Journal of Integer Sequences, Vol. 11 (2008),
Article 08.2.6

On Some Properties of Bivariate Fibonacci and Lucas Polynomials

Hacène Belbachir[1] and Farid Bencherif[2]
USTHB/Faculty of Mathematics
P. O. Box 32
El Alia, 16111
Bab Ezzouar
Algeria
hbelbachir@usthb.dz
hacenebelbachir@gmail.com
fbencherif@usthb.dz
fbencherif@gmail.com

Abstract

In this paper we generalize to bivariate Fibonacci and Lucas polynomials, properties obtained for Chebyshev polynomials. We prove that the coordinates of the bivariate polynomials over appropriate bases are families of integers satisfying remarkable recurrence relations.

1 Introduction

In [4], the authors established that Chebyshev polynomials of the first and second kind admit remarkable integer coordinates in a specific basis. It turns out that this property can be extended to Jacobsthal polynomials [6, 7], Vieta polynomials [18, 10, 14, 15], Morgan-Voyce polynomials [13, 2, 9, 11, 1, 17, 5] and quasi-Morgan-Voyce polynomials [8], and more generally to bivariate polynomials associated with recurrence sequences of order two.

[1]This research is partially supported by the LAID3 Laboratory.
[2]This research is partially supported by the LATN Laboratory.

The bivariate polynomials of Fibonacci and Lucas, denoted respectively by $(U_n) = (U_n(x,y))$ and $(V_n) = (V_n(x,y))$, are polynomials belonging to $\mathbb{Z}[x,y]$ and defined by

$$\begin{cases} U_0 = 0,\ U_1 = 1, \\ U_n = xU_{n-1} + yU_{n-2} \quad (n \geq 2)\,, \end{cases} \qquad \text{and} \qquad \begin{cases} V_0 = 2,\ V_1 = x, \\ V_n = xV_{n-1} + yV_{n-2} \ \ (n \geq 2)\,. \end{cases}$$

It is established, see for example [12, 16, 3], that

$$U_{n+1} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k} y^k, \tag{1}$$

$$V_n = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} y^k \ \ (n \geq 1)\,. \tag{2}$$

Let $\mathcal{E}_n$ be the $\mathbb{Q}$-vector space spanned by the free family $\mathcal{C}_n = (x^{n-2k}y^k)_k$, $(0 \leq k \leq \lfloor n/2 \rfloor)$. Thus the relations (1) and (2) appear as the decompositions of U_{n+1} and V_n over the canonical basis $\mathcal{C}_n$ of $\mathcal{E}_n$.

The goal of this paper is to prove that the families $\mathfrak{U}_n := \left(x^k U_{n+1-k}\right)_k$ and $\mathfrak{V}_n := \left(x^k V_{n-k}\right)_k$ for $n - 2\lfloor n/2 \rfloor \leq k \leq n - \lfloor n/2 \rfloor$ constitute two other bases of $\mathcal{E}_n$ (Theorem 2.1) with respect to which, the polynomials $2U_{n+1}$ and $2V_n$ admit remarkable integer coordinates.

2 Main results

Theorem 2.1. *For any $n \geq 1$, $\mathfrak{U}_n$ and $\mathfrak{V}_n$ are bases of $\mathcal{E}_n$*

As U_{n+1} and V_n belong to $\mathcal{E}_n$, the polynomials U_{2n+1} and V_{2n} are elements of $\mathcal{E}_{2n}$ with basis $\mathfrak{U}_{2n}$ or $\mathfrak{V}_{2n}$. Similarly, U_{2n} and V_{2n-1} belong to $\mathcal{E}_{2n-1}$ with basis $\mathfrak{U}_{2n-1}$ or $\mathfrak{V}_{2n-1}$.

Therefore, there are a priori 8 possible decompositions:

U_{2n+1} $\nearrow$ over $\mathfrak{U}_{2n}$ $\boxed{1} \to$ *trivial*,
U_{2n+1} $\searrow$ over $\mathfrak{V}_{2n}$ $\boxed{2} \to$ *Th. A*,

V_{2n} $\nearrow$ over $\mathfrak{U}_{2n}$ $\boxed{3} \to$ *simple*,
V_{2n} $\searrow$ over $\mathfrak{V}_{2n}$ $\boxed{4} \to$ *trivial*,

U_{2n} $\nearrow$ over $\mathfrak{U}_{2n-1}$ $\boxed{5} \to$ *Th. B*,
U_{2n} $\searrow$ over $\mathfrak{V}_{2n-1}$ $\boxed{6} \to$ *Th. E*,

V_{2n-1} $\nearrow$ over $\mathfrak{U}_{2n-1}$ $\boxed{7} \to$ *Th. C*,
V_{2n-1} $\searrow$ over $\mathfrak{V}_{2n-1}$ $\boxed{8} \to$ *Th. D*,

where the cases 1 and 4 are obvious since $U_{2n+1} \in \mathfrak{U}_{2n}$ and $V_{2n} \in \mathfrak{V}_{2n}$.

The decomposition of V_{2n} in $\mathfrak{U}_{2n}$ is simple: we have $V_{2n} = 2U_{2n+1} - xU_{2n}$.

The remaining cases are established by the five following results.

Theorem 2.2. *($\boldsymbol{A}$). Decomposition of $2U_{2n+1}$ over the basis $\mathfrak{V}_{2n}$.*
For every integer $n \geq 0$, one has

$$2U_{2n+1} = \sum_{k=0}^{n} a_{n,k} x^k V_{2n-k},$$

where

$$a_{n,k} = \sum_{j=0}^{n} (-1)^{j+k} (2 - \delta_{n,j}) \binom{j}{k}.$$

Moreover, $(a_{n,k})_{n,k \geq 0}$ is a family of integers satisfying the following recurrence relation:

$$\begin{cases} a_{n,k} = -a_{n-1,k} + a_{n-1,k-1} \quad (n \geq 1,\ k \geq 1); \\ a_{n,0} = 1 \ \ (n \geq 0)\,; \\ a_{0,k} = \delta_{k,0} \ \ (k \geq 0)\,. \end{cases}$$

($\delta_{i,j}$ being the Kronecker symbol).

The recurrence relation permits us to obtain the following table:

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	0	1						
3	1	1	−1	1					
4	1	0	2	−2	1				
5	1	1	−2	4	−3	1			
6	1	0	3	−6	7	−4	1		
7	1	1	−3	9	−13	11	−5	1	
8	1	0	4	−12	22	−24	16	−6	1

from which it follows that

$$\begin{aligned} 2U_1 &= V_0, \\ 2U_3 &= V_2 + xV_1, \\ 2U_5 &= V_4 + 0V_3 + x^2V_2, \\ 2U_7 &= V_6 + xV_5 - x^2V_4 + x^3V_3. \end{aligned}$$

Theorem 2.3. *($\boldsymbol{B}$). Decomposition of U_{2n} over the basis $\mathfrak{U}_{2n-1}$.*
For every integer $n \geq 1$, one has

$$U_{2n} = \sum_{k=1}^{n} b_{n,k} x^k U_{2n-k}$$

where

$$b_{n,k} = (-1)^{k+1} \binom{n}{k}.$$

Moreover, $(b_{n,k})_{n,k\geq 0}$ *is a family of integers satisfying the following recurrence relation:*

$$\begin{cases} b_{n,k} = b_{n-1,k} - b_{n-1,k-1} \ (n \geq 1, \ k \geq 1); \\ b_{n,0} = -1 \ \ (n \geq 0); \\ b_{0,k} = -\delta_{k,0} \ (k \geq 0). \end{cases}$$

The latter recurrence relation permits us to obtain the following table:

$n \setminus k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
0	−1								
1	−1	1							
2	−1	2	−1						
3	−1	3	−3	1					
4	−1	4	−6	4	−1				
5	−1	5	−10	10	−5	1			
6	−1	6	−15	20	−15	6	−1		
7	−1	7	−21	35	−35	21	−7	1	
8	−1	8	−28	56	−70	56	−28	8	−1

from which it follows that

$$\begin{aligned} U_2 &= xU_1 \\ U_4 &= 2xU_3 - x^2U_2 \\ U_6 &= 3xU_5 - 3x^2U_4 + x^3U_3 \\ U_8 &= 4xU_7 - 6x^2U_6 + 4x^3U_5 - x^4U_4 \end{aligned}$$

Theorem 2.4. *(**C**). Decomposition of* V_{2n-1} *over the basis* $\mathfrak{U}_{2n-1}$*.*
For every integer $n \geq 1$*, one has*

$$V_{2n-1} = \sum_{k=1}^{n} c_{n,k} x^k U_{2n-k}$$

where

$$c_{n,k} = 2\,(-1)^{k+1} \binom{n}{k} - \delta_{k,1}.$$

Moreover, $(c_{n,k})_{n\geq 1,k\geq 0}$ *is a family of integers satisfying the following recurrence relation:*

$$\begin{cases} c_{n,k} = c_{n-1,k} - c_{n-1,k-1} - \delta_{k,2} \ \ (n \geq 2, \ k \geq 1); \\ c_{n,0} = -2 \ \ (n \geq 1); \\ c_{1,k} = -2\delta_{k,0} + \delta_{k,1} \ \ (k \geq 0). \end{cases}$$

The latter recurrence relation permits us to obtain the following table:

$n \setminus k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	−2	1							
2	−2	3	−2						
3	−2	5	−6	2					
4	−2	7	−12	8	−2				
5	−2	9	−20	20	−10	2			
6	−2	11	−30	40	−30	12	−2		
7	−2	13	−42	70	−70	42	−14	2	
8	−2	15	−56	112	−140	112	−56	16	−2

from which we get

$$\begin{cases} V_1 = \quad xU_1 \\ V_3 = 3xU_3 - 2x^2U_2 \\ V_5 = \ 5xU_5 - 6x^2U_4 + \ 2x^3U_3 \\ V_7 = 7xU_7 - 12x^2U_6 + 8x^3U_5 - 2x^4U_4 \end{cases}$$

Theorem 2.5. *(**D**). Decomposition of $2V_{2n-1}$ over the basis $\mathfrak{V}_{2n-1}$.*
For every integer $n \geq 1$, one has

$$2V_{2n-1} = \sum_{k=1}^{n} d_{n,k} x^k V_{2n-1-k}$$

where

$$d_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k}.$$

Moreover, $(d_{n,k})_{n\geq 1,\ k\geq 0}$ is a family of integers satisfying the following recurrence relation:

$$\begin{cases} d_{n,k} = d_{n-1,k} - d_{n-1,k-1} \quad (n \geq 2,\ k \geq 1); \\ d_{n,0} = -2 \quad (n \geq 1)\,; \\ d_{1,k} = -2\delta_{k,0} + \delta_{k,1} \quad (k \geq 0). \end{cases}$$

The latter recurrence relation permits us to obtain the following table:

$n \setminus k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	−2	1							
2	−2	3	−1						
3	−2	5	−4	1					
4	−2	7	−9	5	−1				
5	−2	9	−16	14	−6	1			
6	−2	11	−25	30	−20	7	−1		
7	−2	13	−36	55	−50	27	−8	1	
8	−2	15	−49	91	−105	77	−35	9	−1

from which we obtain

$$\begin{cases} 2V_1 = \ xV_0 \\ 2V_3 = 3xV_2 - x^2V_1 \\ 2V_5 = 5xV_4 \ - 4x^2V_3 + x^3V_2 \\ 2V_7 = 7xV_6 - 9x^2V_5 + 5x^3V_4 - x^4V_3 \end{cases}$$

Theorem 2.6. *(**E**). Decomposition of $2U_{2n}$ over the basis $\mathfrak{V}_{2n-1}$.*

For every integer $n \geq 1$, one has

$$2U_{2n} = \sum_{k=1}^{n} e_{n,k} x^k V_{2n-1-k}$$

where

$$e_{n,k} = (-1)^{k+1} \frac{2n-k}{2n} \binom{n}{k} + \delta_{k,0} + \frac{1}{2} \sum_{j=0}^{n-1} (-1)^{j+k-1} (2 - \delta_{n-1,j}) \binom{j}{k-1}.$$

Moreover, $(e_{n,k})_{n,k\geq 0}$ is a family of integers satisfying the following recurrence relation:

$$\begin{cases} e_{n,k} = e_{n-2,k} - 2e_{n-2,k-1} + e_{n-2,k-2} \quad (n \geq 3, \ k \geq 2); \\ e_{n,0} = 0 \quad \textit{and} \quad e_{n,1} = n \quad (n \geq 1)\,; \\ e_{1,k} = \delta_{k,1} \quad \textit{and} \quad e_{2,k} = 2\delta_{k,1} \quad (k \geq 0). \end{cases}$$

The latter recurrence relation permits us to obtain the following table:

$n \setminus k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	0	1							
2	0	2	0						
3	0	3	−2	1					
4	0	4	−4	2	0				
5	0	5	−8	8	−4	1			
6	0	6	−12	14	−8	2	0		
7	0	7	−18	29	−28	17	−6	1	
8	0	8	−24	44	−48	32	−12	2	0

from which, we have

$$\begin{cases} 2U_2 = \ xV_0 \\ 2U_4 = 2xV_2 + 0x^2V_0 \\ 2U_6 = 3xV_4 - 2x^2V_3 + x^3V_2 \\ 2U_8 = 4xV_6 - 4x^2V_5 + 2x^3V_4 + 0x^4V_3 \end{cases}$$

3 Proof of Theorems

Theorem 1 follows from the following lemma.

Lemma 3.1. *For any integer $n \geq 0$, by setting $m = \lfloor n/2 \rfloor$, we have*

$$\det_{\mathcal{C}_n}(\mathfrak{U}_n) = (-1)^{m(m+1)/2} \quad \textit{and} \quad \det_{\mathcal{C}_n}(\mathfrak{V}_n) = 2(-1)^{m(m+1)/2}.$$

Proof. Let us prove only the first equality as the proof of the other one is similar.

Let $r = n-2m$, $W_k^{(m)} = x^k U_{2m+1-k}$ $(0 \le k \le m)$ and $\Delta_m = \det_{\mathcal{C}_{2m}}(W_0^{(m)}, W_1^{(m)}, \ldots, W_m^{(m)})$, we have

$$\det_{\mathcal{C}_n}(\mathfrak{U}_n) = \det_{\mathcal{C}_{2m+r}}\left(x^{r+k}U_{2m+1-k}\right)_{0\le k\le m} = \Delta_m.$$

The result follows by noticing that $\Delta_0 = 1$ and $\Delta_m = (-1)^m \Delta_{m-1}$ for $m \ge 1$. Indeed, for $m \ge 1$, we have

$$W_{k+1}^{(m)} - W_k^{(m)} = x^k(xU_{2m-k} - U_{2m-k+1}) = -yW_k^{(m-1)} \quad (0 \le k \le m-1).$$

Thus,

$$\begin{aligned}\Delta_m &= \det_{\mathcal{C}_{2m}}(W_0^{(m)}, W_1^{(m)} - W_0^{(m)}, ..., W_{m-1}^{(m)} - W_{m-2}^{(m)}, W_m^{(m)} - W_{m-1}^{(m)}) \\ &= \det_{\mathcal{C}_{2m}}(W_0^{(m)}, -yW_0^{(m-1)}, -yW_1^{(m-1)}, \ldots, -yW_{m-1}^{(m-1)}).\end{aligned}$$

The "component" of $W_0^{(m)} = U_{2m+1}$ over x^{2m} is equal to 1.

The "component" of $-yW_k^{(m-1)}$ over x^{2n}, is equal to 0, for $1 \le k \le m$, so we have

$$\Delta_m = \det_{\mathcal{C}_{2m-2}}(-W_0^{(m-1)}, -W_1^{(m-1)}, \ldots, -W_{m-1}^{(m-1)}) = (-1)^m \Delta_{m-1}.$$ □

Let A_m, B_m, C_m, D_m and E_m be the operators on $(\mathbb{Q}[x,y])^{\mathbb{N}}$ defined by

$$\begin{aligned}A_m &= -(x-E)^m + 2\sum_{k=0}^{m} E^k (x-E)^{m-k} \quad (m \ge 0), \\ B_m &= -(E-x)^m \quad (m \ge 0), \\ C_m &= 2E^m + 2B_m - xE^{m-1} \quad (m \ge 1), \\ D_m &= (E-x)^{m-1}(x-2E) \quad (m \ge 1), \\ E_m &= \frac{1}{2}(xA_{m-1} + D_m) + E^m \quad (m \ge 1),\end{aligned}$$

where E is the forward shift operator given by

$$E\left((W_n)_n\right) = (W_{n+1})_n$$

Then, we have

$$\begin{aligned}A_m &= \sum_{k=0}^{m} a_{m,k}x^k E^{m-k} \quad &\text{with} \quad a_{m,k} &= \sum_{j=0}^{m}(-1)^{j+k}(2-\delta_{m,j})\binom{j}{k} \\ B_m &= \sum_{k=0}^{m} b_{m,k}x^k E^{m-k} \quad &\text{with} \quad b_{m,k} &= (-1)^{k+1}\binom{m}{k} \\ C_m &= \sum_{k=1}^{m} c_{m,k}x^k E^{m-k} \quad &\text{with} \quad c_{m,k} &= 2(-1)^{k+1}\binom{m}{k} - \delta_{k,1} \\ D_m &= \sum_{k=0}^{m} d_{m,k}x^k E^{m-k} \quad &\text{with} \quad d_{m,k} &= (-1)^{k+1}\frac{2m-k}{m}\binom{m}{k} \\ E_m &= \sum_{k=1}^{m} e_{m,k}x^k E^{m-k} \quad &\text{with} \quad e_{m,k} &= \frac{1}{2}(d_{m,k}+a_{m-1,k-1}) + \delta_{k,0}.\end{aligned}$$

With these notations, relations stated by Theorems A, B, C, D and E may be expressed by means of the following relations

$$\begin{array}{llll} \mathbf{a}.\ \forall n & \in & \mathbb{N} & A_nV_n = 2U_{2n+1} \\ \mathbf{b}.\ \forall n & \in & \mathbb{N}^* & B_nU_n = 0 \\ \mathbf{c}.\ \forall n & \in & \mathbb{N}^* & C_nU_n = V_{2n-1} \\ \mathbf{d}.\ \forall n & \in & \mathbb{N}^* & D_nV_{n-1} = 0 \\ \mathbf{e}.\ \forall n & \in & \mathbb{N}^* & E_nV_{n-1} = 2U_n \end{array}$$

which are to be proven. For this, the following lemma will be useful for us.

Lemma 3.2. *For every integers n and m, we have*

1. $V_n = 2U_{n+1} - xU_n \quad (n \geq 0)$ *and* $V_n = U_{n+1} + yU_{n-1} \quad (n \geq 1)$,
2. $(E-x)^n U_m = y^n U_{m-n}$ *and* $(E-x)^n V_m = y^n V_{m-n} \quad (m \geq n \geq 0)$,
3. $\sum_{k=1}^{n} (-y)^{n-k} V_{2k} = U_{2n+1} - (-y)^n \quad (n \geq 0)$.

Proof.

1. See relation (2.9) and (2.8) in [16].
2. We proceed by induction on n.
3. For every integer $n \in \mathbb{N}$, put $T_n := U_{2n+1} - \sum_{k=1}^{n} (-y)^{n-k} V_{2k}$. The relation to be proven is equivalent to $T_n = (-y)^n \quad (n \geq 0)$. Then, we remark that from the first relation of this lemma, we have for every integer $n \geq 1$
$$T_n + yT_{n-1} = U_{2n+1} + yU_{2n-1} - V_{2n} = 0,$$
$(T_n)_{n\geq 0}$ is then a geometric sequence with multiplier $(-y)$ and of first term $T_0 = 1$. It follows that for every integer $n \in \mathbb{N}$, $T_n = (-y)^n$.

□

Proof of relations a., b., c., d. and e. Using the above Lemma, we have

a. $A_nV_n = (-y)^n V_0 + 2\sum_{k=1}^{n} (-y)^{n-k} V_{2k} = 2U_{2n+1}$.
b. $B_nU_n = -(E-x)^n U_n = -y^n U_0 = 0$.
c. $C_nU_n = (2E^n + 2B_n - xE^{n-1}) U_n = 2U_{2n} - xU_{2n-1} = V_{2n-1}$.
d. $D_nV_{n-1} = (E-x)^{n-1} (xV_{n-1} - 2V_n) = y^{n-1} (xV_0 - 2V_1) = 0$.
e. $E_nV_{n-1} = \left(\frac{1}{2}xA_{n-1} + \frac{1}{2}D_n + E^n\right) V_{n-1} = \frac{1}{2}xA_{n-1}V_{n-1} + \frac{1}{2}D_nV_{n-1} + V_{2n-1}$. Using $A_{n-1}V_{n-1} = 2U_{2n-1}$ and $D_nV_{n-1} = 0$, it follows that $E_nV_{n-1} = xU_{2n-1} + V_{2n-1} = 2U_{2n}$ □

Remark 3.1. Theorems A, B, C, D and E generalize results obtained for the Chebyshev polynomials [4], Indeed,

$$\begin{aligned} \frac{1}{2}V_n(2x,-1) &= T_n(x) \text{ is the Chebyshev polynomials of the first kind,} \\ U_{n+1}(2x,-1) &= U_n(x) \text{ is the Chebyshev polynomials of the second kind,} \end{aligned}$$

with

$$\begin{cases} T_n(x) = 2xT_{n-1} - T_{n-2}, \\ T_0 = 1, T_1 = x, \end{cases} \quad \text{and} \quad \begin{cases} U_n(x) = 2xU_{n-1} - U_{n-2}, \\ U_0(x) = 1, U_1 = 2x. \end{cases}$$

References

[1] R. André-Jeannin, A generalization of Morgan-Voyce polynomials, *Fibonacci Quart.* **32** (1994), 228–231.

[2] R. André-Jeannin, Differential Properties of a General Class of Polynomials, *Fibonacci Quart.* **33** (1995), no. 5, 453–458.

[3] H. Belbachir and F. Bencherif, Linear recurrent sequences and powers of a square matrix. *Integers* **6** (2006), A12.

[4] H. Belbachir and F. Bencherif, On some properties of Chebyshev polynomials. *Discuss. Math. Gen. Algebra Appl.* **28** (2), (2008).

[5] R. X. F. Chen and L. W. Shapiro, On sequences G_n satisfying $G_n = (d+2)\, G_{n-1} - G_{n-2}$, *J. Integer Seq.* **10** (2007), Art. 07.8.1.

[6] G. B. Djordjević, Generalized Jacobsthal polynomials, *Fibonacci Quart.* **38**, (2000), 239–243.

[7] A. F. Horadam, Jacobsthal representation polynomials, *Fibonacci Quart.* **35** (1997), 137–148.

[8] A. F. Horadam, Quasi Morgan-Voyce polynomials and Pell convolutions, *Applications of Fibonacci Numbers*, Vol. 8 (Rochester, NY, 1998), pp. 179–193, Kluwer Acad. Publ., Dordrecht, 1999.

[9] A. F. Horadam, Associated Legendre polynomials and Morgan-Voyce polynomials, *Notes Number Theory Discrete Math.* **5** (1999), no. 4, 125-134.

[10] A. F. Horadam, Vieta polynomials, *Fibonacci Quart.* **40** (2002), 223–232.

[11] J. Y. Lee, On the Morgan-Voyce polynomial generalization of the first kind, *Fibonacci Quart.* **40** (2002), 59–65.

[12] E. Lucas, *Théorie des Nombres*, Gauthier-Villars, Paris, 1891.

[13] A. M. Morgan-Voyce, Ladder network analysis using Fibonacci numbers, *IRE Trans. Circuit Theory* 6, no. **3**, (1959), 321–322.

[14] N. Robbins, Vieta triangular array and a related family of polynomials. *Internat. J. Math. Math. Sci.* **14** (2), (1991), 239–244.

[15] A. G. Shannon and A. F. Horadam, Some relationships among Vieta, Morgan-Voyce and Jacobsthal polynomials, *Application of Fibonacci numbers*, Vol. 8, pp. 307–323, Ed. F. Howard, Dordrecht: Kluwer, 1999.

[16] M. N. S. Swamy, Generalized Fibonacci and Lucas polynomials and their associated diagonal polynomials, *Fibonacci Quart.* **37** (1999), 213–222.

[17] M. N. S. Swamy, Generalization of modified Morgan-Voyce polynomials, *Fibonacci Quart.* **38** (2000), 8–16.

[18] F. Vieta. *Opera Mathematica: Ad Angulus Sectiones.* (Theorema VI). Paris, 1615 (see [14]).

2000 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B37 .
Keywords: bivariate Fibonacci and Lucas polynomials, linear recurrences.

(Concerned with sequences A007318, A029653, and A112468.)

Received October 8 2007; revised version received June 3 2008. Published in *Journal of Integer Sequences*, June 27 2008.

Return to Journal of Integer Sequences home page.