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The Hyperrings of Order 3

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Abstract

We first explain the historical and logical relations of hyperstructures introduced by M. Krasner and R. Rota, and generalized by T. Vougiouklis. Then, with our new algorithm based on our previous results on hypergroups and H_v-groups of order 2, 3 and 4, we enumerate hyperrings and H_v-rings. More precisely, we found 63 hyperrings of order 2, 875 H_v-rings of order 2 and 33,277,642 hyperrings of order 3. Finally, in this new context, we study a new connection between groups and hypergroups via the notion of duality.

1 Introduction and Definitions

More than seventy years have gone by since the creation of the concept of hypergroup [21]. M. Krasner and H. S. Wall [28, 35] introduced similar definitions, but only F. Marty's concept is accepted [14, 15, 16, 17, 19, 30, 31, 32].

Definition 1. *A hypergroupoid $\langle H, . \rangle$ is a set H equipped with an hyperoperation $(.) : H \times H \longrightarrow \mathcal{P}(H)$.*

Definition 2. *A quasigroup is an hypergroupoid verifying the axiom of reproduction: $\forall x \in H\ xH = Hx = H$.*

Definition 3. *A semigroup is an hypergroupoid verifying associativity:* $\forall x, y, z \in H\; x(yz) = (xy)z$.

Definition 4 (F. Marty [21, 22, 23])**.** *A hypergroup* $\langle H, .\rangle$ *is a quasigroup verifying associativity:* $\forall x, y, z \in H\; x(yz) = (xy)z$.

In 1991 T. Vougiouklis generalized the definition of F. Marty by weakening associativity [30].

Definition 5. *A hyperoperation is weakly associative if for any* $x, y, z \in H$, $x(yz) \cap (xy)z \neq \emptyset$.

Definition 6 (T. Vougiouklis [30])**.** $\langle H, .\rangle$ *is a* H_v*-group if* $(.) : H \times H \to p(H)$ *is a weakly associative hyperoperation for which the reproduction axiom* $hH = Hh = H$ *is valid for any* h *of* H.

The essential idea which governs the existence of these H_v-groups is a weaken associativity. This weakening simply consists in considering the two terms of the associative law as sets, since this is possible, and in requiring that their intersection shall not be empty.

The H_v-groups have a property of which the hypergroups are deprived. This one is built from the definition of the following partial order.

Definition 7 (T. Vougiouklis [31])**.** *Let* $\langle H, .\rangle$ *and* $\langle H, *\rangle$ *two* H_v*-groups. We say that* $(.)$ *is less or equal than* $(*)$*, and note* $\leq$*, if and only if there exists* $f \in Aut(H, *)$ *such that* $xy \subseteq f(x * y)$ *for any* x, y *of* H.

From this definition we can deduce the following theorem:

Theorem 8 (T. Vougiouklis [31])**.** *If a hyperoperation is weakly associative, then any hyperoperation superior to it and defined on the same set is weakly associative too.*

From this property, we can show the concept of minimality in a natural way.

Definition 9 (T. Vougiouklis [30])**.** $(R, +, .)$ *A hyperstructure is called a hyperring if* $(R, +)$ *is a hypergroup,* $(R, .)$ *is a semigroup and* $(.)$ *is distributive in respect to* $(+)$.

Definition 10 (S. Spartalis, A. Dramalides and T. Vougiouklis [29])**.** $(R, +, .)$ *A hyperstructure is called an* H_v*-ring if* $(R, +)$ *is an* H_v*-group,* $(R, .)$ *is a weak semigroup and* $(.)$ *is weakly distributive in respect to* $(+)$.

A. Dramalidis enumerated a restricted class
of H_v-rings, the dual H_v-rings.

Definition 11 (A. Dramalidis [16])**.** *An* H_v*-ring* $(R, +, .)$ *is dual if* $(R, ., +)$ *is an* H_v*-ring.*

He classified all H_v-ring such that $R = \{0, 1, a\}$ where 0 is the scalar unit of H_v-group $(R, +)$ and absorbing element of semi-hypergroup $(H, .)$ and 1 is the scalar unit of semi-hypergroup $(H, .)$. In the same way, he classified all hyperannoids, where $(.)$ is not distributive in respect to $(+)$. He needed to avoid a maximum of computations because they were done case-by-case. So he tried to minimize the role of associativity because of its high computational cost and use the symmetry of duality.

We enumerate hyperrings and H_v-rings of small orders and this will probably improve our understanding of the hyperannoids [6]. Indeed certain categories of hypergroups were studied because of their low computational cost, but they were useless for the understanding of hyperstructures. Our research [4, 2] already showed the greater importance of cyclic and single-power hypergroups than the canonical hypergroups [25, 26].

From an historical point of view, M. Krasner introduced the notion of hyperring in 1966, ten years after the notion of hyperfield. So the hyperring in M. Krasner's sense generalizes his notion of hyperfield. This one was considered as the natural extension of F. Martys's hypergroups. But this extension is not as natural as it seems. In order to avoid technical problems, Krasner used *ad hoc* properties which were studied by his disciple J. Mittas. Mittas introduced canonical hypergroups which are, in short, a restriction of hyperring and consequently of hyperfield in Krasner's sense. This global schema seemed complete and closed, but in fact was not. The radically different approach of T. Vougiouklis showed this critical point. T. Vougiouklis started his work by weakening associativity in the hypergroup of Marty. It was then easy to extend this notion to hyperring and to hyperfield in a natural way. Moreover, this approach generalizes Krasner's and Rota's approaches. Vougiouklis does not work in a specific case as canonical hypergroups. His approach is based on the hypergroup in Marty's sense and moreover, he introduces H_v-groups. He avoids the pitfall of representativity in the world of hypergroups. Indeed in our research, we show the low importance of canonical hypergroups in the set of hypergroups. From this observation, we easily deduce that M. Krasner's generalization of hyperrings and hyperfields are analogous in the corresponding world. So the generalization of Vougiouklis embraces the whole set of hyperstructures.

2 Enumeration

In enumeration theory we have already obtained some results in different fields (see [8, 9, 11, 12, 18, 20] and A108089, A132590.

In our previous work we enumerate and classify the hypergroups of order 3 [3, 4] and abelian hypergroups of order 4 [5] with Birkhoff's point of view [10]. We then study the H_v-groups of order 3 [7], and abelian H_v-groups of order 4 with Marty-Moufang hypergroups [1].

Thanks to these enumerative results we can characterize some hyperstructures, as shown with rigid hyperstructures or with hypocomplete hypergroups [4]. The obtained results contribute consequently to validate our algorithm. We could also confirm the results of R. Migliorati [24], some results of S-C. Chung and B-M Choi [13] and results of T. Vougiouklis [33, 34] too. Now we present the best computational results in these fields A132591.

Theorem 12 (G. Nordo [27])**.** *There are 3,999 isomorphism classes of hypergroups of order 3.*

We give a more precise presentation with the classification of hypergroups by projectivity and cyclicity (see table 1).

		Classes of isomorphism					
		Abelians			non Abelians		
		Cyclics	non Cyclics		Cyclics	non Cyclics	
			Proj.	non Proj.		Proj.	non Proj.
$\lvert Aut(H)\rvert$	1	4	2	-	-	-	-
	2	3	-	-	6	1	-
	3	70	3	5	154	8	4
	6	360	2	17	3279	20	61

Table 1: Classification of Hypergroups of Order 3

Theorem 13 (R. Bayon & N. Lygeros [7]). *There are 20 isomorphism classes of H_v-groups of order 2 (see table 2), $*$ indicates hypergroups).*

H_v-group	$\lvert Aut(H_v)\rvert$	H_v-group	$\lvert Aut(H_v)\rvert$
$(a;b;b;a)^*$	2	$(H;a;H;b)^*$	2
$(H;b;b;a)$	2	$(a;H;H;b)^*$	1
$(a;H;b;a)$	2	$(H;a;a;H)$	2
$(a;b;H;a)$	2	$(H;b;a;H)$	1
$(H;a;a;b)^*$	2	$(H;a;b;H)$	1
$(H;H;b;a)$	2	$(H;H;H;a)^*$	2
$(H;b;H;a)$	2	$(H;H;H;b)^*$	2
$(a;H;H;a)$	2	$(H;H;a;H)$	2
$(b;H;H;a)$	1	$(H;H;b;H)$	2
$(H;H;a;b)^*$	2	$(H;H;H;H)^*$	1

Table 2: H_v-groups of Order 2 ($H = \{a, b\}$)

Theorem 14 (R. Bayon & N. Lygeros [7]). *There are 1,026,462 isomorphism classes of H_v-groups of order 3 (see table 3).*

		Classes of isomorphism					
		Abelian			non Abelians		
		Cyclics	non Cyclics		Cyclics	non Cyclics	
			Proj.	non Proj.		Proj.	non Proj.
$\lvert Aut(H_v)\rvert$	1	5	2	-	4	2	-
	2	8	1	1	47	5	7
	3	243	8	14	2034	66	76
	6	7439	10	195	1003818	1083	11394

Table 3: Classification of H_v-groups of Order 3

3 Hyperrings

A natural approach to hyperrings is to construct them from their underlying hyperstructures. With this manner, we can easily check intermediate results. Consequently, we use the enumeration of hypergroups, semi-hypergroups, H_v-groups and S_v-groups (which are analogue of H_v-groups for semi-hypergroups).

Fact 1 (R. Bayon & N. Lygeros). *Let $(R, +, .)$ be a hyperring then $Aut(R) = Aut(+) \cup Aut(.)$.*

Corollary 15 (R. Bayon & N. Lygeros). *Let $(R, +, .)$ be a hyperring then $|Aut(R)| \geq max(|Aut(+)|, |Aut(.)|)$.*

Theorem 16 (R. Bayon & N. Lygeros). *There are 63 isomorphism classes of hyperrings of order 2 (see table 4).*

		Classes
$\lvert Aut(R)\rvert$	1	6
	2	114

Table 4: Classification of Hyperrings of Order 2

Theorem 17 (R. Bayon & N. Lygeros). *There are 875 isomorphism classes of H_v-rings of order 2 (see table 5).*

		Classes
$\lvert Aut(R)\rvert$	1	33
	2	1684

Table 5: Classification of H_v-rings of Order 2

Theorem 18 (R. Bayon & N. Lygeros). *There are 33,277,642 isomorphism classes of hyperrings of order 3 (see Table 6).*

		Classes
$\lvert Aut(R)\rvert$	1	31
	2	506
	3	67,857
	6	199,528,434

Table 6: Classification of Hyperrings of Order 3

This global approach generalize the partial results obtained by T. Vougiouklis and A. Dramalidis [16, 31].

3.1 Algorithm

We generate all the simple hyperstructures (hypergroups, semigroups, H_v-groups,...) and for each of them we compute the order of its automorphisms group. The number of hyperstructures, up to isomorphism, p is:

$$p = \sum_{i=1}^{n!} \frac{s_i}{i}$$

where n is the order of the hyperstructures, and s_i is the number of hyperstructures having an automorphisms group of order i.

The algorithm for hyperrings is a natural extension of this one. We generate all simple hyperstructures and check distributivity for each valid pair of hyperstructures. If the hyperringoid verifies distributivity, we compute and we store the order of their automorphisms group. As all paires have been checked, we determine the number of hyperrings, up to isomorphism.

4 Remarks on the notion of the dual of a group

Our research on the notion of the dual of a group is leaded by the program of M. Mizony. We want to establish a structural link between groups and hypergroups. The dual of a group G, denoted $\hat{G}$ is the set of characters which are morphism from the group to the multiplicative group $\mathbb{C}^*$. So $\hat{G}$ is a group for product of morphism. If G is a finite group of order n then the elements of $\hat{G}$ are the morphisms from G to the group of the n^{th} root of unity.

For the cyclic group $\mathbb{Z}/p\mathbb{Z}$ if we denote $\omega = e^{\frac{2\nu x}{p}}$, with $i \in \{0, \ldots, n-1\}$, all the elements of $\hat{G}$ are written:

$$\chi_i : G \to \mathbb{C}^*$$
$$x \mapsto (\omega^i)^x = e^{\frac{2\nu x i}{p}}$$

We have $G \cong \hat{G}$ and $\hat{G}$ is an orthogonal basis of E the set of functions from G to $\mathbb{C}$, vector space with an Hermitian product: $\langle f, g \rangle = \sum_{x \in G} f(\bar{x}).g(x)$ ($f(\bar{x})$ is the conjugate of $f(x)$), and we know that this product definition is important for classical hypergroups. It is a way to catch hyperrings in the sense of F. Marty. It is possible resolve an more generic case, the one of abelian finite group. In this case, the dual of G is an orthogonal basis of E too. We can observe it by considering the decomposition of $\mathbb{Z}$-modules:

$$G \cong \mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_r\mathbb{Z}$$

But this kind of approach is not possible in the non-abelian case. To see this, we can study the symmetric group of order n. Indeed, by choosing two specific transpositions and combining them to have a permutation, we obtain only two characters. This problem in classical group theory is an opening to the theory of hypergroups.

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