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Generating Functions Related to Partition Formulæ for Fibonacci Numbers

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Abstract

The generating functions of 2 double schemes of numbers are explicitly computed using the kernel method, which leads to easy proofs of partition formulæ for Fibonacci numbers.

1 Introduction

The numbers given by the recursions

$$
\begin{aligned}
b_{n+1,k} &= c_{n,k-1} + 2c_{n,k} - b_{n,k},\\
c_{n+1,k} &= b_{n+1,k} + 2b_{n+1,k+1} - c_{n,k}
\end{aligned}
$$

for $n \geq 0$, with $b_{0,0} = c_{0,0} = 1$, $c_{n,-1} = c_{n,0}$ are used by Fahr and Ringel in [1] to partition the Fibonacci numbers. We want to shed new light on these numbers, by computing their (bivariate) generating functions. These lead then also to straight forward proofs of the partition formulæ given by Fahr and Ringel in [1]. Notice, however, that these proofs do not provide any *insight*.

2 The generating functions

Introducing generating functions

$$B(z,x) := \sum_{0\le k\le n} b_{n,k} z^n x^k,$$
$$C(z,x) := \sum_{0\le k\le n} c_{n,k} z^n x^k,$$

these recursions translate into

$$B(z,x) = 1 + zxC(z,x) + 2zC(z,x) - zB(z,x) + zC(z,0),$$
$$C(z,x) = B(z,x) + \frac{2}{x}[B(z,x) - B(z,0)] - zC(z,x).$$

This leads to

$$C(z,x) = -\frac{x + z(x-4)C(z,0)}{zx^2 - z^2x + 2zx - x + 4z}.$$

To solve that, we factor the denominator:

$$C(z,x) = \frac{-x + z(4-x)C(z,0)}{z(x-r_1(z))(x-r_2(z))}$$

with

$$r_{1,2}(z) = \frac{(1-z)^2 \pm (1+z)\sqrt{1-6z+z^2}}{2z}.$$

Since $1/(x - r_2(z))$ has no power series expansion in z and x, the factor must cancel, i.e.,

$$-r_2(z) + z(4 - r_2(z))C(z,0) = 0,$$

whence

$$C(z,0) = \frac{-1 + 4z - z^2 + (1+z)\sqrt{1-6z+z^2}}{2z(1-7z+z^2)}.$$

This is the famous *kernel method*, see, e.g., [2].

After cancellation, this leads to

$$C(z,x) = \frac{4r_2(z)}{2z(1-4xr_2(z))}\frac{1-10z+z^2+(1+z)\sqrt{1-6z+z^2}}{1-7z+z^2},$$

and

$$[x^k]C(z,x) = \frac{(4r_2(z))^{k+1}}{2z}\frac{1-10z+z^2+(1+z)\sqrt{1-6z+z^2}}{1-7z+z^2}.$$

From this we get

$$B(z,x) = \frac{-(1+z)(z^2x - 8xz + 4z + x) + (-z^2x + 12z + 4\,zx - x)\sqrt{1-6z+z^2}}{2(1-7z+z^2)(-z^2x + 2zx - x + 4z + zx^2)}.$$

The formula

$$f_{4(n+1)} = 3 \sum_{k \geq 0} 4^k c_{n,k},$$

given by Fahr and Ringel in [1], can now easily be verified, since the generating function of the right-hand side is

$$3C(z,4) = \frac{3}{1 - 7z + z^2},$$

which is also the generating function of the left-hand side, which can be seen for example from the Binet form of the Fibonacci numbers.

The other formula

$$f_{4n+2} = b_{n,0} + \frac{3}{2} \sum_{k \geq 1} 4^k b_{n,k},$$

follows from the generating function

$$B(z,0) + \frac{3}{2}\Big(B(z,4) - B(z,0)\Big) = \frac{1+z}{1 - 7z + z^2}.$$

References

[1] Ph. Fahr and C.-M. Ringel, A partition formula for Fibonacci numbers, *J. Integer Sequences* **11** (2008), Article 08.1.4.

[2] H. Prodinger, The kernel method: A collection of examples, *Séminaire Lotharingien de Combinatoire*, B50f (2004).

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