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New Upper Bounds for Taxicab and Cabtaxi Numbers

Christian Boyer 53, rue de Mora FR-95880 Enghien-les-Bains France cboyer@club-internet.fr

Abstract

Hardy was surprised by Ramanujan's remark about a London taxi numbered 1729: "it is a very interesting number, it is the smallest number expressible as a sum of two cubes in two different ways". In memory of this story, this number is now called Taxicab(2) = $1729 = 9^3 + 10^3 = 1^3 + 12^3$, Taxicab(n) being the smallest number expressible in n ways as a sum of two cubes. We can generalize the problem by also allowing differences of cubes: Cabtaxi(n) is the smallest number expressible in n ways as a sum or difference of two cubes. For example, Cabtaxi(2) = $91 = 3^3 + 4^3 = 6^3 - 5^3$. Results were only known up to Taxicab(6) and Cabtaxi(9). This paper presents a history of the two problems since Fermat, Frenicle and Viète, and gives new upper bounds for Taxicab(7) to Taxicab(19), and for Cabtaxi(10) to Cabtaxi(30). Decompositions are explicitly given up to Taxicab(12) and Cabtaxi(20).

1 A Fermat problem solved by Frenicle

Our story starts 350 years ago, with letters exchanged between France and England during the reign of Louis XIV and the protectorate of Oliver Cromwell. On August 15th 1657, from Castres (in the south of France), Pierre de Fermat sent to Kenelm Digby some mathematical problems. Translated into English, two of them were:

- 1. Find two cube numbers of which the sum is equal to two other cube numbers.
- 2. Find two cube numbers of which the sum is a cube.

These two statements can be written algebraically as follows:

$$x^3 + y^3 = z^3 + w^3 \tag{1}$$

$$x^3 + y^3 = z^3. (2)$$

Fermat asked Digby, who was living in Paris at that time, to pass the problems on to William Brouncker, John Wallis, and Bernard Frenicle de Bessy, defying them to find solutions. Frenicle succeeded in finding several numerical solutions to (1), as announced in October 1657 in a letter sent by Brouncker to Wallis. The first solutions by Frenicle are:

$$1729 = 9^{3} + 10^{3} = 1^{3} + 12^{3}$$
$$4104 = 9^{3} + 15^{3} = 2^{3} + 16^{3}$$



FIGURE 1: Colbert presenting the founding members of the Académie Royale des Sciences to Louis XIV, in 1666. Bernard Frenicle de Bessy (Paris circa 1605 – Paris 1675), one of the founding members, is probably among the people on the left. [Painting by Henri Testelin, Musée du Château de Versailles, MV 2074].

Treuver deux nombres cubes dont la fumme soit esgal a deux autres nombres cubes. Nempe fic; 1729 = C9 + C10 = C1 + C12. 4104 = C9 + C15 = C2 + C16.

FIGURE 2: The two smallest of Frenicle's solutions found in 1657, as published in Wallis's Commercium Epistolicum, Epistola X, Oxford, 1693.

Brouncker added that Frenicle said nothing about equation (2). Slightly later, in February 1658, Frenicle sent numerous other solutions of (1) to Digby without any explanation of the method used. Fermat himself worked on numbers which are sums of two cubes in more than two ways. Intelligently reusing Viète's formulae for solving $x^3 = y^3 + z^3 + w^3$, he proved in his famous comments on Diophantus that it is possible to construct a number expressible as a sum of two cubes in n different ways, for any n, but his method generates huge numbers. We know now that Fermat's method essentially uses the addition law on an elliptic curve. See also Theorem 412 of Hardy & Wright, using Fermat's method [20, pp. 333–334 & 339].

It was unknown at the time whether equation (2) was soluble; we recognize Fermat's famous last theorem $x^n + y^n = z^n$, when n = 3. This particular case was said to be impossible by Fermat in a letter sent to Digby in April 1658, and proved impossible more than one century later by Euler, in 1770. The general case for any n was also said to be impossible by Fermat in his famous note written in the margin of the Arithmetica by Diophantus, and proved impossible by Andrew Wiles in 1993–1994. For more details on the Fermat /Frenicle/Digby/Brouncker/Wallis letters, see [1], [12, pp. 551–552], [31, 39, 40, 43].

We will now focus our paper on equation (1). Euler worked on it [16], but the first to have worded it exactly as the problem of the "smallest" solution, which is the true Taxicab problem, seems to have been Edward B. Escott. It was published in 1897 in L'Intermédiaire des Mathématiciens [13]:

Quel est le plus petit nombre entier qui soit, de deux façons différentes, la somme de deux cubes? [In English: What is the smallest integer which is, in two different ways, the sum of two cubes?]

Several authors responded [25] to Escott, stating that Frenicle had found 1729 a long time before. A more complete answer was given by C. Moreau [26], listing all the solutions less than 100,000. C. E. Britton [7] listed all the solutions less than 5,000,000. These two lists are given in the Appendix, figures A1a and A1b.

2 Why is 1729 called a "Taxicab" number?

The problem about the number 1729 is now often called the "Taxicab problem", e.g., [18, p. 212], [22, 37, 44], in view of an anecdote, often mentioned in mathematical books, involving the Indian mathematician Srinivasa Ramanujan (1887–1920) and the British mathematician Godfrey Harold Hardy (1877–1947). Here is the story as related by Hardy and given, for example, in [19, p. xxxv]:

I remember once [probably in 1919] going to see him [Ramanujan] when he was lying ill in Putney [in the south-west of London]. I had ridden in taxicab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways." As Euler did, Ramanujan worked on parametric solutions of (1). For example, even if a similar formula had been previously found by Werebrusow [45], Ramanujan found [2, p. 107], [29, p. 387] the very nice condition

If
$$m^2 + mn + n^2 = 3a^2b$$
, then $(m + ab^2)^3 + (bn + a)^3 = (bm + a)^3 + (n + ab^2)^3$. (3)

This equation gives only a small proportion of the solutions. However, with m = 3, n = 0, a = 1, and b = 3, the equation yields $12^3 + 1^3 = 10^3 + 9^3 = 1729$.



FIGURE 3. Equations handwritten by Ramanujan in two different notebooks: [29, p. 225] (left panel), and [30, p. 341] (right panel).

Euler had published the complete parametric solution in rationals of (1), but as Hardy and Wright [20, p. 200] pointed out, "The problem of finding all integral solutions is more difficult". In 1998, Ajai Choudhry published an interesting paper [11] on the parametric solution in integers of (1).

3 Notation used in this paper

In this paper, $\underline{\text{Taxicab}(n)}$ denotes the smallest integer that can be written in n ways as a sum of two cubes of positive integers. Example:

Taxicab(2) =
$$1729 = 12^3 + 1^3 = 10^3 + 9^3$$
.

Fermat proved that Taxicab(n) exists for any n.

We let $\underline{T(n,k)}$ denote the kth smallest primitive solution that can be written in n ways as a sum of two cubes of positive integers, so that

$$Taxicab(n) = T(n, 1) \tag{4}$$

Examples:

$$T(2,1) = 1729 = \text{Taxicab}(2), \quad T(2,2) = 4104.$$

When Taxicab(n) is unknown, however, we let $\underline{T'(n,k)}$ denote the kth smallest known primitive solution (at the time of the article) written in n ways as a sum of two cubes of positive integers, and T'(n, 1) is an upper bound of Taxicab(n):

$$Taxicab(n) \le T'(n, 1) \tag{5}$$

We let $\underline{\text{Cabtaxi}(n)}$ denote the smallest integer that can be written in n ways as a sum of two cubes of positive or negative integers. Example:

Cabtaxi(2) =
$$91 = 3^3 + 4^3 = 6^3 - 5^3$$
.

We let C(n,k) denote the kth smallest primitive solution that can be written in n ways as a sum of two cubes of positive or negative integers.

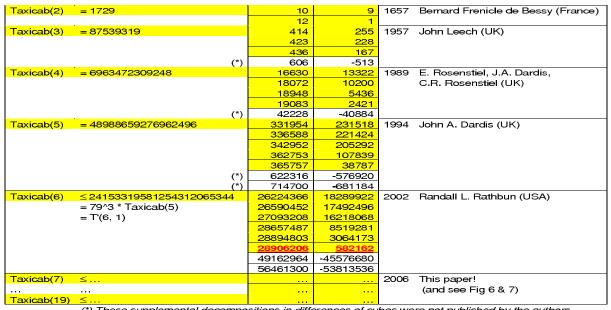
$$Cabtaxi(n) = C(n, 1) \tag{6}$$

When $\operatorname{Cabtaxi}(n)$ is unknown, however, we let $\underline{C'(n,k)}$ denote the kth smallest known primitive solution written in n ways as a sum of two cubes of positive or negative integers. C'(n,1) is an upper bound of $\operatorname{Cabtaxi}(n)$:

$$Cabtaxi(n) \le C'(n, 1). \tag{7}$$

4 1902–2002: from Taxicab(3) to Taxicab(6)

After having asked the question above on Taxicab(2), Escott asked about Taxicab(3) in 1902 [15]. Find the smallest solution of the equation:



$$u^{3} + v^{3} = w^{3} + x^{3} = y^{3} + z^{3}.$$
(8)

(*) These supplemental decompositions in differences of cubes were not published by the authors. Of course, they cannot be "counted" as decompositions in this case of Taxicab numbers.

FIGURE 4. History of Taxicab numbers.

The Euler and Werebrusow [46] parametric solutions of (1) and (8) do not help us find the smallest solution. In 1908 André Gérardin [17] suggested that the solution was probably

$$175959000 = 70^3 + 560^3 = 198^3 + 552^3 = 315^3 + 525^3.$$

An important observation for our study and our future "splitting factors" is that Gérardin's solution is equal to $35^3 * T(2, 2)$. Two out of its three sums come from the second solution 4104 found by Frenicle as

$$70 = 2 * 35, \quad 560 = 16 * 35,$$

 $315 = 9 * 35, \quad 525 = 15 * 35.$

But $198^3 + 552^3$ is not a multiple of 35^3 and can be considered as a "new" decomposition. The true Taxicab(3) was discovered more than 50 years after Escott's question, and exactly 300 years after Frenicle's discovery of Taxicab(2). Using an EDSAC machine, John Leech found, and published in 1957 [21], the five smallest 3-way solutions, the smallest of these five being

$$Taxicab(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3.$$

His results indicated that Gérardin's solution was not Taxicab(3), but T(3, 4) = the fourth smallest primitive 3-way solution.

E. Rosenstiel, J. A. Dardis & C. R. Rosenstiel found Taxicab(4) = 6963472309248, and first announced it in 1989 [27]. They gave more detailed results in [36], along with the next three smallest 4-way solutions.

Until now, David W. Wilson was considered to have been the first to have discovered, in 1997, Taxicab(5) = 48988659276962496, see [47], [3, p. 391], [18, p. 212]. But, as kindly communicated to me by Duncan Moore, this number had been previously found three years earlier in 1994 by John A. Dardis, one of the co-discoverers of Taxicab(4), and published in the February 1995 "Numbers count" column of *Personal Computer World* [28]. After Dardis in 1994 and Wilson in 1997, this same number was again found independently by Daniel J. Bernstein [3] in 1998. Bill Butler also confirmed [8] this number in 2006, while computing the fifteen 5-way solutions $< 1.024 * 10^{21}$.

From 1997 to 2002, the best known upper bound of Taxicab(6) was a 6-way solution found by David W. Wilson. In July 2002, Randall L. Rathbun found [32] a better upper bound of Taxicab(6), 2.42×10^{22} , adding: "I don't believe that this is the lowest value sum for 6 positive cube pairs of equal value". But it seems today that it probably is the lowest value! Calude, Calude & Dinneen claimed in 2003 [9] that this upper bound is the true Taxicab(6) with probability greater than 99%, and further claimed that results in 2005 [10] gave a probability greater than 99.8%, but these claims are not accepted by many mathematicians. And the computations done by Bill Butler proved that Taxicab(6) > 1.024×10^{21} .

5 Splitting factors

We have remarked that Gérardin's solution was equal to $35^3 * T(2,2)$. It is important to note that T'(6,1) is equal to $79^3 * \text{Taxicab}(5)$, as computed by Rathbun. Among the 6 decompositions, only one (underlined in Fig. 4) is a "new" decomposition: the others are 79^3 multiples of the 5 decompositions of Taxicab(5).

Rathbun also remarked that other multiples of Taxicab(5) are able to produce 6-way solutions: 127^3 , 139^3 and 727^3 . I add that they are not the only multiples of Taxicab(5) producing 6-way solutions. The next one is 4622^3 , which indicates again, as for Gérardin's solution, that non-prime numbers do not have to be skipped as we might initially assume: 79, 127, 139 and 727 were primes, but 4622 = 2 * 2311 is not prime.

If N is an n-way sum of two cubes, and if k^3N is an (n + 1)-way sum of two cubes, then k is called a "splitting factor" of N. This means that this k factor "splits" k^3N into a new (n+1)th-way sum of two cubes, the n other sums being directly the k^3 multiples of the already known n ways of N. It was called the "magnification technique" by David W. Wilson.

It is possible that other known 5-way solutions, if they have small splitting factors, may produce smaller 6-way solutions than Rathbun's upper bound. Using the list of 5-way solutions computed by Bill Butler [8], I have computed their splitting factors (Appendix, figure A3). These splitting factors give the smallest known 6-way solutions $< 10^{26}$ (Appendix, figure A4): the first one remains 79^3 *Taxicab(5), which means that it is impossible to do best with this method. We will use this Taxicab(5) number as a basis for our search of upper bounds of Taxicab(n), for larger n.

The method used to find all our decompositions of N into a sum of two cubes is as follows. We first factorize N, then build a list of all its possible pair of factors (r, s) solving N = rs, with r < s. Because any sum of two cubes can be written as

$$N = rs = x^{3} + y^{3} = (x + y)(x^{2} - xy + y^{2}),$$
(9)

any possible sum of two cubes is an integer solution of the system (10) for one of the possible pairs (r, s):

$$x + y = r, \quad x^2 - xy + y^2 = s.$$
 (10)

We search for integer solutions of this system by solving the resulting quadratic equation. Of course, most of the pairs (r, s) do not give an integer solution (x, y).

6 Taxicab(7) and beyond

The first idea is to use several of the existing splitting factors together. When we use n factors together, we add n new ways. For example, $127^3 * \text{Taxicab}(5)$ gives 5 + 1 = 6 way-solutions, and $127^3 * 727^3 * \text{Taxicab}(5)$ gives 5 + 2 = 7 way-solutions. Directly using this idea, the smallest 7-way solution is $79^3 * 127^3 * \text{Taxicab}(5)$.

The second idea is to check, once a splitting factor is used, if a completely new splitting factor is possible on the new number. In our case, yes it is! A very pleasant surprise: $79^3 * \text{Taxicab}(5)$ has a new splitting factor 101, called a "secondary" factor. And because 101 is smaller than 127, we have found a better 7-way solution $79^3 * 101^3 * \text{Taxicab}(5)$ smaller than $79^3 * 127^3 * \text{Taxicab}(5)$. It is possible that some other T(5, i) could produce a smaller 7-way

solution if it has a small secondary factor. This is not the case. For example, using T(5,6), the smallest possible 7-way solution is $25^3 * 367^3 * T(5,6)$, bigger than $79^3 * 101^3 * Taxicab(5)$.

Primary	Secondary	Ternary	
splitting factors < 32,000	splitting factors < 10,000		
79	101	None	
		2971	
127	377 = 13*29	7549	
		8063 = 11*733	
139	4327	None	
727	None		
4622 = 2*2311	None		
14309 = 41*349		_	
16227 = 3*3*3*601			
23035 = 5*17*271			

FIGURE 5. Detailed list of splitting factors of Taxicab(5).

Taxicab(7)	≤ 24885189317885898975235988544
	= 101^3 * T'(6, 1) = 2.49E+28 = T'(7, 1)
Taxicab(8)	≤ 50974398750539071400590819921724352
	= 127^3 * T'(7, 1) = 5.10E+34 = T'(8, 1)
Taxicab(9)	≤ 136897813798023990395783317207361432493888
	= 139^3 * T'(8, 1) = 1.37E+41 = T'(9, 1)
Taxicab(10)	≤ 7335345315241855602572782233444632535674275447104
	= 377^3 * T'(9, 1) = 7.34E+48 = T'(10, 1)
Taxicab(11)	≤ 2818537360434849382734382145310807703728251895897826621632
	= 727 ³ * T'(10, 1) = 2.82E+57 = T'(11, 1)
Taxicab(12)	≤ 73914858746493893996583617733225161086864012865017882136931801625152
	= 2971^3 * T'(11, 1) = 7.39E+67 = T'(12, 1)

FIGURE 6. Best upper bounds, for Taxicab(n), n = 7, 8, ..., 12.

Taxicab(7)	≤ 24885189317885898975235988544	2648660966	1847282122
	= 101^3 * T'(6, 1)	2685635652	1766742096
	= T'(7, 1)	2736414008	1638024868
		2894406187	860447381
		<u>2915734948</u>	<u>459531128</u>
		2918375103	309481473
		2919526806	58798362
		4965459364	-4603244680
		5702591300	-5435167136

FIGURE 7. Upper bound of Taxicab(7) and its 7 decompositions
(2 more decompositions are differences of cubes)

The best upper bounds using this method were computed in November–December 2006, and are listed in Fig. 6. This search needed some hours on a Pentium IV. They are the current records for the upper bounds of the Taxicab numbers.

Fig. 7 gives the full decomposition of the new upper bound of Taxicab(7). Its new 7th decomposition, which is not 101 times one of the 6 decompositions of T'(6, 1) from Fig. 4, is underlined.

The other decompositions of upper bounds up to Taxicab(12) are in the detailed lists of the Appendix. We may continue with the other unused splitting factors of Fig. 5, giving (without explicitly stating their decompositions):

 $\begin{aligned} \text{Taxicab}(13) &\leq T'(13,1) = 4327^3 * T'(12,1) \simeq 5.99 * 10^{78} \\ \text{Taxicab}(14) &\leq T'(14,1) = 4622^3 * T'(13,1) \simeq 5.91 * 10^{89} \\ \text{Taxicab}(15) &\leq T'(15,1) = 7549^3 * T'(14,1) \simeq 2.54 * 10^{101} \\ \text{Taxicab}(16) &\leq T'(16,1) = 8063^3 * T'(15,1) \simeq 1.33 * 10^{113} \\ \text{Taxicab}(17) &\leq T'(17,1) = 14309^3 * T'(16,1) \simeq 3.91 * 10^{125} \\ \text{Taxicab}(18) &\leq T'(18,1) = 16227^3 * T'(17,1) \simeq 1.67 * 10^{138} \\ \text{Taxicab}(19) &\leq T'(19,1) = 23035^3 * T'(18,1) \simeq 2.04 * 10^{151}. \end{aligned}$

7 Cabtaxi numbers

But why should we be limited to sums of positive cubes? We can generalize the problem, allowing sums of positive or negative cubes, these are known as Cabtaxi numbers. Their story starts before that of the Taxicab numbers.



FIGURE 8. François Viète (Fontenay-le-Comte 1540 – Paris 1603)

FIGURE 9. Formula " $6^3 = 3^3 + 4^3 + 5^3$ " by François Viète, as republished in 1646 [41, p. 75].

On 31 July 1589, the French king Henri III was killed by the monk Jacques Clément and was succeeded on the throne by Henri IV. In 1591, François Viète, "one of the most influential men at the court" of Henri IV [42, p. 3] published this very nice formula about his problem XVIII, fourth book of *Zetetica* [41, p. 75] [42, p. 146]:

$$6^3 = 3^3 + 4^3 + 5^3.$$

Moving only one term, we can consider that Viète knew Cabtaxi(2):

$$91 = 3^3 + 4^3 = 6^3 - 5^3.$$

In exactly the same year, 1591, Father Pietro Bongo ("Petrus Bungus" in Latin), canon of Bergamo, independently published this same formula in *Numerorum Mysteria* [12, p. 550]. Bongo is also known to have "demonstrated" that the Antichrist was Martin Luther by using the Hebrew alphabet, the sum of the letters being 666: the number of the Beast. It was an answer to the German mathematician Michael Stifel (1487–1567) who previously proved, using the Latin alphabet, that Pope Leo X was the Antichrist. So strange and mystic the reasoning by some mathematicians at that time ...

Back to mathematics! Viète and Euler worked on parametric solution in rationals of:

$$x^3 = y^3 + z^3 + w^3. (11)$$

In 1756, Euler published [16] the same x = 6 solution of Viète and Bongo, and some other solutions. In 1920 H. W. Richmond published [33] a list of C(2, i) numbers, with a solving method.

Euler was probably the first to have mentioned some 3-way solutions, his smallest being

$$87^3 - 79^3 = 20^3 + 54^3 = 38^3 + 48^3.$$

But the first mention of the true Cabtaxi(3) that I have found was by Escott in 1902 [14]:

$$728 = 12^3 - 10^3 = 9^3 - 1^3 = 8^3 + 6^3.$$

Answering Escott's problem in 1904, Werebrusow published [46], [12, p. 562] this 3-way formula:

If
$$m^2 + mn + n^2 = 3a^2bc$$
, then
 $((m+n)c + ab^2)^3 + (-(m+n)b - ac^2)^3 = (-mc + ab^2)^3 + (mb - ac^2)^3$
 $= (-nc + ab^2)^3 + (nb - ac^2)^3.$ (12)

	01		4	0	1501	
Cabtaxi(2)	= 91		4	3	1591	3 1 1
	700		6	-5	1000	Pietro Bongo (Italy) independently
Cabtaxi(3)	= 728		8	6	1902	E. B. Escott (USA)
	= (``)	2^3 * Cabtaxi(2)		-1		
	074	4050	12	-10	1000	
Cabtaxi(4)	= 274	1256	108	114	~1992	Randall L. Rathbun (USA)
			140	-14		
			168	-126		
	601	7100	207	-183	. 1000	Dendell Dether in (UCA)
Cabtaxi(5)	<mark>= 601</mark>	/ 193	166	113	~1992	Randall L. Rathbun (USA)
			180	<u> </u>		
			185			
			209	-146		
	- 4 -	0774044	246	-207	1000	
Cabtaxi(6)	= 141	2774811	963	804	~1992	Randall L. Rathbun (USA)
			1134	-357 -504		
			1155			
			<u>1246</u> 2115	-805 -2004		
			4746	-4725		
Cohtovi/7)	- 110	02198488	4746	1608	~1002	Randall L. Rathbun (USA)
Cabtaxi(7)		2^3 * Cabtaxi(6)	1920 1939	1589	~1992	Randali L. Ratriburi (USA)
	=()	213 Cablaxi(b)	2268	-714		
			2200	-1008		
			2492	-1610		
			4230	-4008		
			9492	-4008		
Cabtaxi(8)	- 137	513849003496	44298	36984	1998	Daniel J. Bernstein (USA)
		23^3 * Cabtaxi(7)	44597	36547	1990	Daniero, Demstein (OSA)
	-()		50058	22944		
			52164	-16422		
			53130	-23184		
			57316	-37030		
			97290	-92184		
			218316	-217350		
Cabtaxi(9)	= 424	910390480793000	645210	538680	2005	Duncan Moore (UK)
		5^3 * 67^3 * Cabtaxi(7)	649565	532315		
	\ /		752409	-101409		
			759780	-239190		
			773850	-337680		
			834820	-539350		
			1417050	-1342680		
			3179820	-3165750		
			5960010	-5956020		
Cabtaxi(10)	≤				2006	This paper!
					-2007	(and see Fig. 12 & 13)
Cabtaxi(30)	<u>≤</u>					· ····································
		ece relationships were upp				

(*) These relationships were unpublished (and unknown?) by the authors

Werebrusow needed the condition $a^3 = 1$, but his formula is true without this condition. This 3-way formula (12) reuses his previous 2-way formula (3). No example was given by Werebrusow, but we can remark that Cabtaxi(3) can be found, applying (m, n, a, b, c) = (0, 3, 1, 3, 1). Another observation is that Cabtaxi(3) can be deduced from Cabtaxi(2), using a splitting factor 2, which adds one new decomposition $9^3 - 1^3$. The two other decompositions of Cabtaxi(2) are 2^3 multiples of Cabtaxi(2).

Cabtaxi(4), Cabtaxi(5), Cabtaxi(6), Cabtaxi(7) were found by Randall L. Rathbun in the beginning of the 1990s [18, p. 211], while Cabtaxi(8) was discovered by Daniel J. Bernstein in 1998 [3].

In the same month, January 2005, there were two nice results on Cabtaxi(9) from two different people: on the 24th, Jaroslaw Wroblewski found an upper bound of Cabtaxi(9) [22], and one week later, on the 31st January 2005, Duncan Moore found the true Cabtaxi(9) [23] Moore's search also proved that Cabtaxi(10) > 4.6×10^{17} .

Just as Taxicab(5) was a strong basis for Taxicab numbers, we observe in Fig. 10 that Cabtaxi(6) is a strong basis used by bigger Cabtaxi numbers. These interesting relations were never published, and show the strength of splitting factors:

$$Cabtaxi(7) = 2^3 * Cabtaxi(6)$$

$$Cabtaxi(8) = 23^3 * Cabtaxi(7)$$

$$Cabtaxi(9) = (5 * 67)^3 * Cabtaxi(7).$$

Our method is similar to Taxicab numbers, and uses the splitting factors of Cabtaxi(9) given in Fig. 11a. However, because Jaroslaw Wroblewki's number $C'(9,2) = 8.25 * 10^{17}$ is close to $C(9,1) = \text{Cabtaxi}(9) = 4.25 * 10^{17}$, it is interesting also to analyze its splitting factors, as shown in Fig. 11b.

The best upper bounds up to C'(20, 1) using the splitting factors of Cabtaxi(9) were computed in November–December 2006. Three better upper bounds C'(11, 1), C'(17, 1), C'(18, 1)are possible, coming from C'(9, 2): they were found later, in February 2007. All these numbers are listed in Fig. 12 and are the current records for the upper bounds of the Cabtaxi numbers.

Fig. 13 gives the full decomposition of the new upper bound of Cabtaxi(10). Its new 10th decomposition, which is not 13 times one of the 9 decompositions of Cabtaxi(9) from Fig 10, is underlined.

Primary splitting factors < 10,000	Secondary splitting factors < 1,000	Ternary splitting factors < 200
13	29	None
13	127+	None
23	None	
38 = 2*19	37	None
30=2 19	436 = 2*2*109	None
43	None	
74 = 2*37	19	None
183 = 3*61	73	None
193	None	
219 = 3*73	61	None
349	None	
661	None	
859	None	
872 = 2*2*2*109	19	None
872=222109	37	None
	19	None
4036 = 2*2*1009	37	None
	248 = 2*109	None
4367 = 11*397	439	None
4829 = 11*439	397	None

FIGURE 11a. Detailed list of splitting factors of Cabtaxi(9) = 424910390480793000.

Primary splitting factors < 10,000	Secondary splitting factors < 1,000	Ternary splitting factors < 300
	17	None
13	74 = 2*37	5
15	79	7
	417 = 3*139	None
61	11	None
	199	None
185 = 5*37	291 = 3*97	283
105 = 5 57	307	None
	379	None
409	None	
849 = 3*283	485 = 5*97	None
	37	None
995 = 5*199	291 = 3*97	None
	379	None
1021	None	
1153	None	
	37	None
	199	None
1455 = 3*5*97	283	None
	379	None
	481 = 13*37	None
1829 = 31*59	None	
1895 = 5*379	None	
5543 = 23*241	None	
6921 = 3*3*769	None	
8465 = 5*1693	None	

FIGURE 11b. Detailed list of splitting factors of C'(9,2) = 825001442051661504.

Cabtaxi(10)	≤933528127886302221000			
	= 13^3 * Cabtaxi(9) = (2*5*13*67)^3 * Cabtaxi(6)	=	9.34E+20	= C'(10, 1)
Cabtaxi(11)	≤ 8904950890305189093226944			
	= (13*17)^3 * C'(9, 2) (*)	=	8.90E+24	= C'(11, 1)
Cabtaxi(12)	≤ 1912223147184127402358643000			
	= 127^3 * C'(10, 1)	=	1.91E+27	= C'(12, 1)
Cabtaxi(13)	≤ 23266019031789278104497609381000			
	= 23^3 * C'(12, 1)	=	2.33E+31	= C'(13, 1)
Cabtaxi(14)	≤ 567434938166308703690592195193209000			
	= 29^3 * C'(13, 1)	=	5.67E+35	= C'(14, 1)
Cabtaxi(15)	≤ 31136289927061691188910174934641764248000			
	= 38^3 * C'(14, 1)	=	3.11E+40	= C'(15, 1)
Cabtaxi(16)	≤ 1577146493675455843791867090964409284453944000			
	= 37 ³ * C'(15, 1)	=	1.58E+45	= C'(16, 1)
Cabtaxi(17)	≤ 2304515615918039284759197700803079954269924230400	C		
	= (74*5*79*7*61*11)^3 * C'(11, 1) (**)	=	2.30E+49	= C'(17, 1)
Cabtaxi(18)	≤ 1816096345828808446943404864175105108453961062016	600	96000	
	=199^3 * C'(17, 1) (***)	=	1.82E+56	= C'(18, 1)
Cabtaxi(19)	<2989504772369811977234887250705385759929242111342	<mark>998</mark>	7966063200	0
	= (43*183*73)^3 * C'(16, 1)	=	2.99E+62	= C'(19, 1)
Cabtaxi(20)	≤ 2149172021033860338362430683389430843511963750524	516	4899734241	04024000
	= 193^3 * C'(19, 1)	=	2.15E+69	= C'(20, 1)
	Three upper bounds derive from C'(9, 2):			

(*) because it is smaller than 23³ * C'(10, 1)

(**) because it is smaller than 43³ * C'(16, 1)

(***) because it is smaller than (43*183)^3 * C'(16, 1)

FIGURE 12. Best upper bounds for Cabtaxi(10) to Cabtaxi(20).

Cabtaxi(10) ≤ 933528127886302221000	8387730	7002840
= 13^3 * Cabtaxi(9)	8444345	6920095
= C'(10, 1)	<u>9773330</u>	<u>-84560</u>
	9781317	-1318317
	9877140	-3109470
	10060050	-4389840
	10852660	-7011550
	18421650	-17454840
	41337660	<u>-41154750</u>
	77480130	-77428260

FIGURE 13. Upper bound of Cabtaxi(10) and its 10 decompositions.

The other decompositions of upper bounds up to Cabtaxi(20) are presented in the detailed lists of the Appendix. We may continue with the other unused splitting factors of Fig. 11a, giving (without explicitly stating their decompositions):

$$\begin{split} \text{Cabtaxi}(21) &\leq C'(21,1) = 349^3 * C'(20,1) \simeq 9.14 * 10^{76} \\ \text{Cabtaxi}(22) &\leq C'(22,1) = 436^3 * C'(21,1) \simeq 7.57 * 10^{84} \\ \text{Cabtaxi}(23) &\leq C'(23,2) = 661^3 * C'(22,1) \simeq 2.19 * 10^{93} \\ \text{Cabtaxi}(24) &\leq C'(24,2) = 859^3 * C'(23,1) \simeq 1.39 * 10^{102} \\ \text{Cabtaxi}(25) &\leq C'(25,2) = 1009^3 * C'(24,1) \simeq 1.42 * 10^{111} \\ \text{Cabtaxi}(26) &\leq C'(26,2) = (4367 * 439)^3 * C'(24,1) \simeq 9.77 * 10^{120} \\ \text{Cabtaxi}(27) &\leq C'(27,2) = (4367 * 439)^3 * C'(25,1) \simeq 1.00 * 10^{130} \end{split}$$

and of Fig 11b, giving:

 $\begin{aligned} \text{Cabtaxi}(21) &\leq C'(21,2) = (139 * 283 * 291)^3 * C'(18,1) \simeq 2.72 * 10^{77} \\ \text{Cabtaxi}(22) &\leq C'(22,2) = 307^3 * C'(21,1) \simeq 7.88 * 10^{84} \\ \text{Cabtaxi}(23) &\leq C'(23,1) = 379^3 * C'(22,1) \simeq 4.29 * 10^{92} \\ \text{Cabtaxi}(24) &\leq C'(24,1) = 409^3 * C'(23,1) \simeq 2.94 * 10^{100} \\ \text{Cabtaxi}(25) &\leq C'(25,1) = 1021^3 * C'(24,1) \simeq 3.12 * 10^{109} \\ \text{Cabtaxi}(26) &\leq C'(26,1) = 1153^3 * C'(26,1) \simeq 4.79 * 10^{118} \\ \text{Cabtaxi}(27) &\leq C'(27,1) = 1693^3 * C'(26,1) \simeq 2.32 * 10^{128} \\ \text{Cabtaxi}(28) &\leq C'(28,1) = 1829^3 * C'(27,1) \simeq 1.42 * 10^{138} \\ \text{Cabtaxi}(29) &\leq C'(29,1) = 2307^3 * C'(28,1) \simeq 1.75 * 10^{148} \\ \text{Cabtaxi}(30) &\leq C'(30,1) = 5543^3 * C'(29,1) \simeq 2.97 * 10^{159}. \end{aligned}$

8 Unsolved problems

8.1 Are these the true Taxicab and Cabtaxi numbers?

The new upper bounds of Taxicab(7) and Cabtaxi(10) announced in this paper, and detailed in Fig 7 and 13, may have a chance of being the correct Taxicab and Cabtaxi numbers. But the probability decreases as n increases, and is close to 0 for Taxicab(19) and Cabtaxi(30). Who can check if some of these upper bounds are the correct Taxicab and Cabtaxi numbers? Or who will find smaller upper bounds? This is a good subject for mathematical computation.

8.2 Prime versions of Taxicab and Cabtaxi numbers

Our construction with splitting factors generates sums of cubes of non-prime integers: at least n-1 decompositions are k^3 multiples. What about sums of two cubes of primes? The 2-way solutions using only sums of cubed primes are rare. For what we can call "the prime version of Taxicab numbers", the smallest 2-way solutions are

$$6058655748 = 61^3 + 1823^3 = 1049^3 + 1699^3$$
(13a)

$$6507811154 = 31^3 + 1867^3 = 397^3 + 1861^3.$$
(13b)

For the prime version of Cabtaxi numbers, the smallest 2-way solutions are

$$62540982 = 397^3 - 31^3 = 1867^3 - 1861^3 \tag{14a}$$

$$105161238 = 193^3 + 461^3 = 709^3 - 631^3.$$
 (14b)

The solution (14a) is just a different arrangement of (13b).

But nobody has succeeded yet (as far as we know) in constructing a 3-way solution using only sums, or sums and differences, of cubed primes. Who will be the first, or who can prove that it is impossible?

An "easier" question: instead of directly searching for a 3-way solution using 6 cubed primes, is there another 3-way solution using at least 4 cubed primes, different from this one

$$68913 = 40^3 + 17^3 = 41^3 - 2^3 = 89^3 - 86^3$$

(the 4 primes used are 17, 41, 2, 89). See puzzles 90 [34] and 386 [35] of Carlos Rivera.

A supplemental remark: our 3-way problems are unsolved, but are solved for a long time if only coprime pairs are used instead of primes. Several 3-way and 4-way solutions using sums of two coprime cubes are known. The smallest 3-way solution was found by Paul Vojta [18, p. 211] in 1983:

$$15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3.$$

And 3-way, 4-way and 5-way solutions using sums or differences of two coprime cubes are known. It is easy to find the smallest 3-way solution:

$$3367 = 15^3 - 2^3 = 16^3 - 9^3 = 34^3 - 33^3$$

8.3 Who can construct a 4×4 magic square of cubes?

A 3×3 magic square of cubes, using 9 distinct cubed integers, has been proved impossible [18, p. 270], [4, p. 59]: if z^3 is the number in the centre cell, then any line going through the center should have $x^3 + y^3 = 2z^3$. Euler and Legendre demonstrated that such an equation is impossible with distinct integers.

But the question of 4×4 magic squares of cubes, using 16 distinct positive cubed integers, is still open. A breakthrough was made in 2006 by Lee Morgenstern [5] who found a very nice construction method using Taxicab numbers. If

$$a^3 + b^3 = c^3 + d^3 = u \tag{15}$$

and
$$e^3 + f^3 = g^3 + h^3 = v,$$
 (16)

then the 4×4 square of cubes in Fig. 14 is semi-magic, its 4 rows and 4 columns having the same magic sum S = uv.

(af) ³	(de) ³	(ce) ³	(bf) ³
(bh) ³	(cg) ³	(dg) ³	(ah) ³
(bg) ³	(ch) ³	(dh) ³	(ag) ³
(ae) ³	(df) ³	(cf) ³	(be) ³

FIGURE 14. Parametric 4×4 magic square of cubes, Morgenstern's method.

Using u = Taxicab(2) = 1729 and the second smallest 2-way solution v = T(2, 2) = 4104, both found by Frenicle, which implies (a, b, c, d, e, f, g, h) = (1, 12, 9, 10, 2, 16, 9, 15), we find the 4×4 semi-magic square of cubes shown in Fig. 15.

16 ³	20 ³	18 ³	192 ³
180 ³	81 ³	90 ³	15 ³
108 ³	135 ³	150 ³	9 ³
2 ³	160 ³	1 44 ³	24 ³

FIGURE 15. 4×4 semi-magic square of cubes. Magic sum S = 1729 * 4104 = 7,095,816.

This is not a full solution of the problem, because this square is only "semi-magic", in that the diagonals each have a wrong sum. The diagonals (and the square) would be fully magic if a third equation is simultaneously true:

$$(ae)^{3} + (bf)^{3} = (cg)^{3} + (dh)^{3}.$$
(17)

Using 2-way lists kindly provided by Jaroslaw Wroblewski, University of Wrocław, I can say that there is no solution to the system of 3 equations (15), (16) and (17), with a, b, c, d, e, f, g, h < 500,000 or with a, b, c, d < 1,000,000 and e, f, g, h < 25,000. But that does not mean that the system is impossible. The first person who finds a numerical solution of this system of 3 equations will directly get a 4×4 magic square of cubes! But perhaps somebody will succeed in constructing a 4×4 magic square of cubes using a different method. Or somebody will prove that the problem is unfortunately impossible.

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10 Appendix

_	Moreau's list	Diff of cubes (*)	Comments		
T(2, 1)	$1729 = 1^3 + 12^3 = 9^3 + 10^3$		Taxicab(2)		
T(2, 2)	$4104 = 2^3 + 16^3 = 9^3 + 15^3$	$= 18^3 - 12^3$			
	$13832 = 2^3 + 24^3 = 18^3 + 20^3$		non-primitive solution = 2^3 T(2, 1)		
T(2, 3)	$20683 = 10^3 + 27^3 = 19^3 + 24^3$				
	$32832 = 4^3 + 32^3 = 18^3 + 30^3$	$= 36^3 - 24^3$	non-primitive solution = 2^3 T(2, 2)		
T(2, 4)	$39312 = 2^{3} + 34^{3} = 15^{3} + 33^{3}$				
T(2, 5)	$40033 = 9^3 + 34^3 = 16^3 + 33^3$	2 2	2		
	$46683 = 3^3 + 36^3 = 27^3 + 30^3$	$=46^3 - 37^3$	non-primitive solution = 3^3 T(2, 1)		
T(2, 6)	$64232 = 17^3 + 39^3 = 26^3 + 36^3$	2 2			
T(2, 7)	$65728 = 12^3 + 40^3 = 31^3 + 33^3$	$= 76^3 - 72^3$			
	Leech's list				
T(3, 1)	$87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$		Taxicab(3)		
T(3, 2)	$119824488 = 11^{3} + 493^{3} = 90^{3} + 492^{3} = 346^{3} + 428^{3}$	$= 648^3 - 534^3$			
T(3, 3)	$-143604279 = 111^3 + 522^3 = 359^3 + 460^3 = 408^3 + 423^3$		- 2		
T(3, 4)	$175959000 = 70^3 + 560^3 = 198^3 + 552^3 = 315^3 + 525^3$	$= 630^3 - 420^3$	Gérardin's solution = 35^3 T(2, 2)		
$T(3, 5) \frac{327763000}{327763000} = 300^3 + 670^3 = 339^3 + 661^3 = 510^3 + 580^3}{327763000}$					
(*) These supplemental decompositions in differences of cubes were not published by the authors.					

FIGURE A1a. Smallest 2-way solutions¹ listed by Moreau.FIGURE A1b. Smallest 3-way solutions listed by Leech.

n	Taxicab(n) splitting factors < 10,000
2	None
3	794
4	341, 485, 695, 2551
5	79, 127, 139, 727, 4622

FIGURE A2. Splitting factors of Taxicab numbers.

¹All these solutions were previously found by Frenicle.

	Smallest 5-way solutions Splitting factors <	
T(5, 1)	(*) 48988659276962496	79, 127, 139, 727, 4622
T(5, 2)	490593422681271000	139, 377, 1139, 1297
T(5, 3)	6355491080314102272	109, 6159
T(5, 4)	27365551142421413376	67, 6159
T(5, 5)	47893568195858112000	127, 349, 1961, 3197, 5983
T(5, 6)	55634997032869710456	25, 367, 907, 2713, 7747
T(5, 7)	68243313527087529096	849, 1829, 5421
T(5, 8)	265781191139199122625	163, 613, 793, 3889
T(5, 9)	276114357544758340608	485, 695, 2551
T(5, 10)	343978135086713831424	579, 949, 1321, 1393, 3739
T(5, 11)	357230299141507244544	65, 349, 1961, 3197, 5983
T(5, 12)	461725779831883749000	803, 851
T(5, 13)	572219233725765415608	59, 1142, 1591, 2435, 8751
T(5, 14)	653115573732974625000	11, 367, 907, 2713, 7747
T(5, 15)	794421645362287488000	139, 341, 2551
T'(5, 16)	(**) 1199962860219870469632	19, 6159
T'(5, 17)	(**) 2337654192461288064000	97, 341, 2551
T'(5, 18)	(**) 7413331235096863544832	65, 127, 1961, 3197, 5983
T'(5, 19)	(**) 9972542662841658461688	8318

(*) Taxicab(5), first found by J. A. Dardis in 1994, later by D. W. Wilson.

(**) These are the 16th-19th known, but may not be the 16th-19th smallest.

FIGURE A3. Splitting factors of the smallest 5-way solutions.

Sma	allest known 6-way solutions < 10^26	equal to
T'(6, 1)	(*) 24153319581254312065344	= 79^3 T(5, 1)
T'(6, 2)	100347536855722268443968	= 127^3 T(5, 1)
T'(6, 3)	131564874138736741545024	= 139^3 T(5, 1)
T'(6, 4)	869296828638589225875000	= 25^3 T(5, 6) = 11^3 T(5, 14)
T'(6, 5)	1317547017227852341749000	= 139^3 T(5, 2)
T'(6, 6)	(**) 8230545258248091551205888	= 109^3 T(5, 3) = 67^3 T(5, 4) = 19^3 T'(5, 16)
T'(6, 7)	18823431000968427932175168	= 727^3 T(5, 1)
T'(6, 8)	26287287319744419966543000	= 377^3 T(5, 2)
T'(6, 9)	98104370901736427032896000	= 127^3 T(5, 5) = 65^3 T(5, 11)

(*) The upper bound of Taxicab(6) found by Randall L. Rathbun, in 2002. (**) The solution found by David W. Wilson, in 1997.

FIGURE A4. Smallest 6-way solutions derived from 5-way solutions and splitting factors (other 6-way solutions are possible, if they are not derived from 5-way solutions).

n i	Upper bound of Taxicab(n)	а	b
7 1	24885189317885898975235988544	2648660966	1847282122
72		2685635652	1766742096
73		2736414008	1638024868
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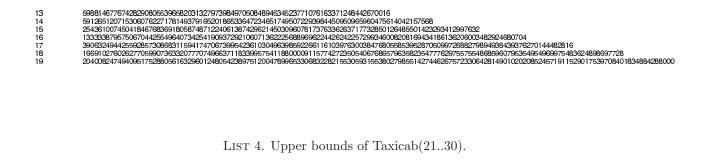
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LIST 2. Upper bounds of Taxicab(10..20) and decompositions.

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LIST 2 (cont'd). Upper bounds of $\mathrm{Taxicab}(10..20)$ and decompositions.

LIST 3. Upper bounds of Taxicab(13..19).



2000 Mathematics Subject Classification: Primary 11D25.

Keywords: Taxicab number, Cabtaxi number, Hardy–Ramanujan number, Bernard Frenicle de Bessy, François Viète, sum of two cubes, difference of two cubes, magic square of cubes.

(Concerned with sequences <u>A011541</u>, <u>A047696</u>.)

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