty} \frac{P_{torus}(n)}{n^2}$ exists and one shows that

$$\rho_{1/2}(grid) = \limsup_{n\to\infty} \frac{P_{grid\ torus}(n)}{n^2}.$$

From Figure 13, we notice that in the toroidal case the following arrangement in GT_3 (the shaded squares are prisoners and the unshaded represent guards) gives

$$\rho_{1/2}(grid) \geq 2/3.$$

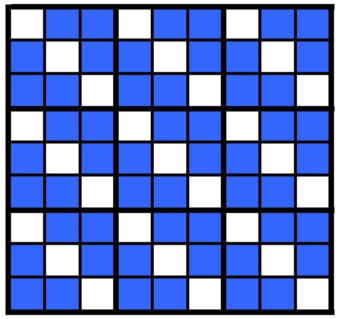

Figure 13: *6 prisoners out of 9*

We next show that the inequality

$$\rho_{1/2}(grid) \le 2/3 \tag{5}$$

must hold true. As before, we associate a binary matrix $X = (x_{ij})$ defined by

$$x_{ij} = \begin{cases} 1, & \text{if a prisoner lies in the } (i,j) \text{ position;} \\ 0, & \text{if a guard lies in the } (i,j) \text{ position.} \end{cases}$$

The problem in the toroidal case is described by

$$2x_{ij} + x^*_{ij} \le 4, \qquad 1 \le i, j \le n,$$

where x^*_{ij} is the number of prisoners lying in squares adjacent to the (i,j) square. If we add the above inequalities we get

$$2\sum_{i,j} x_{ij} + 4\sum_{i,j} x_{ij} \le 4n^2,$$

which gives $P_{grid\ torus}(n) \le \frac{2}{3}n^2$. Hence, we must have

Proposition 15.

$$\rho_{1/2}(grid) = \frac{2}{3}.$$

We calculated some of the values of the sequences $\{P_{grid}(n)\}$, $\{P_{grid\ cylinder}(n)\}$, and $\{P_{grid\ torus}(n)\}$ using LPSolve IDE. These values are listed in the table below. One may try to use our techniques from previous sections in order to prove that the numbers listed below are valid:

n	1	2	3	4	5	6	7	8	9	10	11	12
$P_{grid}(n)$	1	2	5	9	14	20	28	37	47			
$P_{grid\ cylinder}(n)$	0	2	5	8	14	20	28	37	48			
$P_{grid\ torus}(n)$	0	2	6	9	15	24	30	40	54	63	77	96

Some arrangements that give the maximum number of prisoners in the usual grid graph for the half-dependent problem are included in Figure 14.

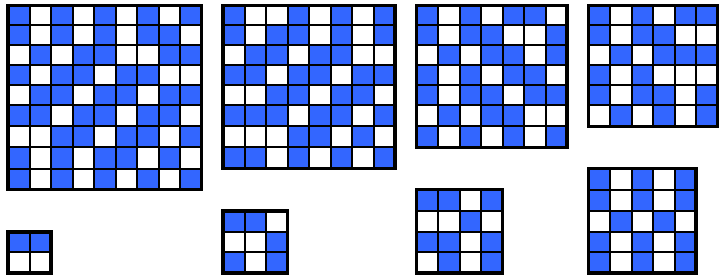

Figure 14: Best half-dependent arrangements for G_n, $n = 2 \ldots 9$

6 Other Results, Conjectures and Open Questions

We believe that the method of finding an upper bound we implemented in Section 4, can be further sharpened. However, we have not found a complete analysis to show that for instance $\Delta \geq -O(n)$ which, we think, would be the sharpest result for this type of domination problem. If this is true, the first approximation for $P(n)$ will be $P(n) = \frac{3n^2}{5} + O(n)$. There is another interesting phenomenon here that we would like to mention. The boundary conditions are in a sense less restrictive than the constraints in the center of the board. As a result it is expected to have quite a good proportion of prisoners on the boundary in a maximum configuration. In support of this we present two examples we found that give "best" (so far) arrangements/proportions in the cases $n = 15$ (Figure 15) and $n = 21$ (Figure 16). It seems to be possible to construct a sequence of arrangements for which

$$\liminf_{n\to\infty} \frac{P(n) - \frac{3n^2}{5}}{n} > 0.$$

Ionascu, Pritikin, and Wright have established values of $P(n)$ for $n \in \{7, 8, 9, 10, 11\}$ [7]. Most of their arrangements were obtained using the LPSolve IDE program in the Lesser GNU public domain for solving integer linear programming problems with Branch-and-Bound and Simplex Methods. The second author used CPLEX while visiting at the Georgia Institute of Technology in the Faculty Development Program in 2005-2006; with the help of Professor William Cook he analyzed the case $n = 11$.

Many interesting questions remain to be answered. What are the values of $P(n)$ for integers n larger than 11? With error-free play, does one particular player enjoy an advantage? Perhaps the advantage varies with the board size.

If $P(n)$ is odd, we conjecture that the game favors the red player, but it is not clear that a winning strategy exists. When $P(n)$ is even, we suspect that error-free play by both players will lead to a tie.

Given that we find several maximal 4×4 board configurations with eight prisoners (an even number), it seems that the second player (blue) will find opportunities to win unless s/he is forced to use Rule I. The question is: can the red player always achieve a win or a tie? We believe there is a strategy for the red player to win despite all of these chances for the blue player. In general, it appears that the final maximal configuration is an important factor in the game, since the number of prisoners in it determines the fate of the game. So it is in the red player's interest to end in a maximal arrangement with an odd number of

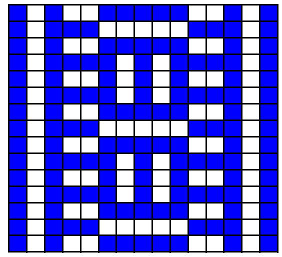

Figure 15: 136 prisoners, $\frac{3}{5}(15^2) = 135$

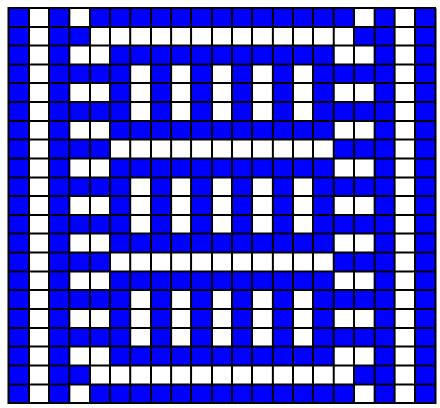

Figure 16: 266 prisoners, $\frac{3}{5}(21^2) = 264.6$

prisoners on the board. Similarly it is part of the blue player's strategy to divert the end configuration to a maximal one that has an even number of prisoners. Each player may change the configuration at only one place at which the opponent has previously placed his two prisoners and leave one of them as it is. As a result, almost half the prisoners on the final board configuration are where each player wanted them to be. So from this perspective the end game is dictated by the parity and the number of maximal configurations with $P(n)-1$, $P(n)-2$, ... prisoners.

In this paper, we have shown that $P(n)$ is bounded above by $\dfrac{7n^2+4n}{11}$, but we conjecture that $P(n) = 3n^2/5 + O(n)$. Computer assisted methods as employed in [7] show that this bound can be improved and that our conjecture is very plausible. It would be great to see a case analysis proof similar to those used here of our conjecture.

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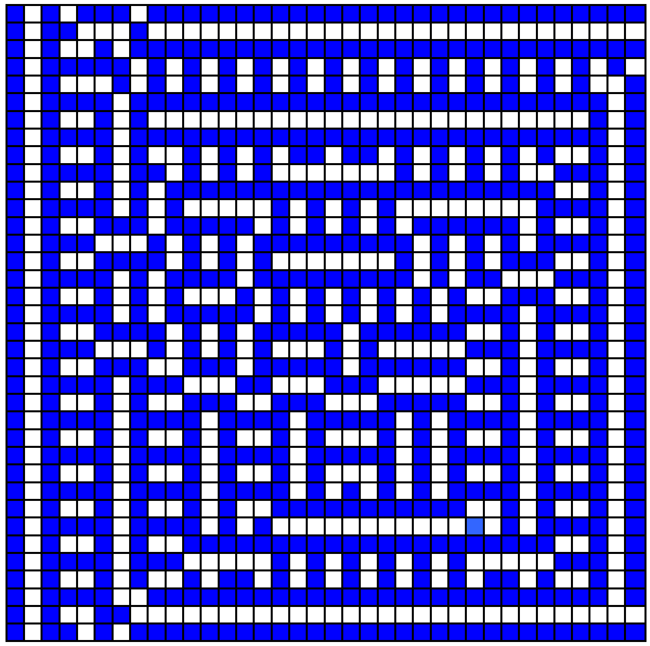

Figure 17: 777 prisoners, $\frac{3}{5}(36^2) = 777.6$

time, even for large size boards, and is producing arrangements that we think are "very close" to maximum arrangements. We include in Figure 17 one of them as a curiosity in the case 36×36 (containing 777 prisoners).

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