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A Natural Extension of Catalan Numbers

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Abstract

A Dyck path is a lattice path in the plane integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of steps $(1,1)$ and $(1,-1)$, each connecting diagonal lattice points, which never passes below the x-axis. The number of all Dyck paths that start at $(0,0)$ and finish at $(2n,0)$ is also known as the nth Catalan number. In this paper we find a closed formula, depending on a non-negative integer t and on two lattice points p_1 and p_2, for the number of Dyck paths starting at p_1, ending at p_2, and touching the x-axis exactly t times. Moreover, we provide explicit expressions for the corresponding generating function and bivariate generating function.

1 Introduction

The *Catalan sequence* is the sequence $\{C_n\}_{n\geq 0} = \{1, 1, 2, 5, 14, 42, 132, 429, \ldots\}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is called the nth *Catalan number*. It is sequence A000108 in Sloane's *Encyclopedia of Integer Sequences*. The generating function for the Catalan numbers is given by $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. The Catalan numbers provide a complete answer to the problem of counting certain properties of more than 165 different combinatorial structures (see [21, p. 219] and [20]). For convenience, the generalization of Catalan numbers presented in this paper is translated in terms of Dyck paths.

A *Dyck path* is a lattice path in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of rise-steps $(1,1)$ and fall-steps $(1,-1)$ that connect diagonal lattice points, which never goes below the x-axis. (See Figure 1 for an illustration.) Let $D((i,j),(i',j'))$ denote the set of all Dyck paths that start at (i,j) and finish at (i',j'). The number of steps in any such Dyck path equals $i'-i$. Notice that $|D((i,j),(i',j'))| = 0$ iff at least one of the following conditions holds:

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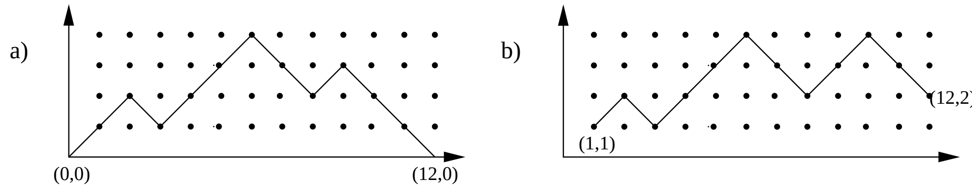

Figure 1: (a) A Dyck path of length 12 from $(0,0)$ to $(12,0)$, having exactly two contact points–the starting point and the finishing point. (b) A Dyck path of length 11 from $(1,1)$ to $(12,2)$, having no contact points.

- $i > i'$.
- $j' - j > i' - i$.
- $j' - j \neq i' - i \pmod 2$.
- $j' < 0$ or $j < 0$.

Otherwise, as a corollary of the Ballot theorem (cf. [10, p. 73]), we get that

$$|D((i,j),(i',j'))| = \binom{i'-i}{\frac{i'-i+|j'-j|}{2}} - \binom{i'-i}{\frac{i'-i+j'+j+2}{2}}. \tag{1}$$

The number of Dyck paths that start at $(0,0)$ and finish at $(2n,0)$ is also known as the nth Catalan number C_n.

There is an extensive literature on Dyck paths, often disguised by means of similar combinatorial objects as Catalan numbers, Motzkin paths, Schröder paths, staircase walks, Ballot-numbers, and more. We mention here but a limited number of examples of previous work.

The notion of a peak on a Dyck path was introduced by Deutsch in [7], where it is shown that the number of Dyck paths of length $2n$ starting and ending on the x-axis with no peaks at height 1 is given by the nth Fine number F_n, $\{F_n\}_{n\geq 0} = \{1,0,1,2,6,18,57,\ldots\}$. In [19], a complete answer for the number of Dyck paths in $D((0,0),(2n,0))$ with no peaks at height k is given. Further, an explicit expression for the generating function for the number of Dyck paths in $D((0,0),(2n,0))$ with exactly r peaks at height k is provided [15].

Connections between Dyck paths and pattern-avoiding permutations have been a subject of ongoing research. Among various works in this context is the work of Knuth [13], where it is shown that $S_n(312)$ satisfy the Catalan recurrence, and the works of Bandlow and Killpatrick [2], Krattenthaler [14], and Mansour et al. [16], where bijections between Dyck paths and permutations that avoid certain patterns of size three are presented.

The close relationship between lattice paths and queueing theory models has been extensively studied. The seminal papers by Mohanty [17] and Flajolet and Guillemin [11] offer lattice path perspectives for the Karlin-McGregor theory of birth-death processes, which is closely related to various queueing theory models. The book by Fayolle et al. on random walks in the plane integer lattice [9] is historically motivated by such queueing theory questions [8]. In [18], a combinatorial technique based on lattice path counting is applied to derive the transient solution of the $M/M/c$ queueing model, and in [3] this result is extended to (almost) arbitrary birth-death processes. In [1], equilibrium probability distributions of the queue length in the $M/M/1$, $M/E_k/1$ and $E_l/E_k/1$ queueing models are presented, based on the generating function for the number of minimal lattice paths. In [6], service times of customers in the $M/M/1$ queueing model are analyzed, and it is shown that a family of polynomial generating sequence associated with Dyck paths of length $2n$ provide the correlation function of the successive services in a busy period with $n+1$ customers.

Brak and Essam [4] considered the case of $k \geq 1$ non-intersecting Dyck paths that start and finish on the x-axis. For the particular case of $k=1$, the *contact polynomial* defined as $P_n(x) := \sum_{t=2}^{n+1} |D_t((0,0),(2n,0))|x^t$ is proved[2] to satisfy $P_n(C(x)) = \sum_{\ell=0}^{\infty} C_{n+\ell}x^\ell$. Via a bijection between *bi-colored* Dyck paths and plain Dyck paths, the analogue of the Chu-Vandemonde summation formula for Dyck paths is derived.

In this paper we study "generalized" Dyck paths, starting and ending at arbitrary points in the non-negative half-plane, "touching" the x-axis any *predetermined* number of times, and never passing below the x-axis. We say that a Dyck path P *touches* the x-axis t times, if exactly t points on P have zero as their y coordinate. Following [4], any point of P that intersects the x-axis is called a *contact*. Denote the set of all Dyck paths starting at (i,j), ending at (i',j'), and touching the x-axis exactly t times by $D_t((i,j),(i',j'))$. We have $D((i,j),(i',j')) = \bigcup_{t\in\mathbb{N}} D_t((i,j),(i',j'))$. Notice that

$$|D_0((0,1),(2n,1))| = |D((0,0),(2n,0))| = C_n.$$

More generally, we have $|D_0((i,j),(i',j'))| = |D((i,j-1),(i',j'-1))|$. In the sequel, we henceforth concentrate on Dyck paths that touch the x-axis *at least once*.

1.1 Main Results

Our main contribution is the following theorem, for which we provide a simple combinatorial proof.

Theorem 1. *For any pair of lattice points (i,j) and (i',j'), and for any integer $t \geq 1$,*

$$|D_t((i,j),(i',j'))| = \begin{cases} 0, & \text{if } j < 0 \text{ or } j' < 0; \\ |D((i,j+t-2),(i'-t,0))|, & \text{if } j' = 0; \\ |D((i,j+t-1),(i'-t,j'-1))| - \\ |D((i,j+t-2),(i'-t,j'-2))|, & \text{if } j' > 0. \end{cases}$$

In the degenerate case $t+|j|+|j'|+|i'-i| = 1$, $|D_t((i,j),(i',j'))| = 1$.

We also find an explicit expression for the corresponding generating function.

[2] In [4], a combinatorial proof for $P_n(C(x)) = \sum_{\ell=0}^{\infty} C_{n+\ell}x^\ell$ is provided. However, this result had already been proved analytically in an earlier work [5].

Theorem 2. *For any non-negative integers j, j' and t, with $t \geq 1$,*

$$D_t^{j,j'}(x) \; := \; \sum_{n\geq 0} d_t^{j,j'}(n)x^n \; = \; \frac{(1-\sqrt{1-4x^2})^{t+j+j'-1}}{2^{(t+j+j'-1)} \cdot x^{(j+j')}},$$

where $D_t^{j,j'}(x)$ is the generating function of the sequence $d_t^{j,j'}(n) := |D_t((0,j),(n,j'))|$.

We derive the bivariate generating function as an easy consequence of Theorem 2.

Corollary 3. *For any non-negative integers j and j',*

$$\Phi^{j,j'}(x,y) \; := \; \sum_{t\geq 1, n\geq 0} d_{j,j'}(n,t)x^n y^t \; = \; \frac{y(1-\sqrt{1-4x^2})^{j+j'}}{2^{(j+j'-1)}x^{(j+j')}(y\cdot\sqrt{1-4x^2}+2-y)},$$

where $\Phi^{j,j'}(x,y)$ is the bivariate generating function of the sequence $d_{j,j'}(n,t) := |D_t((0,j),(n,j'))|$.

1.2 Minor Results

- By Theorem 1 and (1), we determine the coefficients of the contact polynomial, as follows:

$$\begin{aligned} P_n(x) \; &:= \; \sum_{t=2}^{n+1} |D_t((0,0),(2n,0))|x^t \; = \; \sum_{t=2}^{n+1} |D((0,0),(2n-t,t-2))|x^t \\ &= \; \sum_{t=2}^{n+1} \left[\binom{2n-t}{n-1} - \binom{2n-t}{n}\right] x^t \; = \; \sum_{t=2}^{n+1} \frac{t-1}{n-t+1}\binom{2n-t}{n}x^t. \end{aligned}$$

- The following formula, which is the analogue of the Chu-Vandemonde summation formula for Dyck paths, was proved in [4]:

$$\frac{1}{n+b+1}\binom{2n+2b}{n+b} \; = \; \sum_{m=1}^{n} \frac{m}{n}\binom{2n-m-1}{n-1}\frac{m+1}{b+m+1}\binom{2b+m}{b}, \forall b \geq 0, n \geq 1 \tag{2}$$

  In Section 4, we provide a simple proof for (2) using different techniques that involve our new knowledge on the contact polynomial and an interesting identity from [5].

1.3 Notation and Basic Facts

Consider some infinite sequence $\{s_n\}_{n\geq 0} := \{s_0, s_1, s_2, \ldots\}$. It is common to denote the nth element s_n also as $s(n)$ and the entire sequence $\{s_n\}_{n\geq 0}$ also as s. For $n < 0$, define $s_n := 0$. For any integer m, define $s[m]$ as the sequence $\{s_m, s_{m+1}, \ldots\}$, namely, $s[m]_n = s_{n+m}$ if $n \geq 0$, and 0 otherwise. We define $\mathcal{H}(s)$ as the sequence $\{s_0, 0, s_1, 0, s_2, 0 \ldots\}$, namely, $\mathcal{H}(s)_n = s_{\frac{n}{2}}$ if n is even, and 0 otherwise. We define s^k to be the result of applying k convolutions of s with itself, that is, $s^k := s * s \ldots * s$.

Fact 1. *For any sequences of non-negative integers s and t with generating functions $S(x)$ and $T(x)$, respectively, we have*

- *The generating function of their convolution $s * t$ is $S(x) \cdot T(x)$.*
- *The generating function of $s[m]$ is*[3] *$S(x) \cdot x^{-m}$.*
- *The generating function of $\mathcal{H}(s)$ is $S(x^2)$.*
- *The generating function of s^k is $S(x)^k$.*

We will use the following observation in the sequel (often implicitly):

Observation 1. *For any integers i, i', j, j' and t, with $t \geq 1$, there are natural bijections between*

- *$D_t((i,j),(i',j'))$ and $D_t((0,j),(i'-i,j'))$.*
- *$D_t((i,j),(i',j'))$ and $D_t((i,j'),(i',j))$.*

2 Proof of Theorem 1

This section is organized as follows. We start with an investigation of Dyck paths that touch the x-axis exactly once (Section 2.1). Then, we show that there is a bijection between the set of Dyck paths that touch the x-axis any predetermined number of times and the set of Dyck paths that touch it just once (Section 2.2). Equipped with appropriate tools, we conclude with a simple proof of Theorem 1 (Section 2.3).

Remark 4. Observe that there is a natural bijection between

$$D_t((i,j),(i',0))$$

and

$$D_{t-1}((i,j),(i'-1,1)),$$

implying that the case $j' = 0$ of Theorem 1 may be deduced directly from the case $j' > 0$. Nevertheless, we provide an alternative proof for the case $j' = 0$, independent of the case $j' > 0$.

2.1 Touching the x-Axis Just Once

In this section we restrict our attention to Dyck paths that touch the x-axis once.

Lemma 5. *For any integers i, j and i', such that $j > 0$,*

$$|D_1((i,j),(i',0))| = |D((i,j-1),(i'-1,0))|.$$

[3]In the sequel, whenever a shifted sequence $s[m]$ with a positive integer m is used, $S(x)$ will always be divisible by x^m. Thus the notation $S(x) \cdot x^{-m}$ is unambiguous .

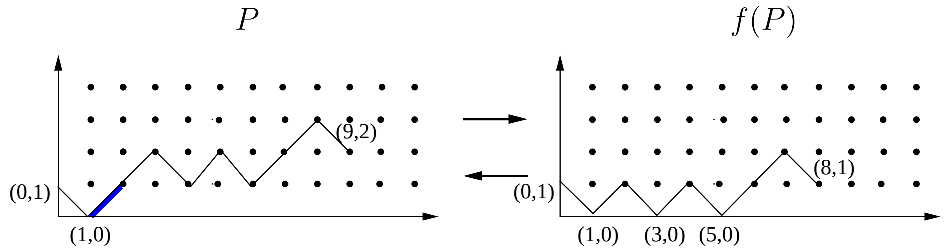

Figure 2: The Dyck path P on the left is in $D_1((0,1),(9,2))$, having $(1,0)$ as a unique contact point. The Dyck path $f(P)$ on the right is in $D_3((0,1),(8,1))$, having in addition to $(1,0)$ also $(3,0)$ and $(5,0)$ as contact points.

Proof. For any Dyck path P in $D_1((i,j),(i',0))$, its last move is a fall-step, connecting $(i'-1,1)$ and $(i',0)$. Omitting this last move from P yields a Dyck path of length $i'-1$ that finishes at $(i'-1,1)$, never passing below $y=1$. This naturally induces a bijection between $D_1((i,j),(i',0))$ and $D_0((i,j),(i'-1,1))$. The proof is concluded by observing that $|D_0((i,j),(i'-1,1))| = |D((i,j-1),(i'-1,0))|$. □

Lemma 6. *For any integers i,j,i' and j', such that $j \geq 0$ and $j' > 0$,*

$$|D_1((i,j),(i',j'))| = |D((i,j),(i'-1,j'-1))| - |D((i,j-1),(i'-1,j'-2))|.$$

Proof. We first claim that there is a bijection between

$$D_1((i,j),(i',j'))$$

and

$$S := \bigcup_{k>0} D_k((i,j),(i'-1,j'-1)).$$

To see this, consider the following map f, taking a Dyck path P in $D_1((i,j),(i',j'))$ and sending it to the one obtained from it by omitting the step which occurs immediately after reaching the contact point. (See Figure 2 for an illustration.) Clearly, $f(P)$ touches the x-axis at least once, starts at (i,j), and finishes at $(i'-1,j'-1)$; that is, $f(P) \in \bigcup_{k>0} D_k((i,j),(i'-1,j'-1))$. It is easy to verify that f is bijective, as required.

Observe that

$$S = D((i,j),(i'-1,j'-1)) \setminus D_0((i,j),(i'-1,j'-1))$$

and

$$|D_0((i,j),(i'-1,j'-1))| = |D((i,j-1),(i'-1,j'-2))|,$$

and we are done. □

2.2 The General Case

We use the next statement in the special case $\ell = t-1$ for proving Theorem 1.

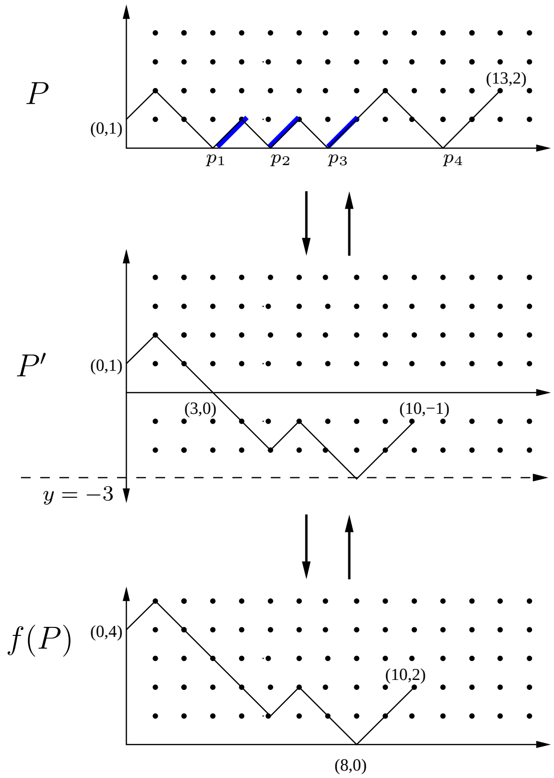

Figure 3: The Dyck path P at the top is in $D_4((0,1),(13,2))$, and has four contact points: $p_1 = (3,0)$, $p_2 = (5,0)$, $p_3 = (7,0)$ and $p_4 = (11,0)$. The path P' at the middle part of the figure is a diagonal lattice path of length 10 from $(0,1)$ to $(10,-1)$, obtained from P by omitting the three rise-steps U_1, U_2 and U_3. The Dyck path $f(P)$ at the bottom is in $D_1((0,4),(10,2))$, having one contact point–$(8,0)$.

Proposition 7. *For any integers i, j, i', j', ℓ, such that $j, j' \geq 0, t \geq 1$ and $0 \leq \ell \leq t-1$, there is a bijection between $D_t((i,j),(i',j'))$ and $D_{t-\ell}((i,j+\ell),(i'-\ell,j'))$.*

Proof. We construct a bijection $f : D_t((i,j),(i',j')) \to D_{t-\ell}((i,j+\ell),(i'-\ell,j'))$. Let P be some path in $D_t((i,j),(i',j'))$. Denote the t contact points of P, from left to right, by $p_1, \ldots, p_t$, and let U_i be the step in P which occurs immediately after reaching p_i, for each $1 \leq i \leq \ell$. As P does not pass below the x-axis, the U_i's are all rise-steps. Consider the Dyck path P' obtained from P by omitting these ℓ rise-steps and define $f(P)$ as the one obtained from it by a shift of all the points in it, ℓ units up. (See Figure 3 for an illustration in the case $\ell = t-1$.)

Observe that $f(P)$ starts at $(i, j+\ell)$, finishes at $(i'-\ell, j')$, touches the x-axis exactly $t-\ell$ times, and never passes below the x-axis. Hence, $f(P) \in D_{t-\ell}((i,j+\ell),(i'-\ell,j'))$. To show that f is bijective, we construct its inverse. Take some Dyck path Q in $D_{t-\ell}((i,j+\ell),(i'-\ell,j'))$. Let Q' be the Dyck path obtained from Q by a shift of all points in it, ℓ units down. Let q_1 be the left-most point in Q' that touches the x-axis and define q_i as the left-most point in Q' with y-coordinate $1-i$, for each $2 \leq i \leq \ell$. We define $f^{-1}(Q)$ to be the Dyck path obtained from Q' by inserting a rise-step U_i immediately after reaching q_i, for each $1 \leq i \leq \ell$. It is easy to verify that $f^{-1}(Q) \in D_t((i,j),(i',j'))$ and $f(f^{-1}(Q)) = Q$. The proposition follows. □

2.3 Proof of Theorem 1

Suppose first that $t+|j|+|j'|+|i'-i| > 1$. The result is immediate if either $j < 0$ or $j' < 0$. We henceforth assume that $j, j' \geq 0$.
Setting $\ell = t-1$ in Proposition 7, we find that

$$|D_t((i,j),(i',j')| = |D_1((i,j+t-1),(i'-t+1,j')|.$$

We distinguish the two following cases:

Case 1: $j' = 0$. We need to show that

$$|D_1((i,j+t-1),(i'-t+1,0))| = |D((i,j+t-2),(i'-t,0))|.$$

If $j+t-1 > 0$, then this equation follows from Lemma 5. Otherwise $t = 1, j = j' = 0$, and so, $|i'-i| > 0$. It is easy to verify that both hand sides of the equation vanish in the latter case.

Case 2: $j' > 0$. By Lemma 6, we have

$$|D_1((i,j+t-1),(i'-t+1,j'))| = |D((i,j+t-1),(i'-t,j'-1))| - |D((i,j+t-2),(i'-t,j'-2))|.$$

In the degenerate case $t+|j|+|j'|+|i'-i| = 1$, we have $t = 1, j = j' = 0$, and $i' = i$. The result in this case follows from the fact that $D_1((i,0),(i,0))$ is comprised of a single Dyck path of length zero that starts and finishes at $(i,0)$.

3 Proof of Theorem 2

Define $\phi(n,k) := |D_{k+2}((0,0),(n,0))|$, for any non-negative integers n and k, and let $a_n := \phi(n,0)$. It is easy to verify that for odd values of n, $a_n = 0$, whereas for even values of n, $a_n = c_{\frac{n}{2}-1}$, except for $n = 0$, where $a_0 = 0$. Equivalently, we have $a_n = \mathcal{H}(C[-1])_n$. Thus, the generating function for the sequence a is given by $A(x) = \frac{1-\sqrt{1-4x^2}}{2}$.

We use Lemmas 8 and 9 to prove Theorem 2.

Lemma 8. *For any non-negative integers n and k, $\phi(n,k) = a^{k+1}(n)$.*

Proof. We construct a bijection

$$g : D_{k+2}((0,0),(n,0)) \to \bigoplus_{\{1 \leq n_i \in \mathbb{N} | n = \sum_{i=1}^{k+1} n_i\}} \prod_{i=1}^{k+1} D_2((0,0),(n_i,0)).$$

Let P be some path in $D_{k+2}((0,0),(n,0))$. Denote the $k+2$ contact points of P, from left to right, by $p_1 = (x_1,0), \ldots, p_{k+2} = (x_{k+2},0)$. (Note that $x_1 = 0, x_{k+2} = n$.) Those points

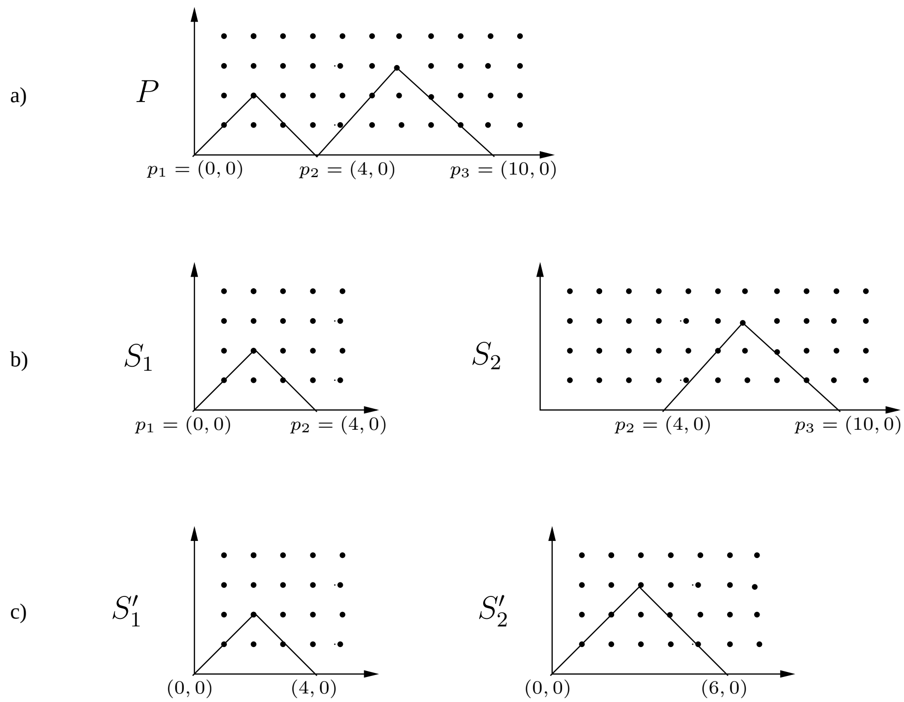

Figure 4: (a) The Dyck path P is in $D_3((0,0),(10,0))$, and has three contact points: $p_1 = (0,0)$, $p_2 = (4,0)$ and $p_3 = (10,0)$. (b) A partition of P according to its contact points into two consecutive sub-paths S_1 and S_2, where S_1 is in $D_2((0,0),(4,0))$ and S_2 is in $D_2((4,0),(10,0))$. (c) The Dyck path S'_1 (respectively, S'_2) is obtained from S_1 (resp., S_2) by a shift of all points in it, $x_1 = 0$ (resp., $x_2 = 4$) units left.

define a partition of P into consecutive sub-paths, namely, for each $1 \le i \le k+1$, we define S_i to be the sub-path of P between p_i and p_{i+1}. For each $1 \le i \le k+1$, define $n_i = x_{i+1} - x_i$, and let S'_i be the Dyck path in $D_2((0,0),(n_i,0))$ obtained from S_i by a shift of all points in it, x_i units left. (See Figure 4 for an illustration.) We define $g(P)$ as $\prod_{i=1}^{k+1} S'_i$. Clearly, this map is bijective (the inverse is given by the concatenation of all sub-paths $\{S'_i\}$ in their respective order). This implies that both sets have the same cardinality, that is,

$$\phi(n,k) = \sum_{\{1 \le n_i \in \mathbb{N} \mid n = \sum_{i=1}^{k+1} n_i\}} \prod_{i=1}^{k+1} |D_2((0,0),(n_i,0))| = \sum_{\{1 \le n_i \in \mathbb{N} \mid n = \sum_{i=1}^{k+1} n_i\}} \prod_{i=1}^{k+1} a_{n_i}. \qquad (3)$$

Since $a_0 = 0$, the right-hand side of (3) is equal to $\sum_{\{0 \le n_i \in \mathbb{N} \mid n = \sum_{i=1}^{k+1} n_i\}} \prod_{i=1}^{k+1} a_{n_i}$. But the latter term is precisely $a^{k+1}(n)$, which completes the proof. $\square$

The following statement implies that the problem of finding the number of Dyck paths that start and finish at points with *arbitrary* y-coordinates is equivalent to the problem of finding the number of Dyck paths that start and finish on the x-axis. It follows as a simple corollary of Proposition 7 and Observation 1.

Lemma 9. *For any integers i, i', j, j' and t, such that $j, j' \geq 0$ and $t \geq 1$, there is a bijection between $D_t((i,j),(i',j'))$ and $D_{t+j+j'}((i,0),(i'+j+j',0))$.*

To complete the proof of Theorem 2, we distinguish the two following cases:
Case 1: $t+j+j' = 1$. In this case $t = 1, j = j' = 0$, and so, $d_j^{j,j'}(n) = d_1^{0,0}(n) = |D_1((0,0),(n,0))|$. By Theorem 1, $d_1^{0,0}(n) = 0$ for all $n > 0$, and $d_1^{0,0}(0) = 1$. It follows that

$$D_1^{0,0}(x) \;=\; \sum_{n\geq 0} d_1^{0,0}(n)x^n \;=\; 1 \;=\; \frac{(1-\sqrt{1-4x^2})^{t+j+j'-1}}{2^{(t+j+j'-1)}\cdot x^{(j+j')}}.$$

Case 2: $t+j+j' \geq 2$. By Lemma 9, we have

$$d_t^{j,j'}(n) = |D_t((0,j),(n,j'))| = |D_{t+j+j'}((0,0),(n+j+j',0))| = \phi(n+j+j', t+j+j'-2).$$

By Lemma 8, we deduce that

$$d_t^{j,j'}(n) = \phi(n+j+j', t+j+j'-2) = a^{t+j+j'-1}(n+j+j') = a^{t+j+j'-1}[j+j'](n).$$

Therefore,

$$D_t^{j,j'}(x) \;=\; \sum_{n\geq 0} d_t^{j,j'}(n)x^n \;=\; \frac{A(x)^{t+j+j'-1}}{x^{(j+j')}} \;=\; \frac{(1-\sqrt{1-4x^2})^{t+j+j'-1}}{2^{(t+j+j'-1)}\cdot x^{(j+j')}}.$$

4 Chu-Vandemonde Summation Formula

In this section we provide a simple proof for the analogue of the Chu-Vandemonde summation formula for Dyck paths, which was already obtained by different techniques in [4, p. 4, Eq. (18)] Refer to [12] for similar results.

Following [4], we define $B_{p,q} := |D((0,0),(p,q))|$, for any non-negative integers p and q. By Theorem 1 and the second part of Observation 1, the equation

$$P_n(x) = \sum_{t=2}^{n+1} |D_t((0,0),(2n,0))|x^t = \sum_{t=2}^{n+1} |D((0,t-2),(2n-t,0))|x^t = \sum_{t=2}^{n+1} B_{2n-t,t-2}x^t$$

holds. In [4, 5], it is proved that $P_n(C(x)) = \sum_{b=0}^{\infty} C_{n+b}x^b$. We obtain the following identity:

$$\begin{aligned}
&\textstyle\sum_{b=0}^{\infty} C_{n+b}x^b \;=\; \sum_{t=2}^{n+1} B_{2n-t,t-2}(\frac{1-\sqrt{1-4x}}{2x})^t \;=\; \sum_{t=2}^{n+1} B_{2n-t,t-2}\frac{(\frac{1-\sqrt{1-4x}}{2})^t}{x^t} \;= \\
&\textstyle\sum_{t=2}^{n+1} B_{2n-t,t-2}\sum_{m=0}^{\infty} a^t(2m+2t)x^m \;=\; \sum_{b=0}^{\infty}(\sum_{t=2}^{n+1} B_{2n-t,t-2}\cdot a^t(2b+2t))x^b.
\end{aligned} \tag{4}$$

By Lemma 8 and Theorem 1, we have

$$a^t(2b+2t) = |D_{t+1}((0,0),(2b+2t,0))| = |D((0,t-1),(2b+t-1,0))| = B_{2b+t-1,t-1}.$$

Equating coefficients in (4) yields

$$C_{n+b} = \sum_{t=2}^{n+1} B_{2n-t,t-2}B_{2b+t-1,t-1} = \sum_{m=1}^{n} B_{2n-m-1,m-1}B_{2b+m,m}, \forall b \geq 0$$

or equivalently,

$$\frac{1}{n+b+1}\binom{2n+2b}{n+b} = \sum_{m=1}^{n} \frac{m}{n}\binom{2n-m-1}{n-1}\frac{m+1}{b+m+1}\binom{2b+m}{b}, \forall b \geq 0, n \geq 1$$

which is the analogue of the Chu-Vandemonde formula for Dyck paths.

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