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A Partition Formula for Fibonacci Numbers

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Abstract

We present a partition formula for the even index Fibonacci numbers. The formula is motivated by the appearance of these Fibonacci numbers in the representation theory of the socalled 3-Kronecker quiver, i.e., the oriented graph with two vertices and three arrows in the same direction.

1 Introduction

Let $f_0, f_1, \dots$ be the Fibonacci numbers, with $f_0 = 0, f_1 = 1$ and $f_{i+1} = f_i + f_{i-1}$ for $i \geq 1$. The aim of this note is to present a partition formula for the even index Fibonacci numbers f_{2n}. Our interest in the even index Fibonacci numbers comes from the fact that the pairs $[f_{4t+2}, f_{4t}]$ provide the dimension vectors of the preprojective representations of 3-Kronecker quiver, and the partition formula relates such a dimension vector to the dimension vector of a corresponding graded representation. The authors are grateful to M. Baake and A. Zelevinsky for useful comments which have been incorporated into the paper.

We start with the 3-regular tree (T, E) with vertex set T and edge set E (3-regularity means that every vertex of T has precisely 3 neighbours). Fix a vertex x_0 as base point. Let T_r be the sets of vertices of T which have distance r to x_0, thus $T_0 = \{x_0\}$, T_1 are the neighbours of x_0, and so on. The vertices in T_r will be called even or odd, depending on r being even or odd. Note that $|T_0| = 1$ and

$$|T_r| = 3 \cdot 2^{r-1} \qquad \text{for } r \geq 1. \tag{1}$$

(The proof for equation (1) is by induction: T_1 consists of the three neighbours of x_0. For $r \geq 1$, any element in T_r has precisely one neighbour in T_{r-1}, thus two neighbours in T_{r+1}, therefore $|T_{r+1}| = 2|T_r|$.)

Given a set S, let $\mathbb{Z}[S]$ be the set of functions $S \to \mathbb{Z}$ with finite support; this is the free abelian group on S (or better, on the set of functions with support being an element of S such that the only non-zero value is 1).

We are interested in certain elements $a_t \in \mathbb{Z}[T]$. Any vertex $y \in T$ yields a reflection σ_y on $\mathbb{Z}[T]$, defined for $b \in \mathbb{Z}[T]$ by

$$(\sigma_y b)(x) = \begin{cases} b(x) & \text{provided } x \neq y \\ -b(y) + \sum_{z \in N(y)} b(z) & \text{provided } x = y, \end{cases}$$

where $N(y)$ denotes the set of neighbours of y in (T, E).

Denote by Φ_0 the product of the reflections σ_y with y even; this product is independent of the order, since these reflections commute: even vertices never are neighbours of each other. Similarly, we denote by Φ_1 the product of the reflections σ_y with y odd.

The elements of $\mathbb{Z}[T]$ we are interested in are labelled a_t with $t \in \mathbb{N}_0$. We will start with the characteristic function a_0 of T_0 (thus, $a_0(x) = 0$ unless $x = x_0$ and $a_0(x_0) = 1$) and look at the functions

$$a_t = (\Phi_0 \Phi_1)^t a_0, \quad \text{with } t \geq 0.$$

Clearly, a_t is constant on T_r, for any $r \geq 0$, so we may define $a_t : \mathbb{N}_0 \to \mathbb{Z}$ by

$$a_t[r] = a_t(x) \quad \text{for } r \in \mathbb{N}_0 \text{ and } x \in T_r.$$

2 The Partition Formula

$$\begin{aligned} f_{4t} &= \sum_{r \text{ odd}} |T_r| \cdot a_t[r] = 3 \sum_{m \geq 0} 2^{2m} \cdot a_t[2m+1], \\ f_{4t+2} &= \sum_{r \text{ even}} |T_r| \cdot a_t[r] = a_t[0] + 3 \sum_{m \geq 1} 2^{2m-1} \cdot a_t[2m]. \end{aligned}$$

The sums exhibited are finite sums, since $a_t[r] = 0$ for $r > 2t$. Note that we have used the equality (1).

For example, for $t = 3$, we obtain the following two equalities:

$$\begin{aligned} 144 = f_{12} &= 3 \cdot 12 + 12 \cdot 5 + 48 \cdot 1 \\ 377 = f_{14} &= 29 + 6 \cdot 18 + 24 \cdot 6 + 96 \cdot 1. \end{aligned}$$

Here is the table for $t \leq 6$.

t	$a_t[0]$	$a_t[1]$	$a_t[2]$	$a_t[3]$	$a_t[4]$	$a_t[5]$	$a_t[6]$	$a_t[7]$	$a_t[8]$	$a_t[9]$	$\cdots$	f_{4t}	f_{4t+2}
0	1											0	1
1	2	1	1									3	8
2	7	3	4	1	1							21	55
3	29	12	18	5	6	1	1					144	377
4	130	53	85	25	33	7	8	1	1			987	2584
5	611	247	414	126	177	42	52	9	10	1	$\cdots$	6765	17711
6	2965	1192	2062	642	943	239	313	63	75	11	$\cdots$	46368	121393

3 Proof of the partition formula

We consider the multigraph $(\overline{T}, \overline{E})$ with two vertices $0, 1$ and three edges between 0 and 1.

Any element $c \in \mathbb{Z}[\overline{T}]$ is determined by the values $c(0)$ and $c(1)$, thus we will write $c = [c(0), c(1)]$.

We define

$$\overline{a}_t(0) = \sum_{r \text{ even}} |T_r| \cdot a_t[r] = a_t[0] + 3 \sum_{m \geq 1} 2^{2m-1} a_t[2m],$$

$$\overline{a}_t(1) = \sum_{r \text{ odd}} |T_r| \cdot a_t[r] = 3 \sum_{m \geq 0} 2^{2m} a_t[2m+1],$$

and have to show:

$$[\overline{a}_t(0), \overline{a}_t(1)] = [f_{4t+2}, f_{4t}].$$

The claim is obviously true for $t = 0$, since

$$[\overline{a}_0(0), \overline{a}_0(1)] = [1, 0] = [f_2, f_0].$$

The general assertion will follow from the following recursion formulae

$$[\overline{a}_t(0), \overline{a}_t(1)] \begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix} = [\overline{a}_{t+1}(0), \overline{a}_{t+1}(1)],$$

$$[f_{m+2}, f_m] \begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix} = [f_{m+6}, f_{m+4}].$$

For the Fibonacci numbers, this is well-known:

$$\begin{aligned}
[f_{m+6}, f_{m+4}] &= [f_{m+5}, f_{m+4}] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
&= [f_{m+1}, f_m] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
&= [f_{m+2}, f_m] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

and the product of the matrices in the last line is just $\begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix}$, which is the Coxeter transformation of the 3-Kronecker quiver.

It remains to deal with $\overline{a}_t$. We may consider (T, E) as the universal covering of $(\overline{T}, \overline{E})$, say with a covering map

$$\pi : (T, E) \to (\overline{T}, \overline{E}),$$

where $\pi(x) = 0$ if x is an even vertex of (T, E), and $\pi(x) = 1$ if x is an odd vertex. Such a covering map will provide a bijection between the three edges in E which contain a given vertex x with the edges in $\overline{E}$.

Given $b \in \mathbb{Z}[T]$, define $\overline{b} \in \mathbb{Z}[\overline{T}]$ by

$$\overline{b}(i) = \sum\nolimits_{x \in \pi^{-1}(i)} b(x) \qquad \text{for } i = 0, 1.$$

Observe that in this way we obtain from $a_t \in \mathbb{Z}[T]$ precisely $\overline{a}_t \in \mathbb{Z}[\overline{T}]$ as defined above.

On $\mathbb{Z}[\overline{T}]$, we also consider reflections, but we have to take into account the multiplicity of edges: There are the two reflections σ_0, σ_1, with

$$\begin{aligned}
(\sigma_0 c)(0) &= -c(0) + 3c(1), & (\sigma_0 c)(1) &= c(1), \\
(\sigma_1 c)(0) &= c(0), & (\sigma_1 c)(1) &= -c(1) + 3c(0),
\end{aligned}$$

for $c \in \mathbb{Z}[\overline{T}]$. Note that this implies that

$$[c(0), c(1)] \begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix} = [(\sigma_0\sigma_1 c)(0), (\sigma_0\sigma_1 c)(1)].$$

We finish the proof by observing that $\overline{a}_t = (\sigma_0\sigma_1)^t \overline{a}_0$. This follows directly from the following consideration: Let $b \in \mathbb{Z}[T]$, then

$$\overline{\Phi_0 b} = \sigma_0 \overline{b}, \quad \overline{\Phi_1 b} = \sigma_1 \overline{b},$$

as it is easily verified.

4 A second approach

A. Zelevinsky has pointed out that it may be worthwhile to stress the following separation: Let us denote by $b_t[r]$ and $c_t[r]$ the sequences

$$b_t[r] = a_t[2r] \quad \text{and} \quad c_t[r] = a_{t+1}[2r + 1].$$

Then we obtain two "Pascal-like" triangles

$$\begin{bmatrix} b_0[0] & b[1]_0 & b_0[2] & \cdots \\ b_1[0] & b_1[1] & b_1[2] & \cdots \\ b_2[0] & b_2[1] & b_2[2] & \cdots \\ b_3[0] & b_3[1] & b_3[2] & \cdots \\ b_4[0] & b_4[1] & b_4[2] & \cdots \\ & \cdots & & \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 7 & 4 & 1 & & \\ 29 & 18 & 6 & 1 & \\ 130 & 85 & 33 & 8 & 1 \\ & \cdots & & & \end{bmatrix}$$

and

$$\begin{bmatrix} c_0[0] & c_0[1] & c_0[2] & \cdots \\ c_1[0] & c_1[1] & c_1[2] & \cdots \\ c_2[0] & c_2[1] & c_2[2] & \cdots \\ c_3[0] & c_3[1] & c_3[2] & \cdots \\ c_4[0] & c_4[1] & c_4[2] & \cdots \\ & \cdots & & \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 12 & 5 & 1 & & \\ 53 & 25 & 7 & 1 & \\ 247 & 126 & 42 & 9 & 1 \\ & \cdots & & & \end{bmatrix}$$

The recursive definition is as follows: One starts with

$$b_0[r] = c_0[r] = \delta_{r,0}$$

(using the Kronecker delta) and one continues for $t \geq 0$ with

$$\begin{aligned} b_{t+1}[r] &= c_t[r-1] + 2c_t[r] - b_t[r], \\ c_{t+1}[r] &= b_{t+1}[r] + 2b_{t+1}[r+1] - c_t[r], \end{aligned}$$

where $c_t[-1] = c_t[0]$.

Using this notation, the partition formula can be written in the following way:

$$\boxed{f_{4t+2} = b_t[0] + 3\sum_{r=1}^{t} 2^{2r-1} \cdot b_t[r], \qquad f_{4(t+1)} = 3\sum_{r=0}^{t} 2^{2r} \cdot c_t[r].}$$

5 Remarks

1. The partition formula presented here seems to be very natural. We were surprised to see that the integer sequence $a_t[0]$ had not been listed in Encyclopedia of Integer Sequences [10]. It is now recorded as sequence A132262. There seems to be numerical evidence that the sequence $a_t[1]$ is just the sequence A110122 of the Encyclopedia of Integer Sequences (it counts the Delannoy paths of length t with no EE's crossing the line $y = x$), but we do not see why these sequences are the same.

2. The 3-regular tree which is used to depict the partition formula plays a prominent role in many parts of mathematics. It is the 3-Cayley tree wihout leafs, and a Bruhat-Tits tree. As M. Baake has pointed out to us, it is called the Bethe lattice [2] with coordination number 3 in the theory of exactly solvable models.

3. A similar method as presented here leads to a partition formula for the odd index Fibonacci numbers, see [5].

4. Our considerations can be generalized to the n-regular tree and the multigraph with two vertices and n edges, i.e., to the n-Kronecker quiver.

6 Motivation and background

Let us recall some basic concepts from the representation theory of quivers (see for example [1]); note that quivers are just directed graphs were mutiple arrows and even loops are allowed. Given any quiver Q, one considers its representations with coefficients in a fixed field k: they are obtained by attaching to each vertex a of Q a (usually finite-dimensional) k-vector space V_a, and to each arrow $\alpha : a \to b$ a k-linear map $V_\alpha : V_a \to V_b$. The function which assigns to the vertex a the k-dimension of V_a is called the dimension vector $\mathbf{dim}\, V$ of V; in case $V_a = 0$ for almost all vertices a, we obtain an element of $\mathbb{Z}[Q_0]$, where Q_0 is the vertex set of Q, and $\mathbb{Z}[Q_0]$ itself may be interpreted as the Grothendieck group of the finite dimensional nilpotent representations of Q with respect to all exact sequences. Observe that the representations of a quiver form an abelian category and one is interested in the indecomposable objects of this category. In case we deal with a finite quiver without loops, Kac [7] has shown that the dimension vectors of the indecomposable objects are precisely the positive roots $\mathbf{v}$ of the Kac-Moody Lie-algebra defined by the underlying graph of the quiver. In case $\mathbf{v}$ is a real root, then there is precisely one isomorphism class, otherwise there are at least two (actually infinitely many, in case k itself is infinite).

The quivers which are of interest in this note are, on the one hand, the 3-Kronecker quiver $\overline{Q}$ (with two vertices, say labelled 0 and 1, and three arrows $1 \to 0$) and, on the other hand, the bipartite quiver Q whose underlying graph (obtained by deleting the orientation of the arrows) is the 3-regular tree. The Kac-Moody Lie-algebra [6] corresponding to the 3-Kronecker quiver is given by the generalized Cartan matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$, its positive real roots are of the form

$$[f_{4t+2}, f_{4t}], \text{ and } [f_{4t}, f_{4t+2}] \text{ with } t \geq 0.$$

Note that the real roots of a Kac-Moody Lie-algebra are always obtained from the simple roots by reflections; in the given case, one uses powers of the Coxeter transformation $\begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix}$ in order to create the positive roots of the form $[f_{4t+2}, f_{4t}]$ starting with the vectors $[1,0]$ and $[3,1]$. Alternatively, one may stress that the square $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ of the matrix which exhibits the Fibonacci sequence, is conjugate to the matrix $\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$ which describes a basic reflection for the 3-Kronecker quiver.

There is a functorial realisation of the reflections as well as of the Coxeter transformation by Bernstein-Gelfand-Ponomarev [3]. The indecomposable representations of the 3-Kronecker quiver with dimension vector $[1,0]$ and $[3,1]$ are just the projective ones, those with dimension vectors of the form $[f_{4t+2}, f_{4t}]$ are said to be preprojective. Instead of dealing with the Coxeter functors of Bernstein-Gelfand-Ponoarev, one also may use the Auslander-Reiten translation functor (see [1]) in order to construct the preprojective representations starting from the projective ones. The preprojective representations have been exhibited in matrix form by one of the authors [9].

It remains to mention the covering theory of Gabriel and his students [4]. The quiver Q may be considered as the universal covering of the quiver $\overline{Q}$. Any representation V of Q gives rise to a representation $\overline{V}$ of $\overline{Q}$ (by attaching to the vertex 0 of $\overline{Q}$ the direct sum of the vector spaces attached to the various sinks of Q, and attaching to 1 the direct sum of the vector spaces attached to the sources). The representations of $\overline{Q}$ obtained in this way are those which are gradable by the free (non-abelian) group in 3 free generators. The covering functor $V \mapsto \overline{V}$ preserves indecomposability and satisfies

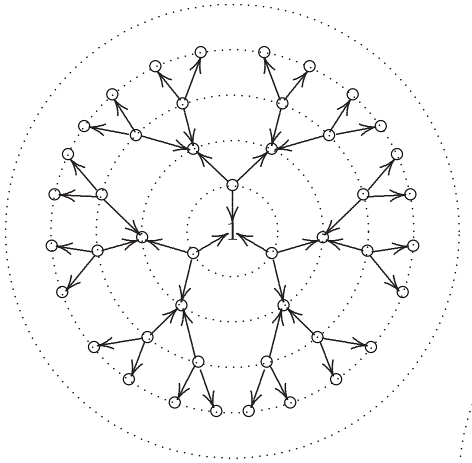
$$\mathbf{dim}\,\overline{V} = \overline{\mathbf{dim}\,V}. \qquad (*)$$

Any preprojective representation of $\overline{Q}$ is of the form $\overline{V}$, where V is an indecomposable representation of Q which is obtained from a simple representation of Q by applying suitable reflection functors, and our partition formula is just the assertion $(*)$.

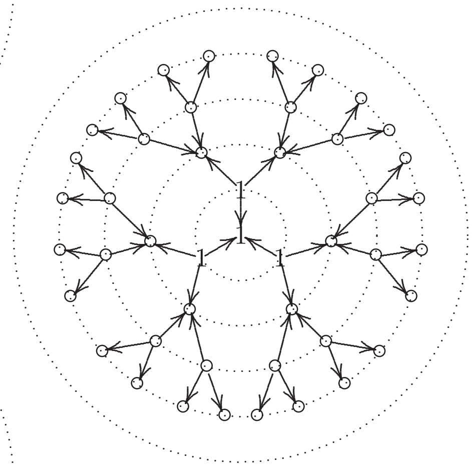
7 Illustrations

The illustrations below show the functions a_t for $t = 0, 1, 2$ as well as the corresponding functions $\Phi_1 a_t$. Here, (T, E) has been endowed with an orientation such that the edges point to the even vertices. In this way, the even vertices are sinks, the odd vertices are sources. The dotted circles indicate the various sets T_r, the center is just $T_0 = \{x_0\}$.

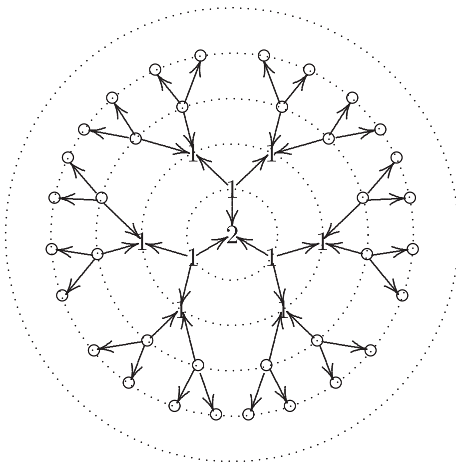
a_0



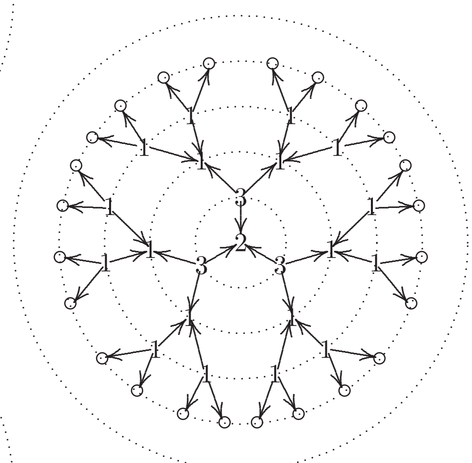
$\Phi_1 a_0$



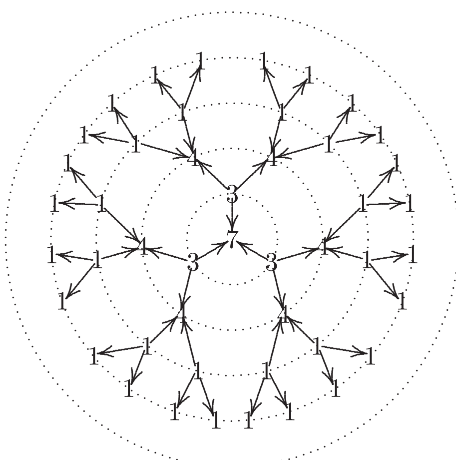
a_1



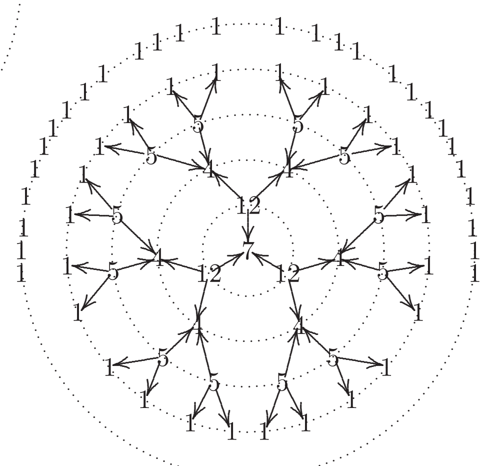
$\Phi_1 a_1$



a_2



$\Phi_1 a_2$



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(Concerned with sequences A110122 and A132262.)

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