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# On Fibonacci-Like Sequences

Le Anh Vinh
School of Mathematics
University of New South Wales
Sydney 2052 NSW
Australia
vinh@maths.unsw.edu.au

**Abstract**

In this note, we study Fibonacci-like sequences that are defined by the recurrence $S_k = a$, $S_{k+1} = b$, $S_{n+2} \equiv S_{n+1} + S_n \pmod{n+2}$ for all $n \geq k$, where $k, a, b \in \mathbb{N}$, $0 \leq a < k$, $0 \leq b < k+1$, and $(a, b) \neq (0, 0)$. We will show that the number $\alpha = 0.S_k S_{k+1} S_{k+2} \cdots$ is irrational. We also propose a conjecture on the pattern of the sequence $\{S_n\}_{n \geq k}$.

## 1 Introduction

Given a sequence of natural numbers $a_1, a_2, \ldots$, the question of determining the irrationality of the number $\alpha = 0.a_1 a_2 \cdots$ is a classical and interesting question. For example, if $a_1, a_2, \ldots$ is the sequence of all prime numbers, then $\alpha$ is irrational ([5]). Another well-known example is the set of generalized Mahler sequences. Let $m \geq 1$, $h \geq 2$ be integers, and

$$(m)_h = m_1 h^{r-1} + m_2 h^{r-2} + \cdots + m_r$$

for some integer $r > 0$ and $0 \leq m_i < h$ for all $1 \leq i \leq r$. Mahler [6] showed that for $t \geq 2$ then the number

$$a(t) = 0.(t^0)_{10}(t^1)_{10}(t^2)_{10} \cdots$$

is irrational. Bundschuh [4] generalized this result to arbitrary bases. More precisely, he showed that for any $t, r \geq 2$ then the number

$$a_r(t) = 0.(t^0)_r(t^1)_r(t^2)_r \cdots$$

is irrational. Readers can find several proofs of this result in [7, 9]. In the most general form, one studies the number

$$a_r(t) = a_r^{(n_i)}(t) = 0.(t^{n_0})_r(t^{n_1})_r(t^{n_2})_r \cdots$$

for given $r, t \geq 2$ and sequence $(n_i)_{i\geq 0}$ of non-negative integers. In [10], Shan and Wang showed that $a_r(t)$ is irrational if $(n_i)$ is an unbounded sequence. Several criteria for irrationality of $a_r(t)$ for bounded $(n_i)$ were obtained by Sander [8], and Shorey and Tijdeman [11]. Motivated by these papers, we will study an analogous result for some Fibonacci-like sequences.

Recall that the Fibonacci sequence is defined by the following recurrence:

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n \ \text{ for all }\ n \geq 0.$$

In this note, we will study some properties of Fibonacci-like sequences that are defined by the following recurrence:

$$S_k = a,\ S_{k+1} = b,\ S_{n+2} \equiv S_{n+1} + S_n \,(\operatorname{mod} n+2\,) \qquad \text{for all} \qquad n \geq k, \tag{1}$$

for some $k, a, b \in \mathbb{N}$, $0 \leq a < k$ and $0 \leq b < k+1$. For any triple $(k, a, b) \in \mathbb{N}^3$ with $0 \leq a < k$ and $0 \leq b < k+1$, we denote $S^k_{a,b} = \{S^k_{a,b}(n)\}_{n=k}^{\infty}$ the sequence defined by recurrence (1).

The main result of this note is the following theorem.

**Theorem 1.** *Suppose that $a, b, k$ are natural numbers with $0 \leq a < k, 0 \leq b < k+1$. Then*

$$\alpha^k_{a,b} = 0.S^k_{a,b}(k)S^k_{a,b}(k+1)S^k_{a,b}(k+2)\ldots \tag{2}$$

*is irrational. Here expression (2) means that the decimal expansion of $\alpha^k_{a,b}$ is obtained by the concatenation of the integers $S^k_{a,b}(n)$ written in decimal form.*

It is worth noticing that most of papers inspired by Mahler deal with exponentially increasing sequences, while $S^k_{a,b}$ is always less than $n$. Furthermore, while the Fibonacci sequence is well-known and has been studied extensively in the literature, it seems that the sequence $S^k_{a,b}$ has not been studied before. The only reference we found about these sequences refers to $S^0_{0,1}$. This sequence is known as sequence A056542 in Sloane's Online Encyclopedia of Integer Sequences [12].

## 2 Irrationality

In order to give a proof for Theorem 1, we first need some lemmas.

**Lemma 2.** *Suppose that $a, b, k \in \mathbb{N}$ such that $0 \leq a < k, 0 \leq b < k+1$, $(a, b) \neq (0, 0)$. Then the sequence $S^k_{a,b}$ is not bounded.*

*Proof.* Suppose that $S^k_{a,b}$ is bounded for some $a, b, k$. Let $M = \max_{n\geq k}\{S^k_{a,b}(n)\}$. Then, for every $n > 2M$, we have $S^k_{a,b}(n) = S^k_{a,b}(n-1) + S^k_{a,b}(n-2)$, since $S^k_{a,b}(n-1) \leq M$ and $S^k_{a,b}(n-2) \leq M$. Thus, the sequence $S^k_{a,b}$ eventually coincides with a usual linear recurrence sequence taking non-negative values. Since $S^k_{a,b}$ is bounded it immediately follows that

$$S^k_{a,b}(n-1) = S^k_{a,b}(n-2) = 0.$$

By backward induction, we have $S^k_{a,b}(n) = 0$ for all $n$, which is a contradiction. This concludes the proof of the lemma. □

**Lemma 3.** *For any sufficiently large $m$, there exists $n$ such that $S^k_{a,b}(n)$ has exactly $m$ digits. In other words, there exists $n$ such that $10^{m-1} \leq S^k_{a,b}(n) < 10^m$.*

*Proof.* From Lemma 2, the sequence $\{S^k_{a,b}(n)\}_{n\geq k}$ is unbounded. Hence there exists $n$ such that $S^k_{a,b}(n) \geq 10^{m-1}$. We choose $n$ as small as possible. Then $S^k_{a,b}(n-1), S^k_{a,b}(n-2) < 10^{m-1}$. This implies that

$$S^k_{a,b}(n) \leq S^k_{a,b}(n-1) + S^k_{a,b}(n-2) < 2 \times 10^{m-1} < 10^m.$$

This concludes the proof of the lemma. □

Using Lemma 2 and Lemma 3, we get the following proof of Theorem 1.

*Proof.* (of Theorem 1) Suppose that $\alpha^k_{a,b}$ is a rational number for some $a, b, k$. Then it has an eventually periodic decimal expansion. Thus we can write

$$\alpha^k_{a,b} = 0.a_1 \dots a_s b_1 \dots b_t b_1 \dots b_t \dots$$

We choose $n$ large enough such that $S^k_{a,b}(n)$ starts from a position after $a_s$. Then for any $r \geq n$, the number $\alpha_r = S^k_{a,b}(r)S^k_{a,b}(r+1)S^k_{a,b}(r+2)\dots$ is periodic of period $wt$ for any positive integer $w$. We choose $m = vt$ for some large positive integer $v$ such that $10^{m-1} > S^k_{a,b}(i)$ for all $i \leq n$. From Lemma 3, there exists $l$ such that $S^k_{a,b}(l)$ has exactly $m$ digits. We choose $l$ to be as small as possible; then $l > n$.

If $S^k_{a,b}(l-1) = 0$, then $S^k_{a,b}(l-2) = S^k_{a,b}(l)$ has exactly $m$ digits, which is a contradiction. Hence $0 < S^k_{a,b}(l-1) < 10^{m-1}$. Similarly, we have $0 < S^k_{a,b}(l-2) < 10^{m-1}$. Hence

$$S^k_{a,b}(l) \leq S^k_{a,b}(l-2) + S^k_{a,b}(l-1) < 2 \times 10^{m-1},$$
$$S^k_{a,b}(l+1) \leq S^k_{a,b}(l-1) + S^k_{a,b}(l) < 3 \times 10^{m-1}.$$

Therefore, $S^k_{a,b}(l+1)$ has no more than $m$ digits. We have two separate cases.

1. Suppose that $S^k_{a,b}(l+1) \equiv S^k_{a,b}(l-1) + S^k_{a,b}(l) \pmod{l+1}$ has $m$ digits. But $\alpha_l = S^k_{a,b}(l)S^k_{a,b}(l+1)S^k_{a,b}(l+2)\dots$ is periodic of period $m = vt$ so $S^k_{a,b}(l+1) = S^k_{a,b}(l)$. This implies that $S^k_{a,b}(l-1) = 0$ which is a contradiction.

2. Suppose that $S^k_{a,b}(l+1) \equiv S^k_{a,b}(l-1) + S^k_{a,b}(l) \pmod{l+1}$ has less than $m$ digits. Let $p = S^k_{a,b}(l+1)$. Since $\alpha_l = S^k_{a,b}(l)S^k_{a,b}(l+1)S^k_{a,b}(l+2)\dots$ is periodic of period $m = vt$ so $S^k_{a,b}(l) = p * q$ for some $q$ where $p * q$ denotes the concatenation of $p$ and $q$. We have

   $$S^k_{a,b}(l+2) \leq S^k_{a,b}(l+1) + S^k_{a,b}(l) < 10^{m-1} + 2 \times 10^{m-1} < 3 \times 10^{m-1}.$$

   So $S^k_{a,b}(l+2)$ has no more than $m$ digits. We have two subcases.

   (a) Suppose that $S^k_{a,b}(l+2)$ has exactly $m$ digits. Then by the periodicity of $\alpha_l$ we have $S^k_{a,b}(l)S^k_{a,b}(l+1)S^k_{a,b}(l+2) = p * q * p * q * p$. If $S^k_{a,b}(l) + S^k_{a,b}(l+1) \geq l+2$ then

   $$S^k_{a,b}(l) + S^k_{a,b}(l+1) < l + 10^{m-1} < l + 2 + 10^{m-1},$$

which implies that $S^k_{a,b}(l+2) < 10^{m-1}$ which is a contradiction. Hence

$$S^k_{a,b}(l) + S^k_{a,b}(l+1) < l+2.$$

This implies that $q * p = S^k_{a,b}(l+2) = S^k_{a,b}(l) + S^k_{a,b}(l+1) = p * q + p$. Suppose that $p * q = 10^h p + q$ and $q * p = 10^z q + p$. Then $q(10^z - 1) = 10^h p$. Thus, $10^h \mid q$. But $p * q = 10^h p + q$ so $q < 10^h$. Hence $q = 0$ and $p = 0$ which is a contradiction.

(b) Suppose that $S^k_{a,b}(l+2)$ has less than $m$ digits. Then we can replace $k$ by $l+1$. And we choose $l'$ to be the smallest $l' > l$ such that $S^l_{a,b}(l')$ has exactly $m$ digits. Apply the above argument for the new sequence $S^l_{a,b}$ until either we come up with a contradiction or we can choose $l'$ large enough such that $l' + 1 > 3 \times 10^{m-1}$. But in this case

$$S^k_{a,b}(l'+1) \leq S^k_{a,b}(l'-1) + S^k_{a,b}(l') < 3 \times 10^{m-1} < l' + 1.$$

So $S^k_{a+b}(l'+1)$ has exactly $m$ digits. And we go to the case 1 which implies a contradiction.

This concludes the proof of the theorem. □

We close this section by an open question.

**Open Problem 1.** *For $a, b, k$ are natural numbers with $0 \leq a < k, 0 \leq b < k+1$. Is $\alpha^k_{a,b}$ an algebraic or transcendental number?*

# 3 Occurrence of zeros

By examining several sequences for small values of $a, b$ and $k$, we notice a curious property of the sequence $S^k_{a,b}$: this sequence always contains many zeros. We are unable to prove this statement. Precisely, we propose the following conjecture.

**Conjecture 4.** *Let $a, b, k$ be natural numbers with $0 \leq a < k, 0 \leq b < k+1$. Then the sequence $S^k_{a,b}$ contains infinitely many zero elements.*

Suppose that the sequence $S^k_{a,b}$ contains only finitely many zero elements for some $a, b, k$. Let $v$ be the largest index such that $S^k_{a,b}(v) = 0$. Let $c = S^k_{a,b}(v+1)$ and $d = S^k_{a,b}(v+2)$. Then the sequence $S^{v+1}_{c,d}$ contains no zero element. Therefore the conjecture is equivalent to the statement "there exists $n$ such that $S^k_{a,b}(n) = 0$ for any $a, b, k$".

If Conjecture 4 holds, let $v_k(a,b)$ be the index of the first zero element in sequence $S^k_{a,b}$. We define

$$v_k = \max_{0 \leq a < k, 0 \leq b < k+1} v_k(a,b).$$

For any $0 \leq a < k$ and $0 \leq b < k+1$ then $S^k_{a,b} = \{a\} \cup S^{k+1}_{b,c}$ for some $0 \leq c < k+2$. Thus, $v_k \leq v_{k+1}$ for any $k$. Furthermore, $v_{v_k+1} \geq v_k + 1 > v_k$ for any $k$. Hence

$$\lim_{k \to \infty} v_k = \infty.$$

Using computer, we computed some values of the sequence $\{v_k\}_{k \in \mathbb{N}}$

$$\{v_k\}_{k \geq 1} = \{28, 28, 108, 108, 130, 130, 184, 184, 184, 1523, 1523, \ldots\}.$$

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## References

[1] M. Amou, Approximation to certain transcendental decimal fractions by algebraic numbers, *J. Number Theory* **37** (1991), 231–241.

[2] P. G. Becker, Exponential diophantine equations and the irrationality of certain real numbers, *J. Number Theory* **39** (1991), 108–116.

[3] P. G. Becker and J. W. Sander, Irrationality and codes, *Semigroup Forum* **51** (1995), 117–124.

[4] P. Bundschuh, Generalization of a recent irrationality result of Mahler, *J. Number Theory* **19** (1984), 248–253.

[5] N. Hegyvari, On some irrational decimal fractions, *Amer. Math. Monthly* **100** (1993), 779–780.

[6] K. Mahler, On some irrational decimal fractions, *J. Number Theory* **13** (1981), 268–269.

[7] H. Niederreiter, On an irrationality theory of Mahler and Bundschuh, *J. Number Theory* **24** (1986), 197–199.

[8] J. W. Sander, Irrationality criteria for Mahler's numbers, *J. Number Theory* **52** (1995), 145–156.

[9] Z. Shan, A note on irrationality of some numbers, *J. Number Theory* **25** (1987), 211–212.

[10] Z. Shan and E. T. H. Wang, Generalization of a theorem of Mahler, *J. Number Theory* **32** (1989), 111–113.

[11] T. N. Shorey and R. Tijdeman, Irrationality criteria for numbers of Mahler's type, in Y. Motohashi, ed., *Analytic Number Theory*, London Mathematical Lecture Note Series **247**, Cambridge, 1997, pp. 343–351.

[12] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://www.research.att.com/~njas.

[13] T. Vijayaraghavan, On the irrationality of a certain decimal, *Proc. Indian Acad. Sci. Sect. A* **10** (1939), 341.

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(Concerned with sequence A056542.)

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