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Bounds for the Eventual Positivity of Difference Functions of Partitions into Prime Powers

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Abstract

In this paper we specialize work done by Bateman and Erdős concerning difference functions of partition functions. In particular we are concerned with partitions into fixed powers of the primes. We show that any difference function of these partition functions is eventually increasing, and derive explicit bounds for when it will attain strictly positive values. From these bounds an asymptotic result is derived.

1 Introduction

Given an underlying set $A \subseteq \mathbb{N}$, we denote the number of partitions of n with parts taken from A by $p_A(n)$. The k-th difference function $p_A^{(k)}(n)$ is defined inductively as follows: for $k = 0$, $p_A^{(k)}(n) = p_A(n)$. If $k > 0$, then

$$p_A^{(k)}(n) = p_A^{(k-1)}(n) - p_A^{(k-1)}(n-1).$$

Let $f_A^{(k)}(x)$ be the generating function for $p_A^{(k)}(n)$. We have [5] the following power series identity:

$$f_A^{(k)}(x) = \sum_{n=0}^{\infty} p_A^{(k)}(n)x^n \tag{1}$$

$$= (1-x)^k \sum_{n=0}^{\infty} p_A(n)x^n \tag{2}$$

$$= (1-x)^k \prod_{a\in A} \frac{1}{1-x^a}. \tag{3}$$

This may be used to define $p_A^{(k)}(n)$ for all $k \in \mathbb{Z}$, including $k < 0$.

Bateman and Erdős [5] characterize the sets A for which $p_A^{(k)}(n)$ is ultimately nonnegative. Note that if $k < 0$, then the power series representation of $(1-x)^k$ has nonnegative coefficients so that $p_A^{(k)}(n) \geq 0$. For $k \geq 0$, they prove the following: if A satisfies the property that whenever k elements are removed from it, the remaining elements have greatest common divisor 1, then

$$\lim_{n\to\infty} p_A^{(k)}(n) = \infty.$$

A simple consequence of this is the fact that $p_A(n)$ is eventually monotonic if $k > 0$.

No explicit bounds for when $p_A^{(k)}(n)$ becomes positive are included with the result of Bateman and Erdős [5]. By following their approach but specializing to the case when

$$A = \{p^\ell : p \text{ is prime}\}, \tag{4}$$

for some fixed $\ell \in \mathbb{N}$, we shall find bounds for n depending on k and ℓ which guarantee that $p_A^{(k)}(n) > 0$. For the remainder, A shall be as in (4), with $\ell \in \mathbb{N}$ fixed.

In a subsequent paper, Bateman and Erdős [6] prove that in the special case when $\ell = 1$, $p_A^{(1)}(n) \geq 0$ for all $n \geq 2$. That is, the sequence A000607 of partitions of n into primes is increasing for $n \geq 1$. Our result pertains to more general underlying sets; partitions into squares of primes (A090677), cubes of primes, etc.

In a series of papers (cf. [10]- [13]), L. B. Richmond studies the asymptotic behaviour for partition functions and their differences for sets satisfying certain stronger conditions. The results none-the-less apply to the cases of interest to us, that is, where A is defined as above. Richmond proves [12] an asymptotic formula for $p_A^{(k)}(n)$. Unfortunately, his formula is not useful towards finding bounds for when $p_A^{(k)}(n)$ must be positive, since, as is customary, he does not include explicit constants in the error term.

Furthermore, his asymptotic formula includes functions such as $\alpha = \alpha(n)$ defined by

$$n = \sum_{a\in A} \frac{a}{e^{\alpha a} - 1} - \frac{k}{e^{\alpha} - 1}.$$

As we are seeking explicit constants, a direct approach will be cleaner than than attempting to adapt the aforementioned formula.

Another result worthy of comment from Richmond [12] pertains to a conjecture of Bateman and Erdős [5]. His result applies to A as defined in (4), and states that

$$\frac{p_A^{(k+1)}(n)}{p_A^{(k)}(n)} = O(n^{-1/2}), \text{ as } n \to \infty.$$

We omit the subscript and write $p^{(k)}(n)$ in case the underlying set is A, and omit the superscript if $k = 0$. The letter B will always be used to denote a finite subset of A. We shall also write ζ_n for the primitive n-th root of unity $e^{2\pi i/n}$. The letters ζ and η shall always denote roots of unity. We shall denote the m-th prime by p_m.

Our main results are Theorems 1.1 and 1.2. The former is established in Sections 2 and 3.

Theorem 1.1. *Let k be a nonnegative integer, let $b_0 = 2\pi\sqrt{1 - \frac{\pi^2}{12}}$ and let*

$$t = \left\lfloor 6\left(\frac{2}{b_0}\right)^{3(k+1)} (k+2)^{8\ell k + 10\ell + 3} \right\rfloor.$$

Then there are positive absolute constants $a_1, \ldots, a_7$ such that if

$$N = N(k, \ell) = a_1 t \left(\frac{a_2 a_3^{\ell \log^2 t}}{a_4^{(k+3)^\ell}}\right)^{t-1} + a_5 t^3 \left(a_6 (a_7 t)^{6\ell}\right)^{t-1},$$

then $n \geq N$ implies that $p^{(k)}(n) > 0$.

Remark 1. *The values of the constants are approximately*

$$\begin{aligned}
a_1 &\approx 1.000148266, \\
a_2 &\approx 2757234.845, \\
a_3 &\approx 1424.848799, \\
a_4 &\approx 2.166322546, \\
a_5 &\approx 1.082709333, \\
a_6 &\approx .0193095561, \\
a_7 &\approx 2.078207555.
\end{aligned}$$

For the remainder of the paper, b_0 shall be as defined in Theorem 1.1. Note that $b_0 \approx 2.6474$. Furthermore, Define

$$F(k, \ell) = \min\{N \in \mathbb{N} : n \geq N \text{ implies that } p^{(k)}(n) > 0\}.$$

We show in Section 4 that Theorem 1.1 yields the following asymptotic result:

Theorem 1.2. *Fix $\ell \in \mathbb{N}$. Then as $k \to \infty$,*

$$\log F(k, \ell) = o\left((k+2)^{8\ell k}\right).$$

Following Bateman and Erdős [5], we first tackle the finite case.

2 Finite subsets of A

Lemma 2.1. *Let B be a finite subset of A of size r, and suppose $k < r$. The function $p_B^{(k)}(n)$ can be decomposed as follows:*

$$p_B^{(k)}(n) = g_B^{(k)}(n) + \psi_B^{(k)}(n),$$

where $g_B^{(k)}(n)$ is a polynomial in n of degree $r-k-1$ with leading coefficient $\left((r-k-1)! \prod_{q \in B} q\right)^{-1}$, and $\psi_B^{(k)}(n)$ is periodic in n with period $\prod_{q \in B} q$.

Proof. We use partial fractions to decompose the generating function $f_B^{(k)}(x)$ as follows:

$$f_B^{(k)}(x) = \frac{1}{(1-x)^{r-k}} \prod_{q \in B} \prod_{j=1}^{q-1} \frac{1}{1 - \zeta_q^j x} \tag{5}$$

$$= \frac{\alpha_1}{1-x} + \frac{\alpha_2}{(1-x)^2} + \ldots + \frac{\alpha_{r-k}}{(1-x)^{r-k}} + \sum_{q \in B} \sum_{j=1}^{q-1} \frac{\beta(\zeta_q^j)}{1 - \zeta_q^j x}, \tag{6}$$

where the α_i, and $\beta(\zeta_q^j)$ are complex numbers that can be determined. Note that

$$\alpha_{r-k} = \left(\prod_{q \in B} q \right)^{-1}.$$

The power series expansion for $(1-x)^{-h}$ is given by

$$\frac{1}{(1-x)^h} = \sum_{n=0}^{\infty} \binom{n+h-1}{h-1} x^n.$$

Hence, if

$$g_B^{(k)}(n) = \sum_{h=1}^{r-k} \alpha_h \binom{n+h-1}{h-1},$$

and

$$\psi_B^{(k)}(n) = \sum_{q \in B} \sum_{j=1}^{q-1} \beta(\zeta_q^j) \zeta_q^{jn},$$

then the lemma is proved. □

For the remainder of this paper, $g_B^{(k)}(n)$ and $\psi_B^{(k)}(n)$ shall be as in Lemma 2.1, for a given finite set $B \subseteq A$ which shall be clear from the context.

Remark 2. *Let B be as in Lemma 2.1. We wish to know the precise value of $\beta(\zeta_q^j)$. To simplify notation a little, we will frequently write β_ζ instead, when ζ is clear from the context.*

In particular, suppose $\eta = \zeta_q^j$, where $q \in B$, and $0 < j < q$. Then

$$\begin{aligned}(1-\eta x) f_B^{(k)}(x) &= (1-x)^k \frac{1}{1+\eta x+\cdots+(\eta x)^{q-1}} \prod_{\substack{p\in B\\ p\neq q}} \frac{1}{1-x^p} \\ &= \beta_\eta + (1-\eta x)\left(\frac{\alpha_1}{1-x}+\cdots+\frac{\alpha_{r-k}}{(1-x)^{r-k}}+\sum_{\zeta\neq\eta}\frac{\beta_\zeta}{1-\zeta x}\right),\end{aligned}$$

hence,

$$\beta_\eta = \frac{(1-\bar\eta)^k}{q} \prod_{\substack{p\in B\\ p\neq q}} \frac{1}{1-\bar\eta^p}.$$

We shall frequently make use of the inequality

$$1-\frac{\theta^2}{2} \le \cos\theta \le 1-\frac{\theta^2}{2}+\frac{\theta^4}{24},$$

which holds for all values of θ. Note that

$$|e^{i\theta}-1| = \sqrt{2(1-\cos\theta)},$$

so for $-2\sqrt{3} \le \theta \le 2\sqrt{3}$,

$$|\theta|\sqrt{1-\frac{\theta^2}{12}} \le |e^{i\theta}-1| \le |\theta|.$$

In particular, for $0 \le \theta \le \pi$,

$$\theta\sqrt{1-\frac{\theta^2}{12}} \le |e^{i\theta}-1| \le \theta. \tag{7}$$

Lemma 2.2. *Suppose that $\zeta \neq 1$ is a q-th root of unity, $q \ge 2$. Then*

$$\frac{b_0}{q} \le |1-\zeta| \le 2.$$

Proof. Clearly $|1-\zeta| \le 2$. On the other hand, by equation (7),

$$\begin{aligned}|1-\zeta| &\ge |1-e^{2\pi i/q}| \\ &\ge \frac{2\pi}{q}\sqrt{1-\frac{4\pi^2}{12q^2}} \\ &\ge \frac{b_0}{q}.\end{aligned}$$

□

Lemma 2.3. *Suppose that $k < r$, and $B \subseteq A$, satisfies $|B| = r$. Suppose further that $\eta = \zeta_q^j$, for some $q \in B$, $j \in \{1, \dots, q-1\}$. Then for $k \geq 0$,*

$$|\beta_\eta| \leq \begin{cases} \frac{2^k q^{r-2}}{b_0^{r-1}}, & \text{if } k \geq 0; \\ \frac{q^{r-k-2}}{b_0^{r-k-1}}, & \text{if } k < 0. \end{cases}$$

Proof. Making use of Remark 2 and Lemma 2.2 we have that for $k \geq 0$:

$$\begin{aligned} |\beta_\eta| &= \frac{|1-\zeta_q^{-j}|^k}{q} \prod_{\substack{p \in B \\ p \neq q}} \frac{1}{|1-\zeta_q^{-jp}|} \\ &\leq \frac{2^k}{q} \prod_{\substack{p \in B \\ p \neq q}} \frac{1}{|1-\zeta_q|} \\ &\leq \frac{2^k}{q} \left(\frac{q}{b_0}\right)^{r-1} \\ &= \frac{2^k q^{r-2}}{b_0^{r-1}}. \end{aligned}$$

A similar arguments works for the case when $k < 0$. □

Theorem 2.1. *Suppose that $k < r$, and $B \subseteq A$, satisfies $|B| = r$. Then*

$$|\psi_B^{(k)}(n)| \leq \begin{cases} \frac{2^k}{b_0^{r-1}} \sum_{q \in B} q^{r-1}, & \text{if } k \geq 0; \\ \frac{1}{b_0^{r-k-1}} \sum_{q \in B} q^{r-k-1}, & \text{if } k < 0. \end{cases}$$

Proof. First assume that $k \geq 0$. By Lemmas 2.3 and 2.1,

$$\begin{aligned} |\psi_B^{(k)}(n)| &= \left| \sum_{q \in B} \sum_{j=1}^{q-1} \beta(\zeta_q^j) \zeta_q^{jn} \right| \\ &\leq \sum_{q \in B} \sum_{j=1}^{q-1} |\beta(\zeta_q^j)| \\ &\leq \sum_{q \in B} \sum_{j=1}^{q-1} \frac{2^k q^{r-2}}{b_0^{r-1}} \\ &= \frac{2^k}{b_0^{r-1}} \sum_{q \in B} (q-1) q^{r-2} \\ &\leq \frac{2^k}{b_0^{r-1}} \sum_{q \in B} q^{r-1}. \end{aligned}$$

A similar argument works for $k < 0$. □

To obtain bounds for the coefficients α_h, we will use Laurent series.

Lemma 2.4. *Suppose that $k < r$, and $B \subseteq A$, satisfies $|B| = r$. Denote the largest element of B by Q, and suppose further that $0 < r_0 < |1 - \zeta_Q|$. Let*

$$d_B(r_0) = \prod_{q \in B} \prod_{j=1}^{q-1} (|\zeta_q^j - 1| - r_0).$$

Then

$$|\alpha_h| \leq \frac{1}{r_0^{r-k-h} d_B(r_0)}.$$

Proof. Let γ be the circle $|z-1| = r_0$. From equations (5) and (6), and the Laurent expansion theorem, we have that

$$\begin{aligned}
|\alpha_h| &= \left| \frac{1}{2\pi i} \int_\gamma f_B^{(k)}(z)(z-1)^{h-1}\, dz \right| \\
&\leq \frac{1}{2\pi} \int_\gamma \frac{1}{|z-1|^{r-k-h+1}} \prod_{q \in B} \prod_{j=1}^{q-1} \frac{1}{|1 - \zeta_q^j z|}\, dz \\
&\leq \frac{1}{r_0^{r-k-h} d_B(r_0)}.
\end{aligned}$$

□

For the following proposition, we define several new constants:

$$\begin{aligned}
c_1 &= \prod_{k=3}^{\infty} \left(1 - \frac{\pi^2}{3 \cdot 4^k}\right), \\
c_2 &= \prod_{k=3}^{\infty} \left(1 - \frac{\pi^2}{3 \cdot 4^k}\right)^{\frac{1}{2^k}}, \\
c_3 &= 4\pi^{-6} c_1, \\
c_4 &= \frac{2 \log \pi}{\log 2} - 1, \\
c_5 &= \left(\frac{\pi}{2}\right)^{1/2} c_2, \\
c_6 &= \left(\frac{\pi}{2}\right)^{1/2} \left(1 - \frac{\pi^2}{48}\right)^{1/4}, \\
c_7 &= \frac{\sin 4\pi/5}{4\pi/5}, \\
b_1 &= c_5 c_6 = 1.471843248\ldots, \\
b_2 &= \frac{c_3}{c_6} = .003278645140\ldots, \\
b_3 &= c_4 = 2.302992260\ldots, \\
b_4 &= c_7 \pi/2 = .3673657828\ldots.
\end{aligned}$$

Proposition 2.1. *Let $B \subseteq A$ satisfy $|B| = r$, and suppose that all the elements of B are odd. Let $Q = \max\{B\}$, and $T = \lceil \log_2 Q \rceil$. If*

$$r_0 = \min\{|\zeta_q^{\lceil \frac{q}{2^j} \rceil} - 1| - |\zeta_{2^j} - 1| : q \in B, j = 2, \ldots, T\},$$

then

$$0 < \frac{b_4}{Q^2} \le r_0 < |1 - \zeta_Q| \le \frac{2\pi}{Q}.$$

Furthermore, if $d_B = d_B(r_0)$, then

$$d_B \ge b_1^{\sum_{q \in B} q} \left(b_2 Q^{b_3} e^{-\frac{\log^2 Q}{\log 2}} \right)^r.$$

Proof. Observe that T satisfies

$$2^{T-1} < Q < 2^T.$$

Now, let

$$\delta_q = \prod_{j=1}^{q-1} (|\zeta_q^j - 1| - r_0),$$

so that

$$d_B = \prod_{q \in B} \delta_q.$$

Note that since each element of B is odd, we have that

$$\delta_q = \prod_{j=1}^{\frac{q-1}{2}} (|\zeta_q^j - 1| - r_0)^2.$$

Consequently,

$$\begin{aligned}
\delta_q &\ge \prod_{k=2}^{T} \prod_{\lceil \frac{q}{2^k} \rceil \le j < \lceil \frac{q}{2^{k-1}} \rceil} |\zeta_{2^k} - 1|^2 \\
&\ge \prod_{k=2}^{T} \left(\frac{\pi^2}{4^{k-1}} \left(1 - \frac{\pi^2}{3 \cdot 4^k} \right) \right)^{\lceil \frac{q}{2^{k-1}} \rceil - \lceil \frac{q}{2^k} \rceil} \\
&= \left(\frac{\pi^2}{4} \left(1 - \frac{\pi^2}{48} \right) \right)^{\lceil \frac{q}{2} \rceil - \lceil \frac{q}{4} \rceil} \times \prod_{k=3}^{T} \left(\frac{\pi^2}{4^{k-1}} \left(1 - \frac{\pi^2}{3 \cdot 4^k} \right) \right)^{\lceil \frac{q}{2^{k-1}} \rceil - \lceil \frac{q}{2^k} \rceil} \\
&\ge \left(\frac{\pi^2}{4} \left(1 - \frac{\pi^2}{48} \right) \right)^{\frac{q-1}{4}} \times \prod_{k=3}^{T} \left(\frac{\pi^2}{4^{k-1}} \left(1 - \frac{\pi^2}{3 \cdot 4^k} \right) \right)^{\frac{q}{2^k} + 1}.
\end{aligned}$$

Now

$$\begin{aligned}
\prod_{k=3}^{T}\left(\frac{\pi^2}{4^{k-1}}\left(1-\frac{\pi^2}{3\cdot 4^k}\right)\right)^{\frac{q}{2^k}+1} &\geq \frac{\pi^{2(T-2)}}{4^{\binom{T}{2}-1}}\left(\frac{\pi^{2(\frac{1}{4}-\frac{1}{2^T})}}{4^{\frac{3}{4}-\frac{T+1}{2^T}}}\right)^q c_1 c_2^q \\
&\geq \frac{4\pi^{-4}Q^{\frac{2\log\pi}{\log 2}}}{Qe^{\frac{\log^2 Q}{\log 2}}}\left(\frac{\pi^{2(\frac{1}{4}-\frac{1}{Q})}}{4^{\frac{1}{4}}}\right)^q c_1 c_2^q \\
&\geq 4\pi^{-4}c_1 Q^{\frac{2\log\pi}{\log 2}-1}e^{-\frac{\log^2 Q}{\log 2}}\left(\frac{\pi^{1/2}c_2}{2^{1/2}}\right)^q \pi^{-2} \\
&= c_3 Q^{c_4} e^{-\frac{\log^2 Q}{\log 2}} c_5^q
\end{aligned}$$

So we have that

$$\begin{aligned}
d_B &\geq \prod_{q\in B}\left(\left(\frac{c_3}{c_6}\right)(c_5 c_6)^q Q^{c_4} e^{-\frac{\log^2 Q}{\log 2}}\right) \\
&= b_1^{\sum_{q\in B} q}\left(b_2 Q^{b_3} e^{-\frac{\log^2 Q}{\log 2}}\right)^r
\end{aligned}$$

Our next task is to bound r_0 from below. For $q \in B$, $j \in \{2, \dots, T\}$, let

$$\begin{aligned}
f(x) &= \sqrt{2(1-\cos x)}, \\
\theta = \theta(j) &= \frac{2\pi}{2^j}, \text{ and} \\
\varepsilon = \varepsilon(q,j) &= \frac{2\pi}{q}\left\lceil \frac{q}{2^j}\right\rceil - \frac{2\pi}{2^j}.
\end{aligned}$$

Then by the mean value theorem,

$$\begin{aligned}
|\zeta_q^{\lceil \frac{q}{2^j}\rceil} - 1| - |\zeta_{2^j} - 1| &= f(\theta+\varepsilon) - f(\theta) \\
&= \frac{\varepsilon \sin c}{\sqrt{2(1-\cos c)}},
\end{aligned}$$

for some $c \in (\theta, \theta+\varepsilon)$.

It is easily seen that $2\pi\lceil q/2^j\rceil/q$ can be no greater than $4\pi/5$. On the interval $[0, 4\pi/5]$, we have $\sin x \geq c_7 x$. Therefore

$$f(\theta+\varepsilon) - f(\theta) \geq \frac{\varepsilon c_7 c}{c} = \varepsilon c_7. \tag{8}$$

Choose $a \in \mathbb{N}$ such that $(a-1)2^j + 1 \leq q \leq a2^j - 1$. Then clearly $a \leq 2^{T-j}$. Furthermore,

$$
\begin{aligned}
\varepsilon &\geq \frac{2\pi a}{a2^j - 1} - \frac{2\pi}{2^j} \\
&= \frac{2\pi}{2^j - 1/a} - \frac{2\pi}{2^j} \\
&\geq \frac{2\pi}{2^j - 2^{j-T}} - \frac{2\pi}{2^j} \\
&\geq \frac{2\pi}{2^T - 1} - \frac{2\pi}{2^T} \\
&\geq \frac{\pi}{2Q^2}.
\end{aligned}
$$

Hence, by (8) we conclude that

$$
r_0 \geq \frac{b_4}{Q^2}.
$$

Bounding r_0 from above is a far simpler matter. By definition,

$$
r_0 < |\zeta_Q - 1| \leq \frac{2\pi}{Q}. \tag{9}
$$

□

3 Infinite subsets of $\mathbb{N}$ and A

Proposition 3.1. *For $k \leq 0$, and $D_1 \subseteq D_2 \subseteq \mathbb{N}$, we have $p_{D_2}^{(k)}(n) \geq p_{D_1}^{(k)}(n) \geq 0$.*

Proof. This follows immediately from equation 2 and the fact that for $k \leq 0$, the power series expansion for $(1-x)^k$ has nonnegative coefficients. □

For the sake of clarity and the comprehensiveness of this exposition we include the following theorem of Bateman and Erdős [5] suitably adapted to our needs.

Theorem 3.1. *Let $D \subseteq \mathbb{N}$ be an infinite set. For any $t \in \mathbb{N}$, we have that*

$$
\frac{p_D(n)}{p_D^{(-1)}(n)} \leq \frac{1}{t+1} + \frac{(t-1)^2}{t+1} \frac{n^{2t-3}}{p_D^{(-1)}(n)}.
$$

Proof. Denote by $P_q(n)$, the number of partitions of n into parts from D such that there are exactly q distinct parts. $P_q(n)$ has generating function

$$
\sum_{n=0}^{\infty} P_q(n) x^n = \sum_{\{a_1, \ldots a_q\} \subseteq D} \frac{x^{a_1}}{1 - x^{a_1}} \cdots \frac{x^{a_q}}{1 - x^{a_q}}.
$$

If $R_q(n)$ is defined by

$$
\sum_{n=0}^{\infty} R_q(n) x^n = \sum_{\{a_1, \ldots a_q\} \subseteq [n]} \frac{x^{a_1}}{1 - x^{a_1}} \cdots \frac{x^{a_q}}{1 - x^{a_q}},
$$

where $[n] = \{1, \ldots, n\}$, then $P_q(n) \le R_q(n)$.

There are $\binom{n}{q}$ subsets of $[n]$ of size q. Also, the coefficient of x^n in

$$(x^{a_1} + x^{2a_1} + \cdots) \cdots (x^{a_q} + x^{2a_q} + \cdots)$$

is less than or equal to the coefficient of x^n in

$$(x + x^2 + \cdots)^q = \sum_{m=q}^{\infty} \binom{m-1}{q-1} x^m,$$

so

$$P_q(n) \le \binom{n}{q}\binom{n-1}{q-1} \le n^{2q-1}.$$

Any partition $n = n_1 a_1 + \cdots n_q a_q$, where $a_1, \ldots, a_q \in A$, gives rise to a partition of $n - a_i$ for $i = 1, \ldots, q$, namely

$$\begin{aligned} n - a_1 &= (n_1 - 1)a_1 + n_2 a_2 + \cdots + n_q a_q, \\ n - a_2 &= n_1 a_1 + (n_2 - 1)a_2 + \cdots + n_q a_q, \\ &\vdots \\ n - a_q &= n_1 a_1 + n_2 a_2 + \cdots + (n_q - 1)a_q. \end{aligned}$$

Note that no two distinct partitions of n can give rise to the same partition of any $m < n$ in this way, and so

$$\sum_{q=1}^{n} q P_q(n) \le \sum_{m=0}^{n-1} p_D(m).$$

Now if $t \in \mathbb{N}$, then

$$\begin{aligned} p_D^{(-1)}(n) &= \sum_{m=0}^{n} p_D(m) \\ &\ge p_D(n) + \sum_{q=1}^{n} q P_q(n) \\ &= (t+1)p_D(n) + \sum_{q=1}^{n} (q-t) P_q(n) \\ &\ge (t+1)p_D(n) - (t-1)\sum_{q=1}^{t-1} P_q(n) \\ &\ge (t+1)p_D(n) - (t-1)^2 n^{2t-3}, \end{aligned}$$

and the theorem is proved. □

For the remainder of this section, we follow the approach of Bateman and Erdős [5] and simultaneously make the results explicit by applying them to the special case under consideration. To be consistent, we shall assume $k \ge 0$, and use the following notation:

Notation 1. *Let B be the least $k+2$ elements of A, and let $C = A \backslash B$. $B_1 = \{p_{k+3}^{\ell}, p_{k+4}^{\ell}, \ldots, p_{k+2t}^{\ell}\}$ be the least $2t-2$ elements of C, where t is determined as in the statement of Theorem 1.1 from the values of k and ℓ in question. Furthermore, let*

$$g = \left(\frac{2}{b_0}\right)^{k+1} (k+2)^{2\ell(k+1)+1}, \text{ and} \tag{10}$$

$$h = 3(k+2)^{2\ell(k+2)} g = 3\left(\frac{2}{b_0}\right)^{k+1} (k+2)^{4\ell k+6\ell+1}.$$

Finally, let us remark that numerous constants shall be defined in the subsequent argument. Their definitions shall remain consistent throughout.

Note that the right hand side of (10) is increasing in k and ℓ, so $g \geq 16/b_0 = 6.04\ldots$.

Lemma 3.1. *For all $n \geq 0$, we have*

$$p_B^{(k)}(n) \geq 1 - g. \tag{11}$$

Proof. Since $|B| = k+2$, $g_B^{(k)}(n)$ is linear in n, and $g_B^{(k+1)}(n)$ is a constant function. By Lemma 2.1,

$$g_B^{(k)}(n) = \sum_{h=1}^{2} \alpha_h \binom{n+h-1}{h-1}$$

where $\alpha_2 = (p_1 \cdots p_{k+2})^{-\ell}$. Substituting $x = 0$ into (5) and (6) gives

$$\alpha_1 + \alpha_2 + \sum_{q \in B} \sum_{j=1}^{q-1} \beta(\zeta_q^j) = 1.$$

Hence,

$$g_B^{(k)}(n) = (p_1 \cdots p_{k+2})^{-\ell} n + 1 - \sum_{q \in B} \sum_{j=1}^{q-1} \beta(\zeta_q^j),$$

so

$$\begin{aligned}
p_B^{(k)}(n) &= g_B^{(k)}(n) + \psi_B^{(k)}(n) \\
&= (p_1 \cdots p_{k+2})^{-\ell} n + 1 - \sum_{q \in B} \sum_{j=1}^{q-1} (1 - \zeta_q^{jn}) \beta(\zeta_q^j) \\
&\geq (p_1 \cdots p_{k+2})^{-\ell} n + 1 - \sum_{q \in B} \sum_{j=1}^{q-1} |1 - \zeta_q^{jn}| |\beta(\zeta_q^j)| \\
&\geq (p_1 \cdots p_{k+2})^{-\ell} n + 1 - 2 \sum_{q \in B} \sum_{j=1}^{q-1} |\beta(\zeta_q^j)| \\
&\geq (p_1 \cdots p_{k+2})^{-\ell} n + 1 - \left(\frac{2}{b_0}\right)^{k+1} \sum_{q \in B} q^{k+1} \\
&\geq 1 - \left(\frac{2}{b_0}\right)^{k+1} \sum_{q \in B} q^{k+1}.
\end{aligned}$$

Now, it is easy to see that

$$\sum_{q \in B} q^{k+1} \leq Q_0^{k+1}(k+2) \leq (k+2)^{2\ell(k+1)+1},$$

where $Q_0 = \max(B)$. From this, the Lemma follows. □

Lemma 3.2. *For all $n \geq 0$, we have*

$$1 + |p_B^{(k+1)}(n)| < g - 1. \tag{12}$$

Proof. Note that

$$g_B^{(k+1)}(n) = (p_1 \cdots p_{k+2})^{-\ell}.$$

Making use of Theorem 2.1, we see that

$$\begin{aligned}
g - 2 - |p_B^{(k+1)}(n)| \geq & g - 2 - (p_1 \cdots p_{k+2})^{-\ell} - |\psi_B^{(k+1)}(n)| \\
\geq & g - 2 - (p_1 \cdots p_{k+2})^{-\ell} - \frac{2^k}{b_0^{k+1}} \sum_{q \in B} q^{k+1} \\
\geq & \sum_{i=1}^{k+2} \left(\left(\frac{2(k+2)^{2\ell}}{b_0} \right)^{k+1} - \frac{1}{2} \left(\frac{2p_i^\ell}{b_0} \right)^{k+1} \right) \\
& - 2 - (p_1 \cdots p_{k+2})^{-1}.
\end{aligned}$$

Observe that for a fixed i, $1 \leq i \leq k+2$, the expression

$$\left(\frac{2(k+2)^{2\ell}}{b_0} \right)^{k+1} - \frac{1}{2} \left(\frac{2p_i^\ell}{b_0} \right)^{k+1},$$

is positive and increasing in ℓ, for $\ell \geq 1$, so,

$$\begin{aligned}\left(\frac{2(k+2)^{2\ell}}{b_0}\right)^{k+1} - \frac{1}{2}\left(\frac{2p_i^\ell}{b_0}\right)^{k+1} &\geq \left(\frac{2(k+2)^2}{b_0}\right)^{k+1} - \frac{1}{2}\left(\frac{2p_i}{b_0}\right)^{k+1} \\ &\geq \left(\frac{2(k+2)^2}{b_0}\right)^{k+1} - \frac{1}{2}\left(\frac{2p_{k+2}}{b_0}\right)^{k+1} \\ &\geq \frac{2(k+2)^2}{b_0} - \frac{p_{k+2}}{b_0} \\ &\geq \frac{4}{b_0} + \frac{(k+2)^2 - p_{k+2}}{b_0} \\ &\geq \frac{4}{b_0}.\end{aligned}$$

Hence,

$$\begin{aligned}g - 2 - |p_B^{(k+1)}(n)| &\geq (k+2)\frac{4}{b_0} - 2 - (p_1 \cdots p_{k+2})^{-1} \\ &\geq \frac{8}{b_0} - \frac{13}{6} = 0.855167405\ldots > 0,\end{aligned}$$

that is,

$$1 + |p_B^{(k+1)}(n)| < g - 1.$$

□

Corollary 3.1.

$$\frac{p_B^{(k)}(n)}{1 + |p_B^{(k+1)}(n)|} \geq 2 \text{ if } n \geq h. \tag{13}$$

Proof. It follows from the proof of Lemma 3.1 that

$$\begin{aligned}p_B^{(k)}(n) &\geq (p_1 \cdots p_{k+2})^{-\ell} n + 1 - g \\ &> (p_1 \cdots p_{k+2})^{-\ell} n - g.\end{aligned}$$

The Corollary follows from this, together with Lemma 3.2 and the fact that

$$p_1 \cdots p_{k+2} \leq (k+2)^{2(k+2)}.$$

□

Lemma 3.3. *There is an $h_1 \in \mathbb{N}$ such that such that $n \geq h_1$ implies*

$$\frac{(t-1)^2}{t+1}\frac{n^{2t-3}}{p_C^{(-1)}(n)} \leq \frac{(t-1)^2}{t+1}\frac{n^{2t-3}}{p_{B_1}^{(-1)}(n)} \leq \frac{1}{t+1}. \tag{14}$$

Proof. The first inequality is a consequence of Proposition 3.1. Note that

$$\begin{aligned} t &\geq 6\left(\frac{2}{b_0}\right)^{3(k+1)}(k+2)^{8\ell k+10\ell+3}-1 \\ &\geq 6\left(\frac{2}{b_0}\right)^{3k+3}2^{8k+13}-1 \\ &\geq \frac{3\cdot 2^{16}}{b_0^3}\left(\frac{2^{11}}{b_0^3}\right)^{k}. \end{aligned}$$

Hence

$$\log t \geq \log\left(\frac{3\cdot 2^{16}}{b_0^3}\right)+k\log\left(\frac{2^{11}}{b_0^3}\right).$$

since $\log t \leq t/e$, if we let $M = e\log(2^{11}/b_0^3)$, then

$$k \leq \frac{t}{M}.$$

Choose r_0 and d_{B_1} with respect to the set B_1 as in Proposition 2.1, and let $Q = p_{k+2t}^{\ell} = \max(B_1)$. It is clear that $t \geq \lfloor 6(2/b_0)^3 2^{13}\rfloor = 21192$, which we denote by t_0. By equation (9), we have that

$$r_0 \leq \frac{2\pi}{Q} \leq \frac{2\pi}{(2t)^{\ell}} \leq \frac{\pi}{t_0}.$$

We also have that

$$r_0 \geq \frac{b_4}{Q^2} \geq \frac{b_4}{(k+2t)^{4\ell}},$$

and

$$\begin{aligned} d_{B_1} &\geq b_1^{\sum_{q\in B_1} q}\left(b_2 Q^{b_3} e^{-\frac{\log^2 Q}{\log 2}}\right)^{2t-2} \\ &\geq b_1^{(k+3)^{\ell}(2t-2)}\left(b_2(k+2t)^{b_3\ell}(k+2t)^{-\frac{2\ell\log(k+2t)}{\log 2}}\right)^{2t-2} \\ &= \left(\frac{b_2\cdot b_1^{(k+3)^{\ell}}(k+2t)^{b_3\ell}}{(k+2t)^{\frac{2\ell\log(k+2t)}{\log 2}}}\right)^{2t-2} \end{aligned}$$

Let us now bound $p_{B_1}^{(-1)}(n)$ from below. Assume that $n \geq 2t$, and let $b_5 = 1/M + 2 = 2.078207555\ldots$, and $b_7 = t_0/(t_0-\pi)$. Making use of Theorem 2.1, we have

$$
\begin{aligned}
p_{B_1}^{(-1)}(n) =& g_{B_1}^{(-1)}(n) + \psi_{B_1}^{(-1)}(n) \\
\geq& \alpha_{2t-1}\binom{n+2t-2}{2t-2} - \sum_{h=1}^{2t-2}|\alpha_h|\binom{n+h-1}{h-1} \\
&- |\psi_{B_1}^{(-1)}(n)| \\
\geq& \frac{(p_{k+3}\cdots p_{k+2t})^{-\ell}n^{2t-2}}{(2t-2)!} - \binom{n+2t-3}{2t-3}\sum_{h=1}^{2t-2}|\alpha_h| \\
&- \frac{1}{b_0^{2t-2}}\sum_{m=k+3}^{k+2t} p_m^{\ell(2t-2)} \\
\geq& \frac{(k+2t)^{-\ell(2t-2)}n^{2t-2}}{(2t-2)!} - \frac{(n+2t-3)^{2t-3}}{(2t-3)!}\sum_{h=1}^{2t-2}\frac{r_0^h}{r_0^{2t-1}d_{B_1}} \\
&- \frac{1}{b_0^{2t-2}}(2t-2)(k+2t)^{2\ell(2t-2)} \\
\geq& \frac{(b_5 t)^{-\ell(2t-2)}n^{2t-2}}{(2t-2)!} - \frac{n^{2t-3}2^{2t-3}}{(2t-3)!}\frac{1}{(1-r_0)r_0^{2t-2}d_{B_1}} \\
&- \frac{(2t-2)(k+2t)^{2\ell(2t-2)}}{b_0^{2t-2}} \\
\geq& \frac{(b_5 t)^{-\ell(2t-2)}n^{2t-2}}{(2t-2)!} - \frac{b_7 n^{2t-3}2^{2t-3}}{(2t-3)!}\left(\frac{(k+2t)^{(4-b_3)\ell+\frac{2\ell\log(k+2t)}{\log 2}}}{b_4 b_2 b_1^{(k+3)^\ell}}\right)^{2t-2} \\
&- \frac{2t(b_5 t)^{2\ell(2t-2)}}{b_0^{2t-2}}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&(k+2t)^{(4-b_3)\ell+\frac{2\ell\log(k+2t)}{\log 2}} \\
&\qquad \leq (b_5 t)^{(4-b_3)\ell+\frac{2\ell\log Ct}{\log 2}} \\
&\qquad = e^{\ell(\log b_5+\log t)\left(\left(4-b_3+\frac{2\log C}{\log 2}\right)+\frac{2\log t}{\log 2}\right)} \\
&\qquad = e^{\ell\left(\frac{2}{\log 2}\log^2 t+\left(4-b_3+\frac{4\log b_5}{\log 2}\right)\log t+\left(4-b_3+\frac{4\log b_5}{\log 2}\right)\log b_5\right)} \\
&\qquad \leq e^{b_6\ell\log^2 t},
\end{aligned}
$$

where b_6 is a constant determined as follows. Let $x_0 = \log t_0$. Then we may take

$$
\begin{aligned}
b_6 &= \frac{2}{\log 2} + \left(4 - b_3 + \frac{4\log b_5}{\log 2}\right)\frac{1}{x_0} + \left(4 - b_3 + \frac{4\log b_5}{\log 2}\right)\frac{\log b_5}{x_0^2} \\
&= 3.523150893\ldots.
\end{aligned}
$$

It suffices to select h_1 such that for $n \geq h_1$,

$$p_{B_1}^{(-1)}(n) n^{-(2t-3)} \geq (t-1)^2. \tag{15}$$

Observe that (15) is implied by

$$\frac{(b_5 t)^{-\ell(2t-2)} n}{(2t-2)!} - \frac{b_7}{2(2t-3)!} \left(\frac{2e^{b_6 \ell \log^2 t}}{b_4 b_2 b_1^{(k+3)^\ell}} \right)^{2t-2} - \frac{2t(b_5 t)^{2\ell(2t-2)}}{n^{2t-3} b_0^{2t-2}} \geq (t-1)^2,$$

which is equivalent to

$$n \geq (b_5 t)^{\ell(2t-2)} \times \left(b_7 (t-1) \left(\frac{2e^{b_6 \ell \log^2 t}}{b_4 b_2 b_1^{(k+3)^\ell}} \right)^{2t-2} + \frac{2t(b_5 t)^{2\ell(2t-2)} (2t-2)!}{n^{2t-3} b_0^{2t-2}} + (t-1)^2 (2t-2)! \right). \tag{16}$$

The inequality

$$e^n n! \leq n^{n+1},$$

holds for $n \geq 7$. This implies that

$$\frac{(2t-2)!}{(2t)^{2t-3}} \leq \frac{(2t)^3}{e^{2t}(2t-1)}.$$

Thus, since $n \geq 2t$ by assumption, (16) is implied by

$$n \geq (b_5 t)^{\ell(2t-2)} (A_1 + A_2 + A_3),$$

where

$$A_1 = b_7 t \left(\frac{2e^{b_6 \ell \log^2 t}}{b_4 b_2 b_1^{(k+3)^\ell}} \right)^{2t-2}$$
$$A_2 = \frac{16 t^4 (b_5 t)^{2\ell(2t-2)}}{(2t-1) e^{2t} b_0^{2t-2}}$$
$$A_3 = \frac{(t-1)^2 (2t)^{2t}}{(2t-1) e^{2t}}.$$

The term A_3 is negligible relative to A_2, yet A_1 and A_2 are not easily compared since one or the other may dominate depending on the values chosen for k, and ℓ. None the less, we may simplify matters a little by absorbing A_3 into A_2 in the following way:

$$\frac{A_2 + A_3}{t^3(b_5 t)^{2\ell(2t-2)}/(e^{2t}b_0^{2t-2})} \leq \frac{16t}{2t-1} + \frac{b_0^{2t-2}(2t)^{2t}}{t(2t-1)(b_5 t)^{2(2t-2)}}$$
$$= \frac{16t}{2t-1} + \left(\frac{4t}{2t-1}\right)\left(\frac{2b_0}{b_5^2 t}\right)^{2t-2}$$
$$\leq \frac{16t_0}{2t_0-1} + \left(\frac{4t_0}{2t_0-1}\right)\left(\frac{2b_0}{b_5^2 t_0}\right)^{2t_0-2}$$
$$\leq 8.000188756.$$

So, let

$$A_2' = 8.000188756\frac{t^3(b_5 t)^{2\ell(2t-2)}}{e^{2t}b_0^{2t-2}}.$$

Then we may take

$$h_1 = (b_5 t)^{\ell(2t-2)}(A_1 + A_2').$$

□

Remark 3. *Note that*

$$t = \left\lfloor 6\left(\frac{2}{b_0}\right)^{3(k+1)}(k+2)^{8\ell k+10\ell+3}\right\rfloor = \lfloor 2g^2 h\rfloor,$$

and so

$$\frac{1}{t+1} \leq \frac{1}{2(g+1)(g-1)h}.$$

Lemma 3.4. *There exists an $N = N(k,\ell) > 0$, such that if $n \geq N$, then*

$$p^{(k)}(n) \geq p_C^{(-1)}(n) > 0.$$

Proof. The second inequality is obvious. For the first, by Proposition 3.1, Theorem 3.1, Lemma 3.3 and Remark 3, we have that for $n \geq h_1$,

$$\frac{p_C(n)}{p_C^{(-1)}(n)} \leq \frac{1}{t+1} + \frac{(t-1)^2}{t+1}\frac{1}{p_{B_1}^{(-1)}(n)n^{-(2t-3)}} \leq \frac{1}{(g+1)(g-1)h}. \tag{17}$$

Now, using, (11), (12), (13), (17), and the identity

$$p^{(k)}(n) = \sum_{m=0}^{n} p_B^{(k)}(n-m)p_C(m),$$

we have that for $n \geq h + h_1 - 1$,

$$
\begin{aligned}
p^{(k)}(n) \geq & 2 \sum_{0\leq m\leq n-h} (1+|p_B^{(k+1)}(n-m)|)p_C(m) \\
& - (g-1) \sum_{n-h<m\leq n} (1+|p_B^{(k+1)}(n-m)|)p_C(m) \\
\geq & 2\sum_{m=0}^{n} (1+|p_B^{(k+1)}(n-m)|)p_C(m) - (g+1)(g-1) \sum_{n-h<m\leq n} p_C(m) \\
\geq & 2\sum_{m=0}^{n} (1+|p_B^{(k+1)}(n-m)|)p_C(m) \\
& - (g^2-1) \left(\sum_{n-h<m\leq n}^{n} \frac{p_C(m)}{p_C^{(-1)}(m)} \right) p_C^{(-1)}(n) \\
\geq & \left(2 - (g^2-1) \sum_{n-h<m\leq n} \frac{p_C(m)}{p_C^{(-1)}(m)} \right) \times \sum_{m=0}^{n} (1+|p_B^{(k+1)}(n-m)|)p_C(m) \\
\geq & \sum_{m=0}^{n} (1+|p_B^{(k+1)}(n-m)|)p_C(m) \\
\geq & \sum_{m=0}^{n} p_C(m) \\
= & p_C^{(-1)}(n).
\end{aligned}
$$

□

Proof of Theorem 1.1.

By the proofs of Lemmas 3.3 and 3.4, it suffices us to choose $N \geq h+h_1 = h+(b_5 t)^{\ell(2t-2)}(A_1+A_2')$. First observe that $A_2' \geq 2t$. The quantity $(b_5 t)^{\ell(2t-2)}A_1$ may be simplified further with an upper bound. Let

$$
b_9 = \frac{\log b_5}{x_0^2} + \frac{1}{x_0} + b_6 = 3.630910490\ldots,
$$

and

$$
b_{10} = \frac{2}{b_2 b_4}.
$$

Then

$$
\begin{aligned}
(b_5t)^{\ell(2t-2)}A_1 &= b_7t\left(\frac{b_{10}(b_5t)^{\ell}e^{b_6\ell\log^2 t}}{b_1^{(k+3)\ell}}\right)^{2t-2}\\
&= b_7t\left(\frac{b_{10}e^{\ell(\log b_5+\log t+b_6\log^2 t)}}{b_1^{(k+3)\ell}}\right)^{2t-2}\\
&\le b_7t\left(\frac{b_{10}e^{b_9\ell\log^2 t}}{b_1^{(k+3)\ell}}\right)^{2t-2}
\end{aligned}
$$

Denote

$$
A_1' = b_7t\left(\frac{b_{10}e^{b_9\ell\log^2 t}}{b_1^{(k+3)\ell}}\right)^{2t-2}.
$$

We will use the fact that $h \le t$ to absorb h into $(b_5t)^{\ell(2t-2)}A_2'$ in the following way:

$$
\begin{aligned}
\frac{h+(b_5t)^{\ell(2t-2)}A_2'}{t^3(b_5t)^{3\ell(2t-2)}/(e^{2t}b_0^{2t-2})} &\le \frac{e^{2t}b_0^{2t-2}}{t^2(b_5t)^{3(2t-2)}} + 8.000188756\\
&= \frac{e^2}{t^2}\left(\frac{eb_0}{b_5^3t^3}\right)^{2t-2} + 8.000188756\\
&\le \frac{e^2}{t_0^2}\left(\frac{eb_0}{b_5^3t_0^3}\right)^{2t_0-2} + 8.000188756\\
&\le 8.000188757
\end{aligned}
$$

Letting $b_8 = 8.0002$, we may take

$$
\begin{aligned}
N &= A_1' + \frac{b_8t^3(b_5t)^{3\ell(2t-2)}}{e^{2t}b_0^{2t-2}}\\
&= b_7t\left(\frac{b_{10}e^{b_9\ell\log^2 t}}{b_1^{(k+3)\ell}}\right)^{2t-2} + \frac{b_8t^3(b_5t)^{3\ell(2t-2)}}{e^{2t}b_0^{2t-2}}.
\end{aligned}
$$

We conclude the proof by restructuring the constants as follows:

$$
\begin{aligned}
a_1 &= b_7\\
a_2 &= b_{10}^2\\
a_3 &= e^{2b_9}\\
a_4 &= b_1^2\\
a_5 &= e^{-2}b_8\\
a_6 &= e^{-2}b_0^{-2}\\
a_7 &= b_5.
\end{aligned}
$$

□

4 Asymptotic results

Observe that as $k \to \infty$, $\log t \asymp k \log k$, when ℓ remains fixed. This implies that if $\ell > 2$, then

$$\frac{a_3^{\ell \log^2 t}}{a_4^{(k+2)^\ell}} \to 0, \text{ as } k \to \infty,$$

and so in this situation, the second term dominates in the expression for $N(k, \ell)$ in Theorem 1.1.

If $\ell = 1$ or 2, then

$$\frac{a_3^{\ell \log^2 t}}{a_4^{(k+2)^\ell}} = e^{\lambda_1(k)},$$

where $\lambda_1(k) \asymp k^2 \log^2 k$, and

$$(a_7 t)^{6\ell} = e^{\lambda_2(k)},$$

where $\lambda_2(k) \asymp k \log k$. In this case the first term dominates in the expression for $N(k, \ell)$. Thus we have proved the following corollary to Theorem 1.1:

Corollary 4.1. *Let $\ell \in \mathbb{N}$ be fixed. Then as $k \to \infty$,*

$$F(k, \ell) = O\left(t \left(\frac{a_2 a_3^{\ell \log^2 t}}{a_4^{(k+3)^\ell}} \right)^{t-1} \right), \text{ if } \ell = 1, 2;$$
$$F(k, \ell) = O\left(t^3 \left(a_6 (a_7 t)^{6\ell} \right)^{t-1} \right), \text{ if } \ell > 2.$$

Theorem 1.2 follows from Corollary 4.1 simply by taking logarithms.

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