Journal of Integer Sequences, Vol. 10 (2007),
Article 07.8.5

Quasi-Fibonacci Numbers of Order 11

Roman Wituła and Damian Słota
Institute of Mathematics
Silesian University of Technology
Kaszubska 23,
Gliwice 44-100
POLAND
r.witula@polsl.pl
d.slota@polsl.pl

Abstract

In this paper we introduce and investigate the so-called quasi-Fibonacci numbers of order 11. These numbers are defined by five conjugate recurrence equations of order five. We study some relations and identities concerning these numbers. We present some applications to the decomposition of some polynomials. Many of the identities presented here are the generalizations of the identities characteristic for general recurrence sequences of order three given by Rabinowitz.

1 Introduction

Wituła, Słota and Warzyński [9] analyzed the relationships between the so-called quasi-Fibonacci numbers of order 7. In this paper we are focused on the generalization of the identities and some facts derived in [9] to the quasi-Fibonacci numbers of order 11 $A_n(\delta)$, $B_n(\delta)$, $C_n(\delta)$, $D_n(\delta)$, and $E_n(\delta)$, $n \in \mathbb{N}$, $\delta \in \mathbb{C}$, defined by the relations (see also Section 3 in this paper for more details):

$$\big(1+\delta\,(\xi^m+\xi^{10m})\big)^n = A_n(\delta) + B_n(\delta)\,(\xi^m+\xi^{10m}) + C_n(\delta)\,(\xi^{2m}+\xi^{9m}) + \\ + D_n(\delta)\,(\xi^{3m}+\xi^{8m}) + E_n(\delta)\,(\xi^{4m}+\xi^{7m})$$

for $m, n \in \mathbb{N}$, $m \le 5$, $\delta \in \mathbb{C}$, where $\xi \in \mathbb{C}$ is a primitive root of unity of order 11.

Additionally, the following important auxiliary sequence is considered:

$$\mathcal{A}_n(\delta) := 5A_n(\delta) - B_n(\delta) - C_n(\delta) - D_n(\delta) - E_n(\delta).$$

Moreover, in order to make shorter expressions and formulas we introduce the following notation:

$$k_m := \cos\left(\frac{m\,\pi}{11}\right), \qquad s_m := \sin\left(\frac{m\,\pi}{11}\right), \qquad \xi := \exp\left(\frac{i\,2\,\pi}{11}\right),$$

$$\sigma_m := \xi^m + \xi^{-m} \qquad \text{and} \qquad \tau_m(\delta) := 1 + \delta\,\sigma_m$$

for $m \in \mathbb{Z}$ and $\delta \in \mathbb{C}$. We note that

$$\sigma_m = 2\,k_{2m} \qquad \text{and} \qquad \tau_m(\delta) = 1 + 2\,\delta\,k_{2m}$$

and

$$\sigma_m\,\sigma_n = \sigma_{m+n} + \sigma_{|m-n|}, \tag{1.1}$$

$$\sigma_{m\,n} = \sigma_{(11-m)n}, \tag{1.2}$$

$$\sigma_{m\,n} = \sigma_{|m|n}, \tag{1.3}$$

$$1 + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 = 0. \tag{1.4}$$

Moreover we have

$$\tau_m\big(\tfrac{1}{2}\big) = 2\,k_m^2, \qquad \tau_m\big(-\tfrac{1}{2}\big) = 2\,s_m^2, \tag{1.5}$$

and

$$\tau_m\left(\frac{T_r(k_{2s})}{2k_{2m}}\right) = \tau_m\left(\frac{T_{2r}(k_s)}{2k_{2m}}\right) = 1 + k_{2rs} = 2\,k_{rs}^2, \tag{1.6}$$

where $T_r(x)$ denote the r-th Chebyshev polynomial of the first kind. For example, from the last formula we get

$$\tau_m\big(2\,k_{2m}^2 - \tfrac{3}{2}\big) = \tau_m\left(\frac{T_3(k_{2m})}{2k_{2m}}\right) = 2\,k_{3m}^2, \tag{1.7}$$

$$\tau_m\big(8\,k_{2m}^4 - 10\,k_{2m}^2 + \tfrac{5}{2}\big) = \tau_m\left(\frac{T_5(k_{2m})}{2k_{2m}}\right) = 2\,k_{5m}^2, \tag{1.8}$$

for every $m \in \mathbb{N}$.

2 Minimal polynomials, linear independence over $\mathbb{Q}$

Let $\Psi_n(x)$ be the minimal polynomial of $\cos(2\pi/n)$ for every $n \in \mathbb{N}$. W. Watkins and J. Zeitlin described in [8] the following identities

$$T_{s+1}(x) - T_s(x) = 2^s \prod_{d|n} \Psi_d(x)$$

if $n = 2s + 1$ and

$$T_{s+1}(x) - T_{s-1}(x) = 2^s \prod_{d|n} \Psi_d(x)$$

if $n = 2s$. In the sequel if $n = 2s + 1$ is a prime number, then we get

$$T_{s+1}(x) - T_s(x) = 2^s \Psi_1(x)\Psi_n(x),$$

which for example implies the formula

$$\begin{aligned}\Psi_{11}(x) &= \frac{1}{32(x-1)}(T_6(x) - T_5(x)) \\ &= \frac{1}{32(x-1)}(32x^6 - 16x^5 - 48x^4 + 20x^3 + 18x^2 - 5x - 1) \\ &= x^5 + \frac{1}{2}x^4 - x^3 - \frac{3}{8}x^2 + \frac{3}{16}x + \frac{1}{32}.\end{aligned}$$

Lemma 1. *Let* $\xi = \exp(i2\pi/11)$. *Then*

$$p_{11}(x) = \prod_{m=1}^{5}(x - \sigma_m) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$$

is a minimal polynomial of the numbers σ_m, $m = 1, 2, \ldots, 5$. *Moreover, we have the identity*

$$\Psi_{11}(x) = \frac{1}{32}p_{11}(2x).$$

Corollary 2. *The numbers* k_{2m}, $m = 0, 1, \ldots, 4$ *are linearly independent over* $\mathbb{Q}$.

Proof. If we suppose that

$$a + b\cos(2\pi/11) + c\cos(4\pi/11) + d\cos(6\pi/11) + e\cos(8\pi/11) = 0$$

for some $a, b, c, d, e \in \mathbb{Q}$, then by the formulas

$$\begin{aligned}\cos 2\alpha &= 2\cos^2\alpha - 1, \\ \cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha, \\ \cos 4\alpha &= 8\cos^4\alpha - 8\cos^2\alpha + 1,\end{aligned}$$

the number $\cos(2\pi/11)$ would be a root of some polynomial $\in \mathbb{Q}[x]$ having degree $\leqslant 4$, which by Lemma 1 is impossible. □

Corollary 3. *Every five numbers which belong to the set* $\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ *are linearly independent over* $\mathbb{Q}$.

Proof. It follows from the identity (1.4). □

The following observation will play a central role in our arguments.

Lemma 4. *Let* $a_1, a_2, \ldots, a_n \in \mathbb{R}$ *be linearly independent over* $\mathbb{Q}$ *and let* $f_k, g_k \in \mathbb{Q}[\delta]$, $k = 1, 2, \ldots, n$. *If the identity holds*

$$\sum_{k=1}^{n} f_k(\delta)\, a_k = \sum_{k=1}^{n} g_k(\delta)\, a_k \tag{2.1}$$

for every $\delta \in \mathbb{Q}$, *then* $f_k(\delta) = g_k(\delta)$ *for every* $k = 1, 2, \ldots, n$ *and* $\delta \in \mathbb{C}$.

Proof. Since $f_k(\delta) \in \mathbb{Q}$ and $g_k(\delta) \in \mathbb{Q}$ for any $\delta \in \mathbb{Q}$ we get from (2.1) that $f_k(\delta) = g_k(\delta)$ for every $k = 1, 2, \ldots, n$ and $\delta \in \mathbb{Q}$. The last equality implies that all, respective coefficients of polynomials f_k and g_k are the same. So $f_k(\delta) = g_k(\delta)$ for all $\delta \in \mathbb{C}$ and $k = 1, 2, \ldots, n$. □

3 Quasi-Fibonacci numbers of order 11

Let us start from the following basic result.

Lemma 5. *Let $\delta \in \mathbb{C}$ and $n \in \mathbb{N}$. Then the following identities hold*

$$\tau_m^n(\delta) = a_n(\delta) + b_n(\delta)\,\sigma_m + c_n(\delta)\,\sigma_{2m} + d_n(\delta)\,\sigma_{3m} + e_n(\delta)\,\sigma_{4m} + f_n(\delta)\,\sigma_{5m} \tag{3.2m-1}$$

$$\tau_m^n(\delta) = A_n(\delta) + B_n(\delta)\,\sigma_m + C_n(\delta)\,\sigma_{2m} + D_n(\delta)\,\sigma_{3m} + E_n(\delta)\,\sigma_{4m} \tag{3.2m}$$

for every $m = 1, 2, \ldots, 5$, i.e.,

$$\begin{aligned} \tau_1^n(\delta) =& a_n(\delta) + b_n(\delta)\,\sigma_1 + c_n(\delta)\,\sigma_2 + d_n(\delta)\,\sigma_3 + e_n(\delta)\,\sigma_4 + f_n(\delta)\,\sigma_5 && (3.1)\\ =& A_n(\delta) + B_n(\delta)\,\sigma_1 + C_n(\delta)\,\sigma_2 + D_n(\delta)\,\sigma_3 + E_n(\delta)\,\sigma_4, && (3.2)\end{aligned}$$

$$\begin{aligned} \tau_2^n(\delta) =& a_n(\delta) + b_n(\delta)\,\sigma_2 + c_n(\delta)\,\sigma_4 + d_n(\delta)\,\sigma_5 + e_n(\delta)\,\sigma_3 + f_n(\delta)\,\sigma_1 && (3.3)\\ =& A_n(\delta) + B_n(\delta)\,\sigma_2 + C_n(\delta)\,\sigma_4 + D_n(\delta)\,\sigma_5 + E_n(\delta)\,\sigma_3, && (3.4)\end{aligned}$$

$$\begin{aligned} \tau_3^n(\delta) =& a_n(\delta) + b_n(\delta)\,\sigma_3 + c_n(\delta)\,\sigma_5 + d_n(\delta)\,\sigma_2 + e_n(\delta)\,\sigma_1 + f_n(\delta)\,\sigma_4 && (3.5)\\ =& A_n(\delta) + B_n(\delta)\,\sigma_3 + C_n(\delta)\,\sigma_5 + D_n(\delta)\,\sigma_2 + E_n(\delta)\,\sigma_1, && (3.6)\end{aligned}$$

$$\begin{aligned} \tau_4^n(\delta) =& a_n(\delta) + b_n(\delta)\,\sigma_4 + c_n(\delta)\,\sigma_3 + d_n(\delta)\,\sigma_1 + e_n(\delta)\,\sigma_5 + f_n(\delta)\,\sigma_2 && (3.7)\\ =& A_n(\delta) + B_n(\delta)\,\sigma_4 + C_n(\delta)\,\sigma_3 + D_n(\delta)\,\sigma_1 + E_n(\delta)\,\sigma_5, && (3.8)\end{aligned}$$

$$\begin{aligned} \tau_5^n(\delta) =& a_n(\delta) + b_n(\delta)\,\sigma_5 + c_n(\delta)\,\sigma_1 + d_n(\delta)\,\sigma_4 + e_n(\delta)\,\sigma_2 + f_n(\delta)\,\sigma_3 && (3.9)\\ =& A_n(\delta) + B_n(\delta)\,\sigma_5 + C_n(\delta)\,\sigma_1 + D_n(\delta)\,\sigma_4 + E_n(\delta)\,\sigma_2, && (3.10)\end{aligned}$$

where

$$a_0(\delta) = 1, \qquad b_0(\delta) = \ldots = f_0(\delta) = 0,$$

$$\left\{\begin{array}{lcl} a_{n+1}(\delta) &=& a_n(\delta) + 2\delta b_n(\delta),\\ b_{n+1}(\delta) &=& \delta a_n(\delta) + b_n(\delta) + \delta c_n(\delta),\\ c_{n+1}(\delta) &=& \delta b_n(\delta) + c_n(\delta) + \delta d_n(\delta),\\ d_{n+1}(\delta) &=& \delta c_n(\delta) + d_n(\delta) + \delta e_n(\delta),\\ e_{n+1}(\delta) &=& \delta d_n(\delta) + e_n(\delta) + \delta f_n(\delta),\\ f_{n+1}(\delta) &=& \delta e_n(\delta) + (1+\delta) f_n(\delta) \end{array}\right. \tag{3.11}$$

and

$$A_n(\delta) = a_n(\delta) - f_n(\delta), \quad B_n(\delta) = b_n(\delta) - f_n(\delta),$$
$$C_n(\delta) = c_n(\delta) - f_n(\delta), \quad D_n(\delta) = d_n(\delta) - f_n(\delta), \quad E_n(\delta) = e_n(\delta) - f_n(\delta),$$

for every $n = 0, 1, 2, \ldots$. Hence, the following identities are derived:

$$\begin{aligned} A_{n+1}(\delta) &= a_n(\delta) + 2\delta b_n(\delta) - \delta e_n(\delta) - (1+\delta) f_n(\delta)\\ &= A_n(\delta) + 2\delta B_n(\delta) - \delta E_n(\delta), \end{aligned}$$

$$\begin{aligned}
B_{n+1}(\delta) &= \delta a_n(\delta) + b_n(\delta) + \delta c_n(\delta) - \delta e_n(\delta) - (1+\delta) f_n(\delta) \\
&= \delta A_n(\delta) + B_n(\delta) + \delta C_n(\delta) - \delta E_n(\delta), \\
C_{n+1}(\delta) &= \delta b_n(\delta) + c_n(\delta) + \delta d_n(\delta) - \delta e_n(\delta) - (1+\delta) f_n(\delta) \\
&= \delta B_n(\delta) + C_n(\delta) + \delta D_n(\delta) - \delta E_n(\delta), \\
D_{n+1}(\delta) &= \delta c_n(\delta) + d_n(\delta) - (1+\delta) f_n(\delta) \\
&= \delta C_n(\delta) + D_n(\delta), \\
E_{n+1}(\delta) &= \delta d_n(\delta) + e_n(\delta) - \delta e_n(\delta) - f_n(\delta) \\
&= \delta D_n(\delta) + (1-\delta) E_n(\delta),
\end{aligned}$$

i.e., the following system of linear equations holds

$$\begin{cases}
A_{n+1}(\delta) = A_n(\delta) + 2\delta B_n(\delta) - \delta E_n(\delta), \\
B_{n+1}(\delta) = \delta A_n(\delta) + B_n(\delta) + \delta C_n(\delta) - \delta E_n(\delta), \\
C_{n+1}(\delta) = \delta B_n(\delta) + C_n(\delta) + \delta D_n(\delta) - \delta E_n(\delta), \\
D_{n+1}(\delta) = \delta C_n(\delta) + D_n(\delta), \\
E_{n+1}(\delta) = \delta D_n(\delta) + (1-\delta) E_n(\delta),
\end{cases} \tag{3.12}$$

where $A_0(\delta) = 1,\ B_0(\delta) = C_0(\delta) = D_0(\delta) = E_0(\delta) = 0.$

Proof. Proofs of the identities $(3.2m-1)$ for every $m = 1, 2, \ldots, 5$ by induction follow. For $n = 1$ we have

$$\tau_m(\delta) = a_1(\delta) + b_1(\delta)\,\sigma_m + c_1(\delta)\,\sigma_{2m} + d_1(\delta)\,\sigma_{3m} + e_1(\delta)\,\sigma_{4m} + f_1(\delta)\,\sigma_{5m}.$$

Suppose that for some $n \in \mathbb{N}$ the equalities $(3.2m-1)$, $m = 1, 2, \ldots, 5$, holds. Then by (1.1), (1.2) and (3.11) we get

$$\begin{aligned}
&\tau_m^{n+1}(\delta) = \tau_m(\delta)\,\tau_m^n(\delta) = \\
&\quad = \big(1 + \delta\,\sigma_m\big)\big(a_n(\delta) + b_n(\delta)\,\sigma_m + c_n(\delta)\,\sigma_{2m} + d_n(\delta)\,\sigma_{3m} + e_n(\delta)\,\sigma_{4m} + f_n(\delta)\,\sigma_{5m}\big) = \\
&\quad = \big(1 + \delta\,\sigma_m\big)\,a_n(\delta) + \big(\sigma_m + \delta\,(\sigma_{2m} + 2)\big)\,b_n(\delta) + \big(\sigma_{2m} + \delta\,(\sigma_{3m} + \sigma_m)\big)\,c_n(\delta) + \\
&+ \big(\sigma_{3m} + \delta\,(\sigma_{4m} + \sigma_{2m})\big)\,d_n(\delta) + \big(\sigma_{4m} + \delta\,(\sigma_{5m} + \sigma_{3m})\big)\,e_n(\delta) + \big(\sigma_{5m} + \delta\,(\sigma_{5m} + \sigma_{4m})\big)\,f_n(\delta) = \\
&\quad = \big(a_n(\delta) + 2\,\delta\,b_n(\delta)\big) + \big(\delta\,a_n(\delta) + b_n(\delta) + \delta\,c_n(\delta)\big)\,\sigma_m + \big(\delta\,b_n(\delta) + c_n(\delta) + \delta\,d_n(\delta)\big)\,\sigma_{2m} + \\
&+ \big(\delta\,c_n(\delta) + d_n(\delta) + \delta\,e_n(\delta)\big)\,\sigma_{3m} + \big(\delta\,d_n(\delta) + e_n(\delta) + \delta\,f_n(\delta)\big)\,\sigma_{4m} + \big(\delta\,e_n(\delta) + (1+\delta)\,f_n(\delta)\big)\,\sigma_{5m} = \\
&\quad = a_{n+1}(\delta) + b_{n+1}(\delta)\,\sigma_m + c_{n+1}(\delta)\,\sigma_{2m} + d_{n+1}(\delta)\,\sigma_{3m} + e_{n+1}(\delta)\,\sigma_{4m} + f_{n+1}(\delta)\,\sigma_{5m},
\end{aligned}$$

which by the principle of mathematical induction means that $(3.2m-1)$ hold for every $n \in \mathbb{N}$ and $m = 1, 2, \ldots, 5$.

Formula $(3.2m)$ from $(3.2m-1)$ and (1.4) follows for every $m = 1, 2, \ldots, 5$. □

Definition 6. *We call all elements of the sequences* $\{A_n(\delta)\}$, $\{B_n(\delta)\}$, …, $\{E_n(\delta)\}$ *the quasi-Fibonacci numbers of order* $(11, \delta)$ *(see Table 1 at the end of the paper). To simplify notation we shall write* $\{A_n\}$, $\{B_n\}$, …, $\{E_n\}$ *instead of* $\{A_n(1)\}$, $\{B_n(1)\}$, …, $\{E_n(1)\}$, *respectively. We call the elements of the sequences* $\{A_n\}$, $\{B_n\}$, …, $\{E_n\}$ *the quasi-Fibonacci numbers of order* 11 *(see Table 2 at the end of the paper).*

Corollary 7. *In the sequel, for $\delta = 1$ we obtain the following recurrence relations:*

$$\begin{cases} A_{n+1} = A_n + 2B_n - E_n, \\ B_{n+1} = A_n + B_n + C_n - E_n, \\ C_{n+1} = B_n + C_n + D_n - E_n, \\ D_{n+1} = C_n + D_n, \\ E_{n+1} = D_n. \end{cases} \tag{3.13}$$

Corollary 8. *Adding identities (3.k) for odd and even $k = 1, \dots, 10$ respectively we obtain the identity:*

$$\begin{aligned} \tau_1^n(\delta) + \tau_2^n(\delta) + \tau_3^n(\delta) + \tau_4^n(\delta) + \tau_5^n(\delta) = \\ &= 5a_n(\delta) - b_n(\delta) - c_n(\delta) - d_n(\delta) - e_n(\delta) - f_n(\delta) \\ &= 5A_n(\delta) - B_n(\delta) - C_n(\delta) - D_n(\delta) - E_n(\delta) \\ &:= \mathcal{A}_n(\delta). \end{aligned} \tag{3.14}$$

Hence, by (1.5) the following identities holds:

$$2^{-n}\mathcal{A}_n\left(\tfrac{1}{2}\right) = \sum_{m=1}^{5} k_m^{2n} \tag{3.15}$$

and

$$2^{-n}\mathcal{A}_n\left(-\tfrac{1}{2}\right) = \sum_{m=1}^{5} s_m^{2n}. \tag{3.16}$$

Moreover, from (3.14) we obtain

$$\begin{aligned} \sum_{m=1}^{5} \operatorname{coeff}\left(\tau_m^n(\delta); \delta^n\right) = \sum_{m=1}^{5} \sigma_m^n = 2^n \sum_{m=1}^{5} k_{2m}^n = \\ = 5 \operatorname{coeff}\left(A_n(\delta); \delta^n\right) - \operatorname{coeff}\left(B_n(\delta); \delta^n\right) - \ldots - \operatorname{coeff}\left(E_n(\delta); \delta^n\right) = \operatorname{coeff}\left(\mathcal{A}_n(\delta); \delta^n\right). \end{aligned}$$

Hence, by (3.15) we get

$$\operatorname{coeff}\left(\mathcal{A}_{2n}(\delta); \delta^{2n}\right) = 2^n \mathcal{A}_n\left(\tfrac{1}{2}\right)$$

and

$$\operatorname{coeff}\left(\mathcal{A}_{2n-1}(\delta); \delta^{2n-1}\right) = 2^{2n-1} \sum_{m=1}^{5} (-1)^m k_m^n.$$

In Table 3 twelve initial values of the sequences $\{\mathcal{A}(1)\}$ (see also A062883 in [6]), $\{\mathcal{A}(1/2)\}$ and $\{\mathcal{A}(-1/2)\}$ are presented.

Corollary 9. *By (1.7) and (3.2m) we have the identity*

$$\begin{aligned} 2^n k_{3m}^{2n} = A_n(\delta) + B_n(\delta)\,\sigma_m + C_n(\delta)\,\sigma_{2m} + D_n(\delta)\,\sigma_{3m} + E_n(\delta)\,\sigma_{4m} = \\ = A_n(\delta) + 2\Big(B_n(\delta)\,k_{2m} + C_n(\delta)\,k_{4m} + D_n(\delta)\,k_{5m} + E_n(\delta)\,k_{3m}\Big) \end{aligned}$$

for $\delta := 2\,k_{2m}^2 - \frac{3}{2}$ *and for every* $m = 1, 2, \ldots, 5$. *Furthermore, by (1.6) and (3.2m) the following general formula hold*

$$2^n\,k_{r\,m}^{2n} = A_n(\delta) + B_n(\delta)\,\sigma_m + C_n(\delta)\,\sigma_{2m} + D_n(\delta)\,\sigma_{3m} + E_n(\delta)\,\sigma_{4m} = \\ = A_n(\delta) + 2\Big(B_n(\delta)\,k_{2m} + C_n(\delta)\,k_{4m} + D_n(\delta)\,k_{5m} + E_n(\delta)\,k_{3m}\Big)$$

for $\delta := T_r(k_{2m})/(2\,k_{2m})$, $m, r \in \mathbb{N}$. *We note that* x *divides* $T_r(x)$ *iff* r *is an odd positive integer.*

Corollary 10. *Since* $\deg_\delta\left(\tau_m^n\right) = n$ *for every* $m = 1, 2, \ldots, 5$, *by (3.2m) for* $m = 1, 2, \ldots, 5$ *it can be easily deduced the following formula*

$$\max\Big\{\deg\big(A_n(\delta)\big), \deg\big(B_n(\delta)\big), \deg\big(C_n(\delta)\big), \deg\big(D_n(\delta)\big), \deg\big(E_n(\delta)\big)\Big\} = n.$$

Hence, by (3.12) and Table 1, if $\deg(D_n(\delta)) = n$ *for* $n = 5, 6, \ldots$ *then also* $\deg(C_n(\delta)) = n$ *for* $n = 5, 6, \ldots$, *and* $\deg(E_n(\delta)) = n$ *for infinite many* $n \in \mathbb{N}$ *(see also the Section 6 in this paper).*

Corollary 11. *Taking into account the equations (from the last to the first one, respectively) of the recurrence system (3.12), we obtain:*

$$\begin{cases} \delta D_n(\delta) = E_{n+1}(\delta) - (1-\delta)E_n(\delta), \\ \delta^2 C_n(\delta) = \delta D_{n+1}(\delta) - \delta D_n(\delta) \\ \qquad = E_{n+2}(\delta) + (\delta-2)E_{n+1}(\delta) + (1-\delta)E_n(\delta), \\ \delta^3 B_n(\delta) = \delta^2 C_{n+1}(\delta) - \delta^2 C_n(\delta) - \delta^3 D_n(\delta) + \delta^3 E_n(\delta) \\ \qquad = E_{n+3}(\delta) + (\delta-3)E_{n+2}(\delta) + (3-2\delta-\delta^2)E_{n+1}(\delta) \\ \qquad +(\delta^2+\delta-1)E_n(\delta), \\ \delta^4 A_n(\delta) = \delta^3 B_{n+1}(\delta) - \delta^3 B_n(\delta) - \delta^4 C_n(\delta) + \delta^4 E_n(\delta) \\ \qquad = E_{n+4}(\delta) + (\delta-4)E_{n+3}(\delta) + (-2\delta^2-3\delta+6)E_{n+2}(\delta) \\ \qquad +(-\delta^3+4\delta^2+3\delta-4)E_{n+1}(\delta) + (\delta^4+\delta^3-2\delta^2-\delta+1)E_n(\delta) \end{cases} \tag{3.17}$$

and, finally:

$$\delta^4 A_{n+1}(\delta) - \delta^4 A_n(\delta) - 2\delta^5 B_n(\delta) + \delta^5 E_n(\delta) = 0, \tag{3.18}$$

i.e.

$$E_{n+5}(\delta) + (\delta-5)E_{n+4}(\delta) + (-4\delta^2-4\delta+10)E_{n+3}(\delta) + \\ + (-3\delta^3+12\delta^2+6\delta-10)E_{n+2}(\delta) + (3\delta^4+6\delta^3-12\delta^2-4\delta+5)E_{n+1}(\delta) + \\ + (\delta^5-3\delta^4-3\delta^3+4\delta^2+\delta-1)E_n(\delta) = 0. \tag{3.19}$$

Immediately from equations (3.13) or (3.17) and (3.19) for $\delta = 1$ we obtain the following formulas:

$$\begin{cases} E_{n+1} = D_n, \\ C_n = D_{n+1} - D_n, \\ B_n = D_{n+2} - 2D_{n+1} + D_{n-1}, \\ A_n = D_{n+3} - 3D_{n+2} + D_{n+1} + 2D_n, \end{cases} \tag{3.20}$$

and

$$D_{n+4} - 4D_{n+3} + 2D_{n+2} + 5D_{n+1} - 2D_n - D_{n-1} = 0. \tag{3.21}$$

The characteristic polynomial $p_{11}(\mathbb{X};\delta)$ of the recurrence equation (3.19) has the following decomposition (see general formula (3.39) below):

$$\begin{aligned} p_{11}(\mathbb{X};\delta) :=\ & \mathbb{X}^5 + (\delta-5)\,\mathbb{X}^4 + (-4\delta^2-4\delta+10)\,\mathbb{X}^3 + (-3\delta^3+12\delta^2+6\delta-10)\,\mathbb{X}^2 + \\ & + (3\delta^4+6\delta^3-12\delta^2-4\delta+5)\,\mathbb{X} + (\delta^5-3\delta^4-3\delta^3+4\delta^2+\delta-1) = \\ & = \prod_{m=1}^{5} \big(\mathbb{X} - \tau_m(\delta)\big). \end{aligned} \tag{3.22}$$

Sketch of the proof: We have

$$\sum_{m=1}^{5} \tau_m(\delta) = \sum_{m=1}^{5} \big(1 + \delta\,\sigma_m\big) = 5 + \delta \sum_{m=1}^{5} \sigma_m \overset{(1.4)}{=} 5 - \delta,$$

$$\begin{aligned} \sum_{1\le m<n\le 5} \tau_m(\delta)\,\tau_n(\delta) &= \frac{1}{2}\left(\Big(\sum_{m=1}^{5}\tau_m(\delta)\Big)^2 - \sum_{m=1}^{5}\tau_m^2(\delta)\right) = \frac{1}{2}\left(\big(5-\delta\big)^2 - \sum_{m=1}^{5}\big(1+\delta\,\sigma_m\big)^2\right) = \\ &= \frac{1}{2}\left(\big(5-\delta\big)^2 - 5 - 2\,\delta\sum_{m=1}^{5}\sigma_m - \delta^2\sum_{m=1}^{5}\sigma_m^2\right) \overset{(1.4),(1.1)}{=} \\ &= \frac{1}{2}\left(25 - 10\,\delta + \delta^2 - 5 + 2\,\delta - \delta^2\sum_{m=1}^{5}\big(\sigma_{2m}+2\big)\right) \overset{(1.4)}{=} 10 - \delta - 4\,\delta^2, \end{aligned}$$

etc. □

It follows from decomposition (3.22) that there exist numbers $\alpha, \beta, \gamma, \varepsilon, \varphi \in \mathbb{R}$ such that:

$$E_n(\delta) = \alpha\,\tau_1^n(\delta) + \beta\,\tau_2^n(\delta) + \gamma\,\tau_3^n(\delta) + \varepsilon\,\tau_4^n(\delta) + \varphi\,\tau_5^n(\delta)$$

for every $n \in \mathbb{N}$. Solving the respective system of linear equations:

$$\begin{cases} E_n(\delta) &=& 0, & n = 1,2,3, \\ E_4(\delta) &=& \delta^4, & \\ E_5(\delta) &=& 5\delta^4 - \delta^5, & \end{cases}$$

we obtain:

$$\begin{aligned} 11\,E_n(\delta) &= \sum_{m=1}^{5}\big(\sigma_{4m}-\sigma_{5m}\big)\,\tau_m^n(\delta) \\ &= \big(\sigma_4-\sigma_5\big)\,\tau_1^n(\delta) + \big(\sigma_3-\sigma_1\big)\,\tau_2^n(\delta) + \big(\sigma_1-\sigma_4\big)\,\tau_3^n(\delta) \\ &\quad + \big(\sigma_5-\sigma_2\big)\,\tau_4^n(\delta) + \big(\sigma_2-\sigma_3\big)\,\tau_5^n(\delta) \\ &= 2\big(k_1-k_3\big)\big(1+2\,\delta\,k_2\big)^n + 2\big(k_6-k_2\big)\big(1+2\,\delta\,k_4\big)^n + 2\big(k_2+k_3\big)\big(1+2\,\delta\,k_6\big)^n \\ &\quad + 2\big(-k_1-k_4\big)\big(1-2\,\delta\,k_3\big)^n + 2\big(k_4-k_6\big)\big(1-2\,\delta\,k_1\big)^n, \end{aligned} \tag{3.23}$$

which implies, the following identities for $\delta = \frac{1}{2}$:

$$\frac{11}{2^{n+1}} E_n\big(\tfrac{1}{2}\big) = \big(k_1 - k_3\big)\, k_1^{2n} + \big(k_6 - k_2\big)\, k_2^{2n} + \big(k_2 + k_3\big)\, k_3^{2n} - \big(k_1 + k_4\big)\, k_4^{2n} + \big(k_4 - k_6\big)\, k_5^{2n} = 2\, s_1\, s_2\, k_1^{2n} - 2\, s_2\, s_4\, k_2^{2n} + 2\, s_3\, s_5\, k_3^{2n} - 2\, s_3\, s_4\, k_4^{2n} + 2\, s_1\, s_5\, k_5^{2n}, \tag{3.24}$$

and for $\delta = -\frac{1}{2}$ (by (1.5)):

$$\frac{11}{2^{n+2}}\, E_n\big(-\tfrac{1}{2}\big) = s_2\, s_1^{2n+1} - s_4\, s_2^{2n+1} + s_5\, s_3^{2n+1} - s_3\, s_4^{2n+1} + s_1\, s_5^{2n+1}. \tag{3.25}$$

Now, from (3.17), (3.23), (1.1) and (1.2) the following decomposition may be generated:

$$\begin{aligned} 11\, D_n(\delta) &= \frac{1}{\delta}\Big(\big(11\, E_{n+1}(\delta)\big) - (1-\delta)\,\big(11\, E_n(\delta)\big)\Big) = \\ &= \frac{1}{\delta} \sum_{m=1}^{5} \big(\sigma_{4m} - \sigma_{5m}\big)\big(\tau_m(\delta) - 1 + \delta\big)\, \tau_m^n(\delta) \\ &= \sum_{m=1}^{5} \big(\sigma_{4m} - \sigma_{5m}\big)\big(\sigma_m + 1\big)\, \tau_m^n(\delta) = \sum_{m=1}^{5} \big(\sigma_{3m} - \sigma_{5m}\big)\tau_m^n(\delta) \\ &= \big(\sigma_3 - \sigma_5\big)\, \tau_1^n(\delta) + \big(\sigma_5 - \sigma_1\big)\, \tau_2^n(\delta) + \big(\sigma_2 - \sigma_4\big)\, \tau_3^n(\delta) \\ &\quad + \big(\sigma_1 - \sigma_2\big)\, \tau_4^n(\delta) + \big(\sigma_4 - \sigma_3\big)\, \tau_5^n(\delta) \\ &= 2\,\big(k_1 + k_6\big)\,\big(1 + 2\,\delta\, k_2\big)^n - 2\,\big(k_1 + k_2\big)\,\big(1 + 2\,\delta\, k_4\big)^n + 2\,\big(k_3 + k_4\big)\,\big(1 + 2\,\delta\, k_6\big)^n \\ &\quad + 2\,\big(k_2 - k_4\big)\,\big(1 - 2\,\delta\, k_3\big)^n - 2\,\big(k_3 + k_6\big)\,\big(1 - 2\,\delta\, k_1\big)^n, \end{aligned} \tag{3.26}$$

which implies the following identity for $\delta = \frac{1}{2}$:

$$\frac{11}{2^{n+1}} D_n\big(\tfrac{1}{2}\big) = \big(k_1 - k_5\big)\, k_1^{2n} - \big(k_1 + k_2\big)\, k_2^{2n} + \big(k_3 + k_4\big)\, k_3^{2n} + \big(k_2 - k_4\big)\, k_4^{2n} - \big(k_3 + k_6\big)\, k_5^{2n} = 2\, s_2\, s_3\, k_1^{2n} - 2\, s_4\, s_6\, k_2^{2n} + 2\, s_2\, s_6\, k_3^{2n} + 2\, s_1\, s_3\, k_4^{2n} - 2\, s_1\, s_4\, k_5^{2n}, \tag{3.27}$$

and for $\delta = -\frac{1}{2}$ (by (1.5)):

$$\frac{11}{2^{n+2}}\, D_n\big(-\tfrac{1}{2}\big) = s_2\, s_3\, s_1^{2n} - s_4\, s_6\, s_2^{2n} + s_2\, s_6\, s_3^{2n} + s_1\, s_3\, s_4^{2n} - s_1\, s_4\, s_5^{2n}. \tag{3.28}$$

Next, by (3.17), (3.26), (1.1) and (1.2), we obtain:

$$\begin{aligned} 11\, C_n(\delta) &= \frac{11}{\delta}\big(D_{n+1}(\delta) - D_n(\delta)\big) = \\ &= \sum_{m=1}^{5} \big(\sigma_{3m} - \sigma_{5m}\big)\, \sigma_m\, \tau_m^n(\delta) = \sum_{m=1}^{5} \big(\sigma_{2m} - \sigma_{5m}\big)\tau_m^n(\delta) \\ &= \big(\sigma_2 - \sigma_5\big)\, \tau_1^n(\delta) + \big(\sigma_4 - \sigma_1\big)\, \tau_2^n(\delta) + \big(\sigma_5 - \sigma_4\big)\, \tau_3^n(\delta) \\ &\quad + \big(\sigma_3 - \sigma_2\big)\, \tau_4^n(\delta) + \big(\sigma_1 - \sigma_3\big)\, \tau_5^n(\delta) \\ &= 2\,\big(k_1 + k_4\big)\,\big(1 + 2\,\delta\, k_2\big)^n - 2\,\big(k_2 + k_3\big)\,\big(1 + 2\,\delta\, k_4\big)^n + 2\,\big(k_3 - k_1\big)\,\big(1 + 2\,\delta\, k_6\big)^n \\ &\quad + 2\,\big(k_6 - k_4\big)\,\big(1 - 2\,\delta\, k_3\big)^n + 2\,\big(k_2 - k_6\big)\,\big(1 - 2\,\delta\, k_1\big)^n, \end{aligned} \tag{3.29}$$

which implies the following identity for $\delta = \frac{1}{2}$:

$$\frac{11}{2^{n+1}} C_n\big(\tfrac{1}{2}\big) = \big(k_1 + k_4\big)\, k_1^{2n} - \big(k_2 + k_3\big)\, k_2^{2n} + \big(k_3 - k_1\big)\, k_3^{2n} + \big(k_6 - k_4\big)\, k_4^{2n} + \big(k_2 - k_6\big)\, k_5^{2n} = 2\, s_3\, s_4\, k_1^{2n} - 2\, s_3\, s_5\, k_2^{2n} - 2\, s_1\, s_2\, k_3^{2n} - 2\, s_1\, s_5\, k_4^{2n} + 2\, s_2\, s_4\, k_5^{2n}, \tag{3.30}$$

and for $\delta = -\frac{1}{2}$ (by (1.5)):

$$\frac{11}{2^{n+2}}\, C_n\big(-\tfrac{1}{2}\big) = s_3\, s_4\, s_1^{2n} - s_3\, s_5\, s_2^{2n} - s_1\, s_2\, s_3^{2n} - s_1\, s_5\, s_4^{2n} + s_2\, s_4\, s_5^{2n}. \tag{3.31}$$

Now, by (3.17), (3.23), (3.26), (3.29), (1.1) and (1.2), we may generate the formula:

$$\begin{aligned} 11\, B_n(\delta) &= \frac{11}{\delta}\big(C_{n+1}(\delta) - C_n(\delta)\big) + 11\,\big(E_n(\delta) - D_n(\delta)\big) = \\ &= \sum_{m=1}^{5} \Big(\big(\sigma_{2m} - \sigma_{5m}\big)\, \sigma_m + \sigma_{4m} - \sigma_{5m} - \sigma_{3m} + \sigma_{5m}\Big)\, \tau_m^n(\delta) = \sum_{m=1}^{5} \big(\sigma_m - \sigma_{5m}\big)\tau_m^n(\delta) \\ &= 2\,\big(k_1 + k_2\big)\,\big(1 + 2\,\delta\, k_2\big)^n + 2\,\big(k_4 - k_2\big)\,\big(1 + 2\,\delta\, k_4\big)^n \\ &\quad + 2\,\big(k_3 + k_6\big)\,\big(1 + 2\,\delta\, k_6\big)^n - 2\,\big(k_3 + k_4\big)\,\big(1 - 2\,\delta\, k_3\big)^n - 2\,\big(k_1 + k_6\big)\,\big(1 - 2\,\delta\, k_1\big)^n, \end{aligned} \tag{3.32}$$

hence, for $\delta = \frac{1}{2}$ we obtain:

$$\frac{11}{2^{n+1}} B_n\big(\tfrac{1}{2}\big) = \big(k_1 + k_2\big)\, k_1^{2n} + \big(k_4 - k_2\big)\, k_2^{2n} + \big(k_3 + k_6\big)\, k_3^{2n} - \big(k_3 + k_4\big)\, k_4^{2n} - \big(k_1 + k_6\big)\, k_5^{2n}, \tag{3.33}$$

and for $\delta = -\frac{1}{2}$ (by (1.5)):

$$\frac{11}{2^{n+2}}\, B_n\big(-\tfrac{1}{2}\big) = s_4\, s_5\, s_1^{2n} - s_1\, s_3\, s_2^{2n} + s_1\, s_4\, s_3^{2n} - s_2\, s_5\, s_4^{2n} - s_2\, s_3\, s_5^{2n}. \tag{3.34}$$

Immediately from Corollary 8 we derive the identity:

$$\begin{aligned} 55\, A_n(\delta) &= 11\,\Big(\sum_{m=1}^{5} \tau_m^n(\delta) + B_n(\delta) + C_n(\delta) + D_n(\delta) + E_n(\delta)\Big) \\ &= \sum_{m=1}^{5} \big(11 + \sigma_m + \sigma_{2m} + \sigma_{3m} + \sigma_{4m} - 4\,\sigma_{5m}\big)\, \tau_m^n(\delta) \\ &= \sum_{m=1}^{5} \Big(\underbrace{\big(1 + \sigma_m + \sigma_{2m} + \sigma_{3m} + \sigma_{4m} + \sigma_{5m}\big)}_{=0} + 10 - 5\,\sigma_{5m}\Big)\, \tau_m^n(\delta) \\ &= 5\,\sum_{m=1}^{5} \big(2 - \sigma_{5m}\big)\tau_m^n(\delta), \end{aligned} \tag{3.35}$$

and thus we obtain the following trigonometrical form of $A_n(\delta)$:

$$11\,A_n(\delta) = s_5^2\left(1+2\,\delta\,k_2\right)^n + s_1^2\left(1+2\,\delta\,k_4\right)^n + s_4^2\left(1+2\,\delta\,k_6\right)^n + \\ + s_2^2\left(1-2\,\delta\,k_3\right)^n + s_3^2\left(1-2\,\delta\,k_1\right)^n. \quad (3.36)$$

Hence, for $\delta=\frac{1}{2}$ we obtain:

$$\frac{11}{2^{n+2}}A_n\left(\tfrac{1}{2}\right) = s_5^2\,k_1^{2n} + s_1^2\,k_2^{2n} + s_4^2\,k_3^{2n} + s_2^2\,k_4^{2n} + s_3^2\,k_5^{2n}, \quad (3.37)$$

and for $\delta=-\frac{1}{2}$ (by (1.5)):

$$\frac{11}{2^{n+2}}\,A_n\left(-\tfrac{1}{2}\right) = s_5^2\,s_1^{2n} + s_1^2\,s_2^{2n} + s_4^2\,s_3^{2n} + s_2^2\,s_4^{2n} + s_3^2\,s_5^{2n}. \quad (3.38)$$

The next lemma is an attempt at generalizing the Lemma 3.13 from [9]. In view of an extensive form of the formulas, only some identities will be presented here.

Lemma 12. *The following identities hold:*
a) (decomposition using Newton's formulas for elementary symmetric polynomials):

$$\begin{aligned}\prod_{m=1}^{5}\left(\mathbb{X}-\tau_m^n(\delta)\right) = \mathbb{X}^5 - \mathcal{A}_n(\delta)\,\mathbb{X}^4 + \frac{1}{2}\left(\mathcal{A}_n^2(\delta)-\mathcal{A}_{2n}(\delta)\right)\mathbb{X}^3 \quad (3.39)\\ -\frac{1}{6}\left(\mathcal{A}_n^3(\delta)+2\,\mathcal{A}_{3n}(\delta)-3\,\mathcal{A}_{2n}(\delta)\,\mathcal{A}_n(\delta)\right)\mathbb{X}^2\\ +\frac{1}{24}\left(\mathcal{A}_n^4(\delta)-6\,\mathcal{A}_{2n}(\delta)\,\mathcal{A}_n^2(\delta)+8\,\mathcal{A}_{3n}(\delta)\,\mathcal{A}_n(\delta)-6\,\mathcal{A}_{4n}(\delta)+3\,\mathcal{A}_{2n}^2(\delta)\right)\mathbb{X}\\ +\left(\delta^5-3\,\delta^4-3\,\delta^3+4\,\delta^2+\delta-1\right)^n.\end{aligned}$$

b)

$$2\,\mathcal{A}_n(\delta) - 11\,A_n(\delta) = \sum_{m=1}^{5}\sigma_{5m}\tau_m^n(\delta), \quad (3.40)$$

$$11\left(B_n(\delta)-A_n(\delta)\right) + 2\,\mathcal{A}_n(\delta) = \sum_{m=1}^{5}\sigma_m\tau_m^n(\delta) = \frac{1}{\delta}\left(\mathcal{A}_{n+1}(\delta)-\mathcal{A}_n(\delta)\right), \quad (3.41)$$

which implies also the recurrence formula:

$$\mathcal{A}_{n+1}(\delta) = (2\,\delta+1)\,\mathcal{A}_n(\delta) + 11\,\delta\left(B_n(\delta)-A_n(\delta)\right); \quad (3.42)$$

$$11\left(C_n(\delta)-A_n(\delta)\right) + 2\,\mathcal{A}_n(\delta) = \sum_{m=1}^{5}\sigma_{2m}\tau_m^n(\delta), \quad (3.43)$$

$$11\left(D_n(\delta)-A_n(\delta)\right) + 2\,\mathcal{A}_n(\delta) = \sum_{m=1}^{5}\sigma_{3m}\tau_m^n(\delta), \quad (3.44)$$

$$11\left(E_n(\delta)-\mathcal{A}_n(\delta)\right)+2\,\mathcal{A}_n(\delta)=\sum_{m=1}^{5}\sigma_{4m}\tau_m^n(\delta). \tag{3.45}$$

c) (we set here $\mathcal{A}_n\equiv\mathcal{A}_n(1)$ *and* $E_n\equiv E_n(1)$*)*

$$\begin{aligned}
11\big(E_{n+1}\,\mathcal{A}_n-E_n\,\mathcal{A}_{n+1}\big)=&\sum_{m=1}^{5}\big(5\,\sigma_{5m}+3\,(\sigma_m+\sigma_{4m})\big)(-\sigma_m)^n \qquad (3.46)\\
&+\sum_{m=1}^{5}\big(2+\sigma_{3m}-\sigma_{2m}-\sigma_{4m})\big)\big(\sigma_m-\sigma_{4m}-\sigma_{5m}\big)^n\\
=&2^{n+1}\Big((-5\,k_1+3\,k_2-3\,k_3)\,k_9^n+(5\,k_2+3\,k_4+3\,k_6)\,k_7^n+(-5\,k_3+3\,k_2+3\,k_6)\,k_5^n\\
&+(5\,k_4-3\,k_3-3\,k_1)\,k_3^n+(5\,k_6+3\,k_4-3\,k_1)\,k_1^n+\big(1+k_6\,(1-2\,k_2)\big)\big(k_2\,(1+2\,k_1)\big)^n\\
&+\big(1-k_1\,(1-2\,k_4)\big)\big(k_4\,(1-2\,k_2)\big)^n+\big(1+k_4\,(1+2\,k_5)\big)\big(k_6\,(1+2\,k_3)\big)^n\\
&+\big(1+k_2\,(1+2\,k_3)\big)\big(k_8\,(1-2\,k_4)\big)^n+\big(1-k_3\,(1+2\,k_1)\big)\big(k_{10}\,(1-2\,k_5)\big)^n\Big).
\end{aligned}$$

Proof. (a) The decompositions (3.39) from Vieta's formulas, formula (3.22) and the following (version) of Newton's formula for elementary symmetric polynomials [2] follows:

$$\chi_m=\frac{1}{m!}\begin{vmatrix}\eta_1 & 1 & 0 & \dots & 0\\ \eta_2 & \eta_1 & 2 & \dots & 0\\ \eta_3 & \eta_2 & \eta_1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \eta_m & \eta_{m-1} & \eta_{m-2} & \dots & \eta_1\end{vmatrix} \tag{3.47}$$

where $\eta_j=x_1^j+x_2^j+\ldots+x_n^j$, $\chi_j=\sum\limits_{1\le m_1<m_2<\ldots<m_j\le n}x_1x_2\ldots x_j$. Indeed, for $x_m=\tau_m^n(\delta)$, $m=1,2,\ldots,5$, from (3.47) we get

$$\chi_m=\frac{1}{m!}\begin{vmatrix}\mathcal{A}_n(\delta) & 1 & 0 & 0 & \dots & 0\\ \mathcal{A}_{2n}(\delta) & \mathcal{A}_n(\delta) & 2 & 0 & \dots & 0\\ \mathcal{A}_{3n}(\delta) & \mathcal{A}_{2n}(\delta) & \mathcal{A}_n(\delta) & 3 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathcal{A}_{mn}(\delta) & \mathcal{A}_{(m-1)n}(\delta) & \mathcal{A}_{(m-2)n}(\delta) & \mathcal{A}_{(m-3)n}(\delta) & \dots & \mathcal{A}_n(\delta)\end{vmatrix}=$$

$$=\begin{cases}\frac{1}{2!}\left(\mathcal{A}_n^2(\delta)-\mathcal{A}_{2n}(\delta)\right), & \text{for } m=2,\\ \frac{1}{3!}\left(\mathcal{A}_n^3(\delta)+2\,\mathcal{A}_{3n}(\delta)-3\,\mathcal{A}_n(\delta)\,\mathcal{A}_{2n}(\delta)\right), & \text{for } m=3,\\ \text{etc.}\end{cases}$$

We note that for this case we have

$$\prod_{m=1}^{5}\left(\mathbb{X}-\tau_m^n(\delta)\right)=\mathbb{X}^5-\chi_1\,\mathbb{X}^4+\chi_2\,\mathbb{X}^3-\chi_3\,\mathbb{X}^2+\chi_4\,\mathbb{X}-\chi_5.$$

Moreover, from (3.22) we have

$$\chi_5 = \prod_{m=1}^{5} \tau_m(\delta) = -\delta^5 + 3\,\delta^4 - 4\,\delta^2 - \delta + 1.$$

(b) (3.40): By (3.14) and (3.35) we have:

$$2\,\mathcal{A}_n(\delta) - 11\,A_n(\delta) = 2\sum_{m=1}^{5} \tau_m^n(\delta) - \sum_{m=1}^{5} \big(2-\sigma_m\big)\,\tau_m^n(\delta) = \sum_{m=1}^{5} \sigma_{5m}\tau_m^n(\delta).$$

(3.41): From (3.14) we obtain

$$\frac{1}{\delta}\big(\mathcal{A}_{n+1}(\delta) - \mathcal{A}_n(\delta)\big) = \frac{1}{\delta}\sum_{m=1}^{5}\big(\tau_m^{n+1}(\delta) - \tau_m^n(\delta)\big) = \sum_{m=1}^{5}\frac{1}{\delta}\big(\tau_m(\delta)-1\big)\,\tau_m^n(\delta) = \sum_{m=1}^{5}\sigma_m\tau_m^n(\delta).$$

On the other hand, by (3.32) and (3.14) we get

$$\sum_{m=1}^{5}\sigma_m\tau_m^n(\delta) = \sum_{m=1}^{5}\sigma_{5m}\tau_m^n(\delta) + \sum_{m=1}^{5}\big(\sigma_m - \sigma_{5m}\big)\tau_m^n(\delta) = 11\,\big(B_n(\delta) - A_n(\delta)\big) + 2\,\mathcal{A}_n(\delta).$$

Formulas (3.43), (3.44) and (3.45) from (3.40) and formulas $(3.2m-1)$, $m = 2, 3, 4$ can be easily deduced.

(c) The following ten simple equalities form the technical base (by hand calculation) of the proof of identity (3.46):

$$\begin{aligned}
&(1+\sigma_1)(1+\sigma_3) = -\sigma_5, &\qquad &(1+\sigma_1)(1+\sigma_2) = \sigma_1 - \sigma_4 - \sigma_5,\\
&(1+\sigma_1)(1+\sigma_4) = -\sigma_2, & &(1+\sigma_1)(1+\sigma_5) = \sigma_5 - \sigma_2 - \sigma_3,\\
&(1+\sigma_2)(1+\sigma_3) = -\sigma_4, & &(1+\sigma_2)(1+\sigma_4) = \sigma_2 - \sigma_1 - \sigma_3,\\
&(1+\sigma_2)(1+\sigma_5) = -\sigma_1, & &(1+\sigma_3)(1+\sigma_4) = \sigma_4 - \sigma_2 - \sigma_5,\\
&(1+\sigma_4)(1+\sigma_5) = -\sigma_3, & &(1+\sigma_3)(1+\sigma_5) = \sigma_3 - \sigma_1 - \sigma_4.
\end{aligned}$$

□

4 Summation formulas

Lemma 13. *We have:*

$$\delta\sum_{n=1}^{N} C_n(\delta) = D_{N+1}(\delta), \tag{4.1}$$

$$\delta\sum_{n=1}^{N} B_n(\delta) = C_{N+1}(\delta) + \delta\sum_{n=1}^{N}\big(E_n(\delta) - D_n(\delta)\big) = C_{N+1}(\delta) - E_{N+1}(\delta), \tag{4.2}$$

$$\delta\sum_{n=1}^{N} E_n(\delta) = A_1(\delta) - A_{N+1}(\delta) + 2\delta\sum_{n=1}^{N} B_n(\delta) \tag{4.3}$$

$$= 1 - A_{N+1}(\delta) + 2C_{N+1}(\delta) - 2E_{N+1}(\delta), \tag{4.4}$$

and, from (3.19) we obtain:

$$\delta^5 \sum_{n=1}^{N} E_n(\delta) = -E_{N+5}(\delta) + (4-\delta)E_{N+4}(\delta) + (4\delta^2+3\delta-6)E_{N+3}(\delta) + \\ + (3\delta^3-8\delta^2-3\delta+4)E_{N+2}(\delta) + (-3\delta^4-3\delta^3+4\delta^2+\delta-1)E_{N+1}(\delta) + \delta^4; \quad (4.5)$$

$$\delta \sum_{n=1}^{N} D_n(\delta) = E_{N+1}(\delta) + \delta \sum_{n=1}^{N} E_n(\delta), \quad (4.6)$$

$$\delta \sum_{n=0}^{N-1} (1-\delta)^n D_{N-n}(\delta) = E_{N+1}(\delta) \quad (4.7)$$

or

$$\frac{E_N(\delta)}{(1-\delta)^{N-1}} = \delta \sum_{n=0}^{N-1} \frac{D_n(\delta)}{(1-\delta)^n}, \quad (4.8)$$

$$\delta \sum_{n=1}^{N} A_n(\delta) = -A_{N+1}(\delta) + B_{N+1}(\delta) + 2C_{N+1}(\delta) - D_{N+1}(\delta) - 2E_{N+1}(\delta) + 1 - \delta. \quad (4.9)$$

and

$$\delta^2 \sum_{n=1}^{N} (N-n+1)\big(2\,B_n(\delta) - E_n(\delta)\big) = \delta \sum_{n=1}^{N+1} A_n(\delta) - (N+1)\,\delta. \quad (4.10)$$

Proof. (4.1): By (3.17) and Table 1 we get

$$\delta \sum_{n=1}^{N} C_n(\delta) = \sum_{n=1}^{N} \big(D_{n+1}(\delta) - D_n(\delta)\big) = D_{N+1}(\delta) - D_1(\delta) = D_{N+1}(\delta).$$

(4.2): By (3.17) and Table 1 we get

$$\delta \sum_{n=1}^{N} B_n(\delta) = \sum_{n=1}^{N} \big(C_{n+1}(\delta) - C_n(\delta)\big) + \delta \sum_{n=1}^{N} \big(E_n(\delta) - D_n(\delta)\big) = \\ = C_{N+1}(\delta) - C_1(\delta) + \delta \sum_{n=1}^{N} \big(E_n(\delta) - E_{n+1}(\delta)\big) = \\ = C_{N+1}(\delta) - C_1(\delta) + E_1(\delta) - E_{N+1}(\delta) = C_{N+1}(\delta) - E_{N+1}(\delta).$$

(4.3): Follows from (3.18).

(4.4): Can be derived from (4.3) and (4.2). The other proof from formulas (3.23), (3.29) and (3.35) follows: first, by (3.23) we get

$$\delta \sum_{n=1}^{N} E_n(\delta) = \frac{\delta}{11} \sum_{n=1}^{N} \sum_{m=1}^{5} \big(\sigma_{4m} - \sigma_{5m}\big)\, \tau_m^n(\delta) = \frac{\delta}{11} \sum_{m=1}^{5} \sum_{n=1}^{N} \big(\sigma_{4m} - \sigma_{5m}\big)\, \tau_m^n(\delta) = \\ = \frac{\delta}{11} \sum_{m=1}^{5} \big(\sigma_{4m} - \sigma_{5m}\big) \frac{\tau_m^{N+1}(\delta) - \tau_m(\delta)}{\tau_m(\delta) - 1} = \frac{1}{11} \sum_{m=1}^{5} \frac{\sigma_{4m} - \sigma_{5m}}{\sigma_m} \big(\tau_m^{N+1}(\delta) - \tau_m(\delta)\big), \quad (4.11)$$

but we have

$$\frac{\sigma_{4m}-\sigma_{5m}}{\sigma_m} = \frac{\sigma_{4m}+\sigma_{5m}}{\sigma_m} - 2\,\frac{\sigma_{5m}}{\sigma_m} = \sigma_{5m} - 2\left(\sigma_{4m}-\sigma_{2m}+1\right) =$$
$$= 2\left(\sigma_{2m}-\sigma_{5m}\right) - 2\left(\sigma_{4m}-\sigma_{5m}\right) - \left(2-\sigma_{5m}\right),$$

which, by (4.11), (3.29), (3.23) and (3.35) implies

$$\delta \sum_{n=1}^{N} E_n(\delta) = 2\left(C_{N+1}(\delta)-C_1(\delta)\right) - 2\left(E_{N+1}(\delta)-E_1(\delta)\right) - \left(A_{N+1}(\delta)-A_1(\delta)\right).$$

Hence, by Table 1 the formula (4.4) follows.

(4.7): By (3.18) we obtain

$$\begin{aligned} \delta\, D_N(\delta) &= E_{N+1}(\delta) - (1-\delta)\, E_N(\delta), \\ \delta\,(1-\delta)\, D_{N-1}(\delta) &= (1-\delta)\, E_N(\delta) - (1-\delta)^2\, E_{N-1}(\delta), \\ \delta\,(1-\delta)^2\, D_{N-2}(\delta) &= (1-\delta)^2\, E_{N-1}(\delta) - (1-\delta)^3\, E_{N-2}(\delta), \\ &\vdots \\ \delta\,(1-\delta)^{N-1}\, D_1(\delta) &= (1-\delta)^{N-1}\, E_2(\delta) - (1-\delta)^N\, E_1(\delta), \end{aligned}$$

which, after summation implies the formula (4.7).

(4.9): By (3.18) we have

$$\delta \sum_{n=1}^{N} A_n(\delta) = \sum_{n=1}^{N} \left(B_{n+1}(\delta)-B_n(\delta)\right) + \delta \sum_{n=1}^{N} E_n(\delta) - \delta \sum_{n=1}^{N} C_n(\delta).$$

Hence, by (4.1) and (4.3) we get the desired formula

$$= \left(B_{N+1}(\delta)-B_1(\delta)\right) + \left(1 - A_{N+1}(\delta) + 2\, C_{N+1}(\delta) - 2\, E_{N+1}(\delta)\right) - D_{N+1}(\delta).$$

(4.10): From (3.18) we deduce the following system of equalities

$$\begin{aligned} A_{N+1}(\delta) - A_N(\delta) &= 2\,\delta\, B_N(\delta) - \delta\, E_N(\delta), \\ 2\, A_N(\delta) - 2\, A_{N-1}(\delta) &= 4\,\delta\, B_{N-1}(\delta) - 2\,\delta\, E_{N-1}(\delta), \\ 3\, A_{N-1}(\delta) - 3\, A_{N-2}(\delta) &= 6\,\delta\, B_{N-2}(\delta) - 3\,\delta\, E_{N-2}(\delta), \\ &\vdots \\ N\, A_2(\delta) - N\, A_1(\delta) &= 2\,N\,\delta\, B_1(\delta) - N\,\delta\, E_1(\delta), \end{aligned}$$

which, after summation implies the desired formulas. □

Corollary 14. *(See also Corollary 23.) From (4.7) for* $\delta = \pm\frac{1}{2}$ *we get*

$$2^{N+1}\, E_{N+1}\left(\tfrac{1}{2}\right) = \sum_{n=0}^{N-1} 2^n\, D_n\left(\tfrac{1}{2}\right)$$

and

$$-2\left(\tfrac{2}{3}\right)^N E_{N+1}\left(-\tfrac{1}{2}\right) = \sum_{n=0}^{N-1} \left(\tfrac{2}{3}\right)^n D_n\left(-\tfrac{1}{2}\right).$$

5 Reduction formulas for indices

Lemma 15. *The following identities hold:*

$$\begin{cases} A_{m+n} = A_m A_n + 2B_m B_n + 2C_m C_n + 2D_m D_n + 2E_m E_n, \\ B_{m+n} = A_m B_n + B_m A_n + B_m C_n + C_m B_n + C_m D_n + D_m C_n + D_m E_n + E_m D_n, \\ C_{m+n} = A_m C_n + B_m B_n + B_m D_n + D_m B_n + C_m A_n + C_m E_n + E_m C_n, \\ D_{m+n} = A_m D_n + B_m C_n + B_m E_n + D_m A_n + C_m B_n + E_m b_n + E_m E_n, \\ E_{m+n} = A_m E_n + B_m D_n + E_m A_n + D_m B_n + C_m C_n + D_m E_n + E_m D_n, \end{cases} \tag{5.1}$$

and (the special cases when $m = n$):

$$\begin{cases} A_{2n} = A_n^2 + 2B_n^2 + 2C_n^2 + 2D_n^2 + 2E_n^2, \\ B_{2n} = 2A_n B_n + 2B_n C_n + 2C_n D_n + 2D_n E_n, \\ C_{2n} = 2A_n C_n + 2C_n E_n + 2D_n B_n + B_n^2, \\ D_{2n} = 2A_n D_n + 2C_n B_n + 2B_n E_n + E_n^2, \\ E_{2n} = 2A_n E_n + 2E_n D_n + 2D_n B_n + C_n^2. \end{cases} \tag{5.2}$$

The proof runs similar to the proof of Lemma 3.16 from [9].

Lemma 16. *We have the identity:*

$$\Delta := \begin{vmatrix} A_k(\delta) & 2B_k(\delta) - E_k(\delta) & 2C_k(\delta) - D_k(\delta) - E_k(\delta) & -C_k(\delta) + D_k(\delta) & -B_k(\delta) - C_k(\delta) + 2E_k(\delta) \\ B_k(\delta) & A_k(\delta) + C_k(\delta) - E_k(\delta) & B_k(\delta) - E_k(\delta) & -D_k(\delta) + E_k(\delta) & -B_k(\delta) - C_k(\delta) + 2E_k(\delta) \\ C_k(\delta) & B_k(\delta) + D_k(\delta) - E_k(\delta) & A_k(\delta) - D_k(\delta) & B_k(\delta) - C_k(\delta) - D_k(\delta) & -B_k(\delta) \\ D_k(\delta) & C_k(\delta) & B_k(\delta) - D_k(\delta) - E_k(\delta) & A_k(\delta) - C_k(\delta) - D_k(\delta) & -C_k(\delta) + E_k(\delta) \\ E_k(\delta) & D_k(\delta) - E_k(\delta) & C_k(\delta) - D_k(\delta) - E_k(\delta) & B_k(\delta) - C_k(\delta) - D_k(\delta) + E_k(\delta) & A_k(\delta) - B_k(\delta) - C_k(\delta) + D_k(\delta) \end{vmatrix} =$$

$$= \big(- \delta^5 + 3\,\delta^4 + 3\,\delta^3 - 4\,\delta^2 - \delta + 1 \big)^k. \tag{5.3}$$

The proof of this lemma runs like the proof of Lemma 3.21 from [9].

Lemma 17. *The following identities hold:*

$$A_{n-k}(\delta) = \frac{\Delta_1}{\Delta}, \quad B_{n-k}(\delta) = \frac{\Delta_2}{\Delta}, \quad C_{n-k}(\delta) = \frac{\Delta_3}{\Delta}, \quad D_{n-k}(\delta) = \frac{\Delta_4}{\Delta}, \quad E_{n-k}(\delta) = \frac{\Delta_5}{\Delta},$$

where Δ_i arise from the determinant Δ by replacing its i-th column by the vector $(A_n, B_n, C_n, D_n, E_n)^T$ (the Cramers rule for respective system of equations).

The proof is similar to the proof of Lemma 3.20 from [9].

Remark 18. *After multiplication we obtain*

$$\big(\tau_2(1)\big)^{2n}\big(\tau_5(1)\big)^{2n} = \big(2+\sigma_2\big)^n = 2^n\big(\tau_2\big(\tfrac12\big)\big)^n \overset{\text{by (3.4)}}{=}$$
$$= 2^n\big(A_n\big(\tfrac12\big) + B_n\big(\tfrac12\big)\,\sigma_2 + C_n\big(\tfrac12\big)\,\sigma_4 + D_n\big(\tfrac12\big)\,\sigma_5 + E_n\big(\tfrac12\big)\,\sigma_3\big). \quad (5.4)$$

On the other hand, by (3.4) and (3.10) we get:

$$\begin{aligned}
&\big(\tau_2(1)\big)^{2n}\big(\tau_5(1)\big)^{2n} = \\
&= \big(A_{2n} + B_{2n}\,\sigma_2 + C_{2n}\,\sigma_4 + D_{2n}\,\sigma_5 + E_{2n}\,\sigma_3\big)\big(A_{2n} + B_{2n}\,\sigma_5 + C_{2n}\,\sigma_1 + D_{2n}\,\sigma_4 + E_{2n}\,\sigma_2\big) = \\
&= \sigma_2\,\big(A_{2n}\,\big(B_{2n} - C_{2n} + E_{2n}\big) - B_{2n}\,\big(C_{2n} - E_{2n}\big) + E_{2n}\,\big(2\,C_{2n} - D_{2n} - E_{2n}\big)\big) + \\
&+ \sigma_3\,\big(C_{2n}\,\big(C_{2n} - A_{2n} - B_{2n} + D_{2n}\big) + B_{2n}\,\big(B_{2n} - D_{2n}\big) + E_{2n}\,\big(A_{2n} + B_{2n}\big) - D_{2n}^2 - E_{2n}^2\big) + \\
&+ \sigma_4\,\big(B_{2n}\,\big(B_{2n} - 2\,C_{2n} - D_{2n} + E_{2n}\big) + C_{2n}\,\big(D_{2n} + E_{2n}\big) + D_{2n}\,\big(A_{2n} - D_{2n} + E_{2n}\big) - E_{2n}^2\big) + \\
&+ \sigma_5\,\big(A_{2n}\,\big(B_{2n} + D_{2n} - C_{2n}\big) + C_{2n}\,\big(C_{2n} - 2\,B_{2n} + D_{2n} + E_{2n}\big) - D_{2n}\,\big(D_{2n} + E_{2n}\big)\big) + \\
&+ \big(A_{2n}\,\big(A_{2n} - C_{2n}\big) - B_{2n}\,\big(2\,C_{2n} - D_{2n} - 2\,E_{2n}\big) + D_{2n}\,\big(2\,C_{2n} - D_{2n} - E_{2n}\big) - E_{2n}^2\big).
\end{aligned}$$

Hence, comparing the coefficients in the expression σ_m, $m = 0,2,3,4,5$ we obtain, by Lemma 4, the following interesting identities:

$$\begin{aligned}
2^n\,A_n\big(\tfrac12\big) &= A_{2n}\,\big(A_{2n} - C_{2n}\big) - B_{2n}\,\big(2\,C_{2n} - D_{2n} - 2\,E_{2n}\big) && (5.5)\\
&\quad + D_{2n}\,\big(2\,C_{2n} - D_{2n} - E_{2n}\big) - E_{2n}^2, \\
2^n\,B_n\big(\tfrac12\big) &= A_{2n}\,\big(B_{2n} - C_{2n} + E_{2n}\big) - B_{2n}\,\big(C_{2n} - E_{2n}\big) && (5.6)\\
&\quad + E_{2n}\,\big(2\,C_{2n} - D_{2n} - E_{2n}\big), \\
2^n\,C_n\big(\tfrac12\big) &= B_{2n}\,\big(B_{2n} - 2\,C_{2n} - D_{2n} + E_{2n}\big) + C_{2n}\,\big(D_{2n} + E_{2n}\big) && (5.7)\\
&\quad + D_{2n}\,\big(A_{2n} - D_{2n} + E_{2n}\big) - E_{2n}^2, \\
2^n\,D_n\big(\tfrac12\big) &= A_{2n}\,\big(B_{2n} + D_{2n} - C_{2n}\big) && (5.8)\\
&\quad + C_{2n}\,\big(C_{2n} - 2\,B_{2n} + D_{2n} + E_{2n}\big) - D_{2n}\,\big(D_{2n} + E_{2n}\big), \\
2^n\,E_n\big(\tfrac12\big) &= C_{2n}\,\big(C_{2n} - A_{2n} - B_{2n} + D_{2n}\big) + B_{2n}\,\big(B_{2n} - D_{2n}\big) && (5.9)\\
&\quad + E_{2n}\,\big(A_{2n} + B_{2n}\big) - D_{2n}^2 - E_{2n}^2.
\end{aligned}$$

6 Some new recurrence formulas

The following recurrence formulas determined the coefficients of the polynomials $A_n(\Delta)$, $B_n(\Delta)$, …, $E_n(\Delta)$.

Lemma 19. *Let $\Delta \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$. Then the following recurrence formulas hold*

$$\begin{aligned}
(-1)^n\,A_n(\Delta) &= \Delta^n\,\big(A_n\,(A_n - B_n - D_n) + B_n\,(D_n + 2\,E_n) + C_n\,(D_n - C_n - E_n) - E_n^2\big) \\
&\quad + \sum_{k=0}^{n-1}\binom{n}{k}\,(-1)^{k+1}\,A_k(\Delta); && (6.1)
\end{aligned}$$

$$(-1)^n\, B_n(\Delta) = \Delta^n \big(C_n\,(A_n + 2\,B_n - C_n - D_n - E_n) + D_n\,(D_n + E_n - A_n) - A_n\,B_n\big) + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{k+1}\, B_k(\Delta); \tag{6.2}$$

$$(-1)^n\, C_n(\Delta) = \Delta^n \big(C_n\,(B_n - C_n - D_n + E_n) + D_n\,(D_n - A_n) + E_n\,(A_n + B_n - E_n)\big) + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{k+1}\, C_k(\Delta); \tag{6.3}$$

$$(-1)^n\, D_n(\Delta) = \Delta^n \big(B_n\,(B_n + C_n + E_n - A_n - D_n) + E_n\,(A_n + D_n - C_n - E_n) - A_n\,D_n\big) + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{k+1}\, D_k(\Delta); \tag{6.4}$$

$$(-1)^n\, E_n(\Delta) = \Delta^n \big(B_n\,(B_n + E_n - A_n - D_n) + C_n\,(A_n - C_n) + E_n\,(2\,D_n - E_n)\big) + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{k+1}\, E_k(\Delta). \tag{6.5}$$

Proof. We have the equality:

$$\tau_1(1)\,\tau_3(1) = -\sigma_5.$$

Hence we get ($\delta \in \mathbb{C} \setminus \{0\}$):

$$\tau_1^n(1)\,\tau_3^n(1) = \big(\delta - (\delta + \sigma_5)\big)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, \delta^{n-k}\,(\delta + \sigma_5)^k = \delta^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, \tau_5^k\big(\tfrac{1}{\delta}\big),$$

which by (3.2), (3.6) and (3.10), implies the identity

$$\big(A_n + B_n\,\sigma_1 + C_n\,\sigma_2 + D_n\,\sigma_3 + E_n\,\sigma_4\big)\big(A_n + B_n\,\sigma_3 + C_n\,\sigma_5 + D_n\,\sigma_2 + E_n\,\sigma_1\big) = \\ = \delta^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \big(A_n(\tfrac{1}{\delta}) + B_n(\tfrac{1}{\delta})\,\sigma_5 + C_n(\tfrac{1}{\delta})\,\sigma_1 + D_n(\tfrac{1}{\delta})\,\sigma_4 + E_n(\tfrac{1}{\delta})\,\sigma_2\big).$$

After the calculation of the left side of the above identity and the following substitution:

$$\sigma_3 = -1 - \sigma_1 - \sigma_2 - \sigma_4 - \sigma_5$$

and making use of the linear independence over $\mathbb{Q}$ of the numbers 1, σ_1, σ_2, σ_4, σ_5 (see Corollary 3), we obtain:

$$\delta^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, A_k(\tfrac{1}{\delta}) = A_n\,(A_n - B_n - D_n) + B_n\,(D_n + 2\,E_n) + \\ + C_n\,(D_n - C_n - E_n) - E_n^2$$

etc., which, after the substitution of $\Delta := 1/\delta$ and minor transformations, implies identities (6.1)–(6.5). □

Corollary 20. *If* $\deg\big(A_n(\Delta)\big)=n$ *then*

$$\operatorname{coeff}\big(A_n(\Delta);\Delta^n\big)=(-1)^n\big(A_n\,\big(A_n-B_n-D_n\big)+B_n\,\big(D_n+2\,E_n\big)+C_n\,\big(D_n-C_n-E_n\big)-E_n^2\big)$$

and

$$\operatorname{coeff}\big(B_n(\Delta);\Delta^n\big)=(-1)^n\big(C_n\,\big(A_n+2\,B_n-C_n-D_n-E_n\big)+D_n\,\big(D_n+E_n-A_n\big)-A_n\,B_n\big).$$

Problem. *Is it true that*

$$\deg\big(A_n(\Delta)\big)=\deg\big(B_n(\Delta)\big)=\deg\big(C_n(\Delta)\big)=\deg\big(D_n(\Delta)\big)=\deg\big(E_n(\Delta)\big)=n$$

for every $n=5,6,\ldots$*?*

7 Some convolution type identities

Lemma 21. *We have*

$$\tau_1(1)\,\tau_2(1)\,\tau_3(1)\;=1+\sigma_1+\sigma_2.$$

Hence, for $\delta\in\mathbb{C}$, $\delta\,(1-\delta)\neq 0$, *we obtain*

$$\tau_1^n(1)\,\tau_2^n(1)\,\tau_3^n(1)\;=\Big(\big(1-\delta+\sigma_1\big)+\big(\delta+\sigma_2\big)\Big)^n=$$
$$=\sum_{k=0}^{n}\binom{n}{k}(1-\delta)^k\,\delta^{n-k}\,\tau_1^k\big(1/(1-\delta)\big)\,\tau_2^{n-k}\big(1/\delta\big),$$

which by (3.2), (3.4) and (3.6), implies the identity:

$$\big(A_n+B_n\,\sigma_1+C_n\,\sigma_2+D_n\,\sigma_3+E_n\,\sigma_4\big)\,\big(A_n+B_n\,\sigma_2+C_n\,\sigma_4+D_n\,\sigma_5+E_n\,\sigma_3\big)\times$$
$$\times\,\big(A_n+B_n\,\sigma_3+C_n\,\sigma_5+D_n\,\sigma_2+E_n\,\sigma_1\big)=$$
$$=\sum_{k=0}^{n}\binom{n}{k}(1-\delta)^k\,\delta^{n-k}\,\Big(A_k\big(\tfrac{1}{1-\delta}\big)+B_k\big(\tfrac{1}{1-\delta}\big)\,\sigma_1+C_k\big(\tfrac{1}{1-\delta}\big)\,\sigma_2+D_k\big(\tfrac{1}{1-\delta}\big)\,\sigma_3$$
$$+\,E_k\big(\tfrac{1}{1-\delta}\big)\,\sigma_4\Big)\,\Big(A_{n-k}\big(\tfrac{1}{\delta}\big)+B_{n-k}\big(\tfrac{1}{\delta}\big)\,\sigma_2+C_{n-k}\big(\tfrac{1}{\delta}\big)\,\sigma_4+D_{n-k}\big(\tfrac{1}{\delta}\big)\,\sigma_5+E_{n-k}\big(\tfrac{1}{\delta}\big)\,\sigma_3\Big).$$

Hence, after some calculation, by comparing the absolute terms and the coefficients in the expressions σ_1, σ_2, σ_3 *and* σ_4 *we get the following five independent identities:*

$$A_n^3+B_n^3+2\,C_n^3+D_n^3+E_n^3-A_n^2\,\big(C_n+D_n\big)-B_n^2\,\big(A_n+D_n\big)+C_n^2\,\big(D_n-A_n-B_n-2\,E_n\big)+$$
$$+\,D_n^2\,\big(2\,E_n-A_n-B_n-3\,C_n\big)-E_n^2\,\big(A_n+2\,B_n\big)+A_n\,B_n\,\big(D_n+2\,E_n-C_n\big)+$$
$$+\,A_n\,D_n\,\big(3\,C_n-2\,E_n\big)+B_n\,C_n\,\big(2\,D_n+E_n\big)+D_n\,E_n\,\big(B_n-2\,C_n\big)=$$
$$=\sum_{k=0}^{n}\binom{n}{k}(1-\delta)^k\,\delta^{n-k}\,\Big[A_k\big(\tfrac{1}{1-\delta}\big)\,\Big(A_{n-k}\big(\tfrac{1}{\delta}\big)-D_{n-k}\big(\tfrac{1}{\delta}\big)\Big)+$$
$$+\,C_k\big(\tfrac{1}{1-\delta}\big)\,\Big(2\,B_{n-k}\big(\tfrac{1}{\delta}\big)-C_{n-k}\big(\tfrac{1}{\delta}\big)-E_{n-k}\big(\tfrac{1}{\delta}\big)\Big)-B_k\big(\tfrac{1}{1-\delta}\big)\,\Big(D_{n-k}\big(\tfrac{1}{\delta}\big)+C_{n-k}\big(\tfrac{1}{\delta}\big)\Big)+$$
$$+\,D_k\big(\tfrac{1}{1-\delta}\big)\,\Big(E_{n-k}\big(\tfrac{1}{\delta}\big)-B_{n-k}\big(\tfrac{1}{\delta}\big)\Big)+E_k\big(\tfrac{1}{1-\delta}\big)\,\Big(2\,C_{n-k}\big(\tfrac{1}{\delta}\big)-B_{n-k}\big(\tfrac{1}{\delta}\big)\Big)\Big],\quad (7.1)$$

$$
\begin{gathered}
B_n^3 - C_n^3 - D_n^3 + E_n^3 + A_n^2 \left(C_n + D_n - E_n\right) + B_n^2 \left(C_n - A_n\right) + C_n^2 \left(A_n + D_n - E_n\right) + \\
+ D_n^2 \left(C_n - B_n + E_n\right) - E_n^2 \left(2\,B_n + 2\,D_n\right) + A_n\,B_n \left(E_n - 2\,C_n - D_n\right) + \\
+ D_n\,E_n \left(A_n + 3\,B_n + C_n\right) - C_n \left(A_n\,E_n + B_n\,D_n\right) = \\
= \sum_{k=0}^{n} \binom{n}{k} (1-\delta)^k\,\delta^{n-k} \Big[A_k\big(\tfrac{1}{1-\delta}\big)\,B_{n-k}\big(\tfrac{1}{\delta}\big) + B_k\big(\tfrac{1}{1-\delta}\big) \Big(E_{n-k}\big(\tfrac{1}{\delta}\big) - A_{n-k}\big(\tfrac{1}{\delta}\big) - B_{n-k}\big(\tfrac{1}{\delta}\big)\Big) + \\
+ C_k\big(\tfrac{1}{1-\delta}\big) \Big(A_{n-k}\big(\tfrac{1}{\delta}\big) + C_{n-k}\big(\tfrac{1}{\delta}\big) - E_{n-k}\big(\tfrac{1}{\delta}\big)\Big) + D_k\big(\tfrac{1}{1-\delta}\big) \Big(D_{n-k}\big(\tfrac{1}{\delta}\big) - B_{n-k}\big(\tfrac{1}{\delta}\big) - C_{n-k}\big(\tfrac{1}{\delta}\big)\Big) + \\
+ E_k\big(\tfrac{1}{1-\delta}\big) \Big(B_{n-k}\big(\tfrac{1}{\delta}\big) - E_{n-k}\big(\tfrac{1}{\delta}\big)\Big)\Big], \quad (7.2)
\end{gathered}
$$

$$
\begin{gathered}
E_n^3 - D_n^3 + A_n^2\,D_n + B_n^2 \left(3\,C_n - A_n - 2\,E_n\right) - C_n^2\,B_n + D_n^2 \left(A_n + B_n + C_n + E_n\right) + \\
+ A_n\,E_n \left(C_n - 2\,D_n\right) - E_n^2 \left(3\,C_n + 2\,D_n\right) + B_n\,E_n \left(2\,C_n + 4\,D_n - A_n\right) + C_n\,D_n \left(2\,E_n - 5\,B_n\right) = \\
= \sum_{k=0}^{n} \binom{n}{k} (1-\delta)^k\,\delta^{n-k} \Big[A_k\big(\tfrac{1}{1-\delta}\big)\,E_{n-k}\big(\tfrac{1}{\delta}\big) + B_k\big(\tfrac{1}{1-\delta}\big) \Big(C_{n-k}\big(\tfrac{1}{\delta}\big) - A_{n-k}\big(\tfrac{1}{\delta}\big)\Big) + \\
+ C_k\big(\tfrac{1}{1-\delta}\big) \Big(D_{n-k}\big(\tfrac{1}{\delta}\big) - E_{n-k}\big(\tfrac{1}{\delta}\big)\Big) + D_k\big(\tfrac{1}{1-\delta}\big) \Big(A_{n-k}\big(\tfrac{1}{\delta}\big) - B_{n-k}\big(\tfrac{1}{\delta}\big) - C_{n-k}\big(\tfrac{1}{\delta}\big) + D_{n-k}\big(\tfrac{1}{\delta}\big)\Big) + \\
+ E_k\big(\tfrac{1}{1-\delta}\big) \Big(C_{n-k}\big(\tfrac{1}{\delta}\big) - D_{n-k}\big(\tfrac{1}{\delta}\big) - E_{n-k}\big(\tfrac{1}{\delta}\big)\Big)\Big], \quad (7.3)
\end{gathered}
$$

$$
\begin{gathered}
B_n^3 + A_n^2 \left(C_n - B_n\right) + B_n^2 \left(C_n - A_n - D_n - E_n\right) + C_n^2 \left(D_n - 3\,B_n\right) + D_n^2 \left(E_n - C_n\right) + \\
+ E_n^2 \left(A_n - 3\,D_n\right) + A_n\,C_n \left(2\,B_n + D_n - 3\,E_n\right) + B_n\,C_n \left(E_n - D_n\right) + \\
+ D_n\,E_n \left(5\,B_n + C_n\right) = \sum_{k=0}^{n} \binom{n}{k} (1-\delta)^k\,\delta^{n-k} \Big[A_k\big(\tfrac{1}{1-\delta}\big)\,C_{n-k}\big(\tfrac{1}{\delta}\big) + \\
+ B_k\big(\tfrac{1}{1-\delta}\big) \Big(D_{n-k}\big(\tfrac{1}{\delta}\big) - A_{n-k}\big(\tfrac{1}{\delta}\big) - B_{n-k}\big(\tfrac{1}{\delta}\big) + E_{n-k}\big(\tfrac{1}{\delta}\big)\Big) + \\
+ C_k\big(\tfrac{1}{1-\delta}\big) \Big(B_{n-k}\big(\tfrac{1}{\delta}\big) + D_{n-k}\big(\tfrac{1}{\delta}\big) - E_{n-k}\big(\tfrac{1}{\delta}\big)\Big) - D_k\big(\tfrac{1}{1-\delta}\big)\,B_{n-k}\big(\tfrac{1}{\delta}\big) + \\
+ E_k\big(\tfrac{1}{1-\delta}\big) \Big(A_{n-k}\big(\tfrac{1}{\delta}\big) - D_{n-k}\big(\tfrac{1}{\delta}\big)\Big)\Big], \quad (7.4)
\end{gathered}
$$

etc.

8 Polynomials associated with quasi-Fibonacci numbers of order 11

By taking into account the Chebyshev polynomials $T_n(x)$ again we obtain, for $\tau_k = \tau_k(\delta)$, $k = 1, 2, \ldots, 5$:

$$\begin{cases} \sigma_k \ = 2\cos\frac{2k\pi}{11} = \frac{1}{\delta}(\tau_k - 1), \\ \sigma_{2k} = 2\cos\frac{4k\pi}{11} = 2T_2\big(\cos\frac{2k\pi}{11}\big) = 2T_2\big(\frac{1}{2\delta}(\tau_k-1)\big) = \frac{1}{\delta^2}(\tau_k^2 - 2\tau_k + 1 - 2\delta^2), \\ \sigma_{3k} = 2\cos\frac{6k\pi}{11} = 2T_3\big(\cos\frac{2k\pi}{11}\big) = 2T_3\big(\frac{1}{2\delta}(\tau_k-1)\big) = \\ \quad = \frac{1}{\delta^3}\big(\tau_k^3 - 3\tau_k^2 + (3-3\delta^2)\tau_k - 1 + 3\delta^2\big), \\ \sigma_{4k} = 2\cos\frac{8k\pi}{11} = 2T_4\big(\cos\frac{2k\pi}{11}\big) = 2T_4\big(\frac{1}{2\delta}(\tau_k-1)\big) = \\ \quad = \frac{1}{\delta^4}\big(\tau_k^4 - 4\tau_k^3 + (6-4\delta^2)\tau_k^2 + (-4+8\delta^2)\tau_k + 1 - 4\delta^2 + 2\delta^4\big), \\ \sigma_{5k} = 2\cos\frac{10k\pi}{11} = 2T_5\big(\cos\frac{2k\pi}{11}\big) = \frac{1}{\delta^5}\big(\tau_k^5 - 5\tau_k^4 + (10-5\delta^2)\tau_k^3 + \\ \qquad + (-10+15\delta^2)\tau_k^2 + (10 - 30\delta^2 + 5\delta^4)\tau_k - 6 + 20\delta^2 - 5\delta^4\big). \end{cases} \tag{8.1}$$

Therefore, immediately from equation (3.1) we obtain the equality:

$$\begin{aligned} \tau_k^n = & A_n(\delta) + \frac{1}{\delta}B_n(\delta)(\tau_k - 1) + \frac{1}{\delta^2}C_n(\delta)(\tau_k^2 - 2\tau_k + 1 - 2\delta^2) \\ & + \frac{1}{\delta^3}D_n(\delta)(\tau_k^3 - 3\tau_k^2 + (3-3\delta^2)\tau_k - 1 + 3\delta^2) \\ & + \frac{1}{\delta^4}E_n(\delta)(\tau_k^4 - 4\tau_k^3 + (6-4\delta^2)\tau_k^2 + (-4+8\delta^2)\tau_k + 1 - 4\delta^2 + 2\delta^4), \end{aligned}$$

i.e.,

$$\begin{aligned} W_{n,11}(\tau_k;\delta) := & \ \tau_k^n - \tfrac{1}{\delta^4}E_n(\delta)\tau_k^4 + \big[-\tfrac{1}{\delta^3}D_n(\delta) + \tfrac{4}{\delta^4}E_n(\delta)\big]\tau_k^3 \\ & + \big[\tfrac{1}{\delta^4}(-6+4\delta^2)E_n(\delta) + \tfrac{3}{\delta^3}D_n(\delta) - \tfrac{1}{\delta^2}C_n(\delta)\big]\tau_k^2 \\ & + \big[\tfrac{1}{\delta^4}(4-8\delta^2)E_n(\delta) + \tfrac{1}{\delta^3}(-3+3\delta^2)D_n(\delta) + \tfrac{2}{\delta^2}C_n(\delta) - \tfrac{1}{\delta}B_n(\delta)\big]\tau_k \\ & + \big[\tfrac{1}{\delta^4}(-1+4\delta^2-2\delta^4)E_n(\delta) + \tfrac{1}{\delta^3}(1-3\delta^2)D_n(\delta) + \tfrac{1}{\delta^2}(-1+2\delta^2)C_n(\delta) \\ & + \tfrac{1}{\delta}B_n(\delta) - A_n(\delta)\big] = 0, \end{aligned}$$

which means that:

$$p_{11}(x;\delta)\Big| W_{n,11}(x;\delta), \quad n \geqslant 5.$$

For example for $\delta = 1$, we have the relation:

$$\begin{aligned} (x^5 - 4x^4 + 2x^3 + 5x^2 - 2x - 1)\Big|\Big[& x^n - E_n x^4 + (4E_n - D_n)x^3 + \\ & + (-2E_n + 3D_n - C_n)x^2 + (-4E_n + 2C_n - B_n)x + E_n - 2D_n + C_n - A_n\Big] \end{aligned}$$

for all $n \geqslant 5$.

More precisely, the following decomposition can be deduced:

Lemma 22. *We have:*

$$p_{11}(x;\delta)\Big(\sum_{k=1}^{n-4}E_k(\delta)x^{n-4-k}\Big)=W_{n,11}(x;\delta). \tag{8.2}$$

For $\delta=1$ we obtain special decomposition:

$$\begin{aligned}(x^5-4x^4+2x^3+5x^2-2x-1)\Big(\sum_{k=3}^{n-2}D_kx^{n-2-k}\Big)&=\\ =x^n-E_nx^4+(4E_n-D_n)x^3&+(-2E_n+3D_n-C_n)x^2+\\ &+(-4E_n+2C_n-B_n)x+E_n-2D_n+C_n+B_n-A_n,\end{aligned} \tag{8.3}$$

and, after differentiating of (8.3):

$$\begin{aligned}(5x^4-16x^3+6x^2+10x-2)\Big(\sum_{k=3}^{n-2}D_kx^{n-2-k}\Big)+&\\ +(x^5-4x^4+2x^3+5x^2-2x-1)\Big(\sum_{k=3}^{n-2}D_kx^{n-2-k}\Big)'&=\\ =n\,x^{n-1}-4E_nx^3+3(4E_n-D_n)x^2+2(-C_n+3D_n-2E_n)x-B_n+2C_n-4E_n.&\end{aligned} \tag{8.4}$$

Hence, by applying (8.3), we get

$$\begin{aligned}(x^5-4x^4+2x^3+5x^2-2x-1)^2\Big(\sum_{k=3}^{n-3}(n-2-k)D_kx^{n-3-k}\Big)&=\\ =\big(x^5-4x^4+2x^3+5x^2-2x-1\big)\Big(n\,x^{n-1}-4E_nx^3+3(4E_n-D_n)x^2+&\\ +2(-C_n+3D_n-2E_n)x-B_n+2C_n-4E_n\Big)+&\\ +\big(-5x^4+16x^3-6x^2-10x+2\big)\Big(x^n-E_nx^4+(4E_n-D_n)x^3+(-2E_n+3D_n-C_n)x^2+&\\ +(-4E_n+2C_n-B_n)x+E_n-2D_n+C_n+B_n-A_n\Big).&\end{aligned} \tag{8.5}$$

Corollary 23. *We have:*

$$\sum_{k=3}^{n-2}D_k=-A_n+2C_n-2E_n+1, \tag{8.6}$$

$$(-1)^{n+1}\sum_{k=3}^{n-2}(-1)^kD_k=(-1)^n-A_n+2B_n-2C_n+2D_n-2E_n, \tag{8.7}$$

$$\sum_{k=3}^{n-3}(n-2-k)D_k=n-3+3A_n-B_n-6C_n+3D_n+6E_n, \tag{8.8}$$

$$\begin{aligned}(-1)^{n+1}\sum_{k=3}^{n-3}(-1)^k(n-2-k)D_k=(-1)^n(n-15)+15A_n-29B_n+&\\ +26C_n-21D_n+14E_n,&\end{aligned} \tag{8.9}$$

and

$$\sum_{k=3}^{n-2} kD_k = 1-(n+1)A_n + B_n + 2(n+1)C_n - 3D_n - 2(n+1)E_n. \qquad (8.10)$$

9 Some properties of zeros of polynomials $\mathcal{D}_n(x)$

Let us set:

$$\mathcal{D}_n(x) = \sum_{k=1}^{n} D_k x^{n-k}, \quad n \in \mathbb{N}.$$

Then,

$$\mathcal{D}_1(x) \equiv 0, \qquad \mathcal{D}_2(x) \equiv 0, \qquad \mathcal{D}_3(x) \equiv 1,$$
$$\mathcal{D}_4(x) = x+4, \qquad \mathcal{D}_5(x) = x^2+4x+14,$$

and we obtain the recurrence relations (see (3.21)):

$$\begin{aligned}\mathcal{D}_{n+5}(x) = x^{n+2} + 4\mathcal{D}_{n+4}(x) - 2\mathcal{D}_{n+3}(x) - 5\mathcal{D}_{n+2}(x) + \\ + 2\mathcal{D}_{n+1}(x) + \mathcal{D}_n(x), \qquad n \geqslant 1.\end{aligned}$$

All polynomials $\mathcal{D}_n(x)$ for $n = 2k+1$, $k \in \mathbb{N}$, $k \geqslant 2$, have an even degree, for example:

$$\begin{aligned}\mathcal{D}_7(x) &= x^4 + 4x^3 + 14x^2 + 43x + 126,\\ \mathcal{D}_9(x) &= x^6 + 4x^5 + 14x^4 + 43x^3 + 126x^2 + 357x + 993.\end{aligned}$$

As follows from calculation, these polynomials have no real roots for $n \leqslant 200$.

However, all polynomials $\mathcal{D}_n(x)$ for $n = 2k$, $k \in \mathbb{N}$, $k \geqslant 2$, have an odd degree, for example:

$$\begin{aligned}\mathcal{D}_6(x) &= x^3 + 4x^2 + 14x + 43,\\ \mathcal{D}_8(x) &= x^5 + 4x^4 + 14x^3 + 43x^2 + 126x + 357.\end{aligned}$$

Remark 24. *Polynomials $\mathcal{D}_{2k}(x)$ have exactly one real root s_k, which is less than zero (for $k \leqslant 100$). The roots s_k form an increasing sequence from $s_2 = -4$ to $s_{100} = -2.699746$ (by numerical calculation). All numerical calculations were performed using Mathematica.*[1]

10 Jordan decomposition

For sequences $A_n(\delta)$, $B_n(\delta)$, $C_n(\delta)$, $D_n(\delta)$ and $E_n(\delta)$ we have the identity:

$$\begin{bmatrix} A_{n+1}(\delta) \\ B_{n+1}(\delta) \\ C_{n+1}(\delta) \\ D_{n+1}(\delta) \\ E_{n+1}(\delta) \end{bmatrix} = \mathcal{W}(\delta) \begin{bmatrix} A_n(\delta) \\ B_n(\delta) \\ C_n(\delta) \\ D_n(\delta) \\ E_n(\delta) \end{bmatrix}, \qquad n \in \mathbb{N}, \qquad (10.1)$$

[1] *Mathematica* is registered trademark of Wolfram Research Inc.

where

$$\mathcal{W}(\delta)=\begin{bmatrix} 1 & 2\,\delta & 0 & 0 & -\delta \\ \delta & 1 & \delta & 0 & -\delta \\ 0 & \delta & 1 & \delta & -\delta \\ 0 & 0 & \delta & 1 & 0 \\ 0 & 0 & 0 & \delta & 1-\delta \end{bmatrix}. \tag{10.2}$$

Matrix $\mathcal{W}(\delta)$ is diagonalizable, because of the following decomposition:

$$\mathcal{W}(\delta)=A\cdot \operatorname{diag}\big[\tau_1(\delta),\tau_2(\delta),\ldots,\tau_5(\delta)\big]\cdot A^{-1}, \tag{10.3}$$

where:

$$A=\big[K_1,K_2,K_3,K_4,K_5\big]$$

and

$$K_m=\big[\sigma_m^{-4}+2\,\sigma_m^{-3}-2\,\sigma_m^{-2}-2\,\sigma_m^{-1},\sigma_m^{-2}-\sigma_m^{-1}-1,-\sigma_m^{-1}-\sigma_m^{-2},-\sigma_m^{-3}-\sigma_m^{-2},-\sigma_m^{-3}\big]^T,$$

for every $m=1,2,\ldots,5$. We note, that:

$$A^{-1}=\frac{1}{11}\big[L_1,L_2,L_3,L_4,L_5\big]\cdot\begin{bmatrix} 11 & -3 & -20 & 8 & 16 \\ -2 & 2 & 7 & -7 & -11 \\ -13 & 7 & 22 & -15 & -17 \\ 1 & -1 & -3 & 3 & 4 \\ 3 & -2 & -5 & 4 & 4 \end{bmatrix},$$

where

$$L_m=\big[\sigma_1^{m+3},\sigma_2^{m+3},\sigma_3^{m+3},\sigma_4^{m+3},\sigma_5^{m+3}\big]^T,$$

for $m=1,2,\ldots,5$.

Notice, that the characteristic polynomial $w(\lambda)$ of the matrix $\mathcal{W}(\delta)$ has the form:

$$w(\lambda)=p_{11}\Big(\lambda;\delta\Big).$$

11 Final remarks

Quasi-Fibonacci numbers of order (7, δ) and (11, δ) constitute a special case of the so-called quasi-Fibonacci numbers of order (k, δ), where k is an odd positive integer, $k\geq 5$ and $\delta\in\mathbb{C}$. These numbers are defined to be the elements of the following sequences of polynomials:

$$\big\{a_{n,i}(\delta)\big\}_{n\in\mathbb{N}}\subset\mathbb{Z}[\delta],\qquad i=1,2,\ldots,\frac{1}{2}\,\varphi(k),$$

where φ is the Euler's totient function, which are determined by the following relations:

$$\big(1+\delta\,(\xi^l+\xi^{k-l})\big)^n=a_{n,1}(\delta)+\sum_{\substack{i\in\{2,3,\ldots,(k-1)/2\}\\(i,k)=1}}a_{n,f(i)}(\delta)\,(\xi^{i\,l}+\xi^{i\,(k-l)}) \tag{11.1}$$

for $l=1,2,\ldots,\frac{k-1}{2}$, $(l,k)=1$, $n\in\mathbb{N}$, where $\xi\in\mathbb{C}$ is a primitive root of unity of order k and where f is the increasing bijection of the set $\{i\in\mathbb{N}:\ 2\le i\le (k-1)/2 \ \text{ and } \ (i,k)=1\}$ onto the set $\{2,3,\ldots,\varphi(k)/2\}$. In the sequel, if k is a prime number, relations (11.1) have the following simpler form:

$$\big(1+\delta\,(\xi^{l}+\xi^{k-l})\big)^{n} = a_{n,1}(\delta) + \sum_{i=1}^{(k-3)/2} a_{n,i+1}(\delta)\,(\xi^{i\,l}+\xi^{i\,(k-l)}) \tag{11.2}$$

for $l=1,2,\ldots,\frac{k-1}{2}$ and $n\in\mathbb{N}$.

Quasi-Fibonacci numbers of order $(k,\,\delta)$ possess many interesting properties and natural applications. We remark that these are for the most cases derived immediately from relations (11.1). For example the quasi-Fibonacci numbers enable us to describe the coefficients of the following polynomials in the pure algebraic language:

$$\prod_{\substack{l\in\{1,2,\ldots,k\}\\ (l,2k+1)=1}} \Big(\mathbb{X}-\cos^{n}\big(\tfrac{2\,l\,\pi}{2\,k+1}\big)\Big), \qquad \prod_{\substack{l\in\{1,2,\ldots,k\}\\ (l,2k+1)=1}} \Big(\mathbb{X}-\sin\big(\tfrac{2\,l\,\pi}{2\,k+1}\big)\,\cos^{n}\big(\tfrac{2\,l\,\pi}{2\,k+1}\big)\Big),$$

etc. (see for example [9, 10]). It is possible also to do it for many other trigonometric sums, for example, in [10] the following identity was derived (plus twelve others like this one):

$$2\,\cos(\tfrac{2\pi}{7})\,\sqrt[3]{2\cos(\tfrac{4\pi}{7})}+2\,\cos(\tfrac{4\pi}{7})\,\sqrt[3]{2\cos(\tfrac{8\pi}{7})}+2\,\cos(\tfrac{8\pi}{7})\,\sqrt[3]{2\cos(\tfrac{2\pi}{7})}=\sqrt[3]{-2-3\,\sqrt[3]{49}};$$

which is a variation of the known identity of Ramanujan (see [1]):

$$\sqrt[3]{\cos\big(\tfrac{2\pi}{7}\big)}+\sqrt[3]{\cos\big(\tfrac{4\pi}{7}\big)}+\sqrt[3]{\cos\big(\tfrac{8\pi}{7}\big)}=\sqrt[3]{\tfrac{1}{2}\big(5-3\,\sqrt[3]{7}\big)}\,.$$

It seems almost definite that some types of the identities will be characteristic of only a certain kind of the quasi-Fibonacci numbers of order $(k,\,\delta)$, with respect to the odd $k\in\mathbb{N}$, $k\ge 5$. Accordingly, it is relevant to analyze these numbers separately and independently, for different values of k.

We note that quasi-Fibonacci numbers of order $(5,\,1)$ are equal to the classical Fibonacci numbers. Moreover, for simplicity of notation, the quasi-Fibonacci numbers of order $(11,\,\delta)$ discussed in our paper are denoted by (see the relations (11.2) above for $k=11$ and the relations $(3.2m)$ for $m=1,2,3,4,5$ in Section 3):

$$\begin{array}{lll} A_n(\delta):=a_{n,1}(\delta) & B_n(\delta):=a_{n,2}(\delta) & C_n(\delta):=a_{n,3}(\delta)\\ D_n(\delta):=a_{n,4}(\delta) & E_n(\delta):=a_{n,5}(\delta). & \end{array}$$

12 Acknowledgements

The authors wish to thank the reviewer for his valuable criticisms and suggestions, leading to the present improved version of our paper.

Table 1:

n	$A_n(\delta) = A_{11,n}(\delta)$	$D_n(\delta) = D_{11,n}(\delta)$
0	1	0
1	1	0
2	$2\delta^2 + 1$	0
3	$6\delta^2 + 1$	δ^3
4	$6\delta^4 + 12\delta^2 + 1$	$4\delta^3$
5	$-\delta^5 + 30\delta^4 + 20\delta^2 + 1$	$4\delta^5 + 10\delta^3$
6	$19\delta^6 - 6\delta^5 + 90\delta^4 + 30\delta^2 + 1$	$-\delta^6 + 24\delta^5 + 20\delta^3$
7	$-7\delta^7 + 133\delta^6 - 21\delta^5 + 210\delta^4 + 42\delta^2 + 1$	$14\delta^7 - 7\delta^6 + 84\delta^5 + 35\delta^3$
n	$B_n(\delta) = B_{11,n}(\delta)$	$E_n(\delta) = E_{11,n}(\delta)$
0	0	0
1	δ	0
2	2δ	0
3	$3\delta^3 + 3\delta$	0
4	$12\delta^3 + 4\delta$	δ^4
5	$9\delta^5 + 30\delta^3 + 5\delta$	$-\delta^5 + 5\delta^4$
6	$-\delta^6 + 54\delta^5 + 60\delta^3 + 6\delta$	$5\delta^6 - 6\delta^5 + 15\delta^4$
7	$28\delta^7 - 7\delta^6 + 189\delta^5 + 105\delta^3 + 7\delta$	$-6\delta^7 + 35\delta^6 - 21\delta^5 + 35\delta^4$
n	$C_n(\delta) = C_{11,n}(\delta)$	
0	0	
1	0	
2	δ^2	
3	$3\delta^2$	
4	$4\delta^4 + 6\delta^2$	
5	$-\delta^5 + 20\delta^4 + 10\delta^2$	
6	$14\delta^6 - 6\delta^5 + 60\delta^4 + 15\delta^2$	
7	$-7\delta^7 + 98\delta^6 - 21\delta^5 + 140\delta^4 + 21\delta^2$	

Table 2:

n	A_n	B_n	C_n	D_n	E_n
0	1	0	0	0	0
1	1	1	0	0	0
2	3	2	1	0	0
3	7	6	3	1	0
4	19	16	10	4	1
5	50	44	29	14	4
6	134	119	83	43	14
7	358	322	231	126	43
8	959	868	636	357	126
9	2569	2337	1735	993	357
10	6886	6284	4708	2728	993
11	18461	16885	12727	7436	2728

Table 3:

n	$\mathcal{A}_n(1)$	$\mathcal{A}_n(1/2)$	$\mathcal{A}_n(-1/2)$
0	5	5	5
1	4	9/2	11/2
2	12	25/4	33/4
3	25	39/4	55/4
4	64	257/16	385/16
5	159	437/16	693/16
6	411	1517/32	2541/32
7	1068	2671/32	4719/32
8	2808	38017/256	70785/256
9	7423	68169/256	133705/256
10	19717	245935/512	508079/512
11	52529	1782735/2048	3879865/2048

Table 4:

n	$A_n(1/2)$	$B_n(1/2)$	$C_n(1/2)$	$D_n(1/2)$	$E_n(1/2)$
0	1	0	0	0	0
1	1	1/2	0	0	0
2	3/2	1	1/4	0	0
3	5/2	15/8	3/4	1/8	0
4	35/8	7/2	7/4	1/2	1/16
5	251/32	209/32	119/32	11/8	9/32
6	911/64	779/64	241/32	207/64	53/64
7	3327/128	1449/64	1897/128	7	65/32
8	6095/128	1345/32	229/8	3689/256	289/64
9	5593/64	19941/256	27949/512	7353/256	4845/512
10	164407/1024	36903/256	105641/1024	57361/1024	19551/1024
11	604487/2048	545721/2048	12397/64	220363/2048	4807/128

Table 5:

n	$A_n(-1/2)$	$B_n(-1/2)$	$C_n(-1/2)$	$D_n(-1/2)$	$E_n(-1/2)$
0	1	0	0	0	0
1	1	-1/2	0	0	0
2	3/2	-1	1/4	0	0
3	5/2	-15/8	3/4	-1/8	0
4	35/8	-7/2	7/4	-1/2	1/16
5	253/32	-209/32	121/32	-11/8	11/32
6	935/64	-781/64	253/32	-209/64	77/64
7	3509/128	-1463/64	2079/128	-231/32	55/16
8	6655/128	-1375/32	33	-3927/256	561/64
9	3179/32	-20757/256	34067/512	-8151/256	10659/512
10	195415/1024	-39325/256	136609/1024	-66671/1024	48279/1024
11	753709/2048	-598345/2048	136367/512	-269951/2048	52877/512

References

[1] R. Grzymkowski and R. Wituła, *Calculus Methods in Algebra, Part One*, WPKJS, 2000 (in Polish).

[2] A. Mostowski and M. Stark, *The Elements of Higher Algebra*, PWN, 1974 (in Polish).

[3] S. Paszkowski, *Numerical Applications of Chebyshef Polynomials and Series*, PWN, 1975 (in Polish).

[4] S. Rabinowitz, Algorithmic manipulation of third-order linear recurrences, *Fib. Quart.* **34** (1996), 447–464.

[5] T. J. Rivlin, *Chebyshef Polynomials from Approximation Theory to Algebra and Number Theory*, 2nd ed., Wiley, 1990.

[6] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2007.

[7] D. Surowski and P. McCombs, Homogeneous polynomials and the minimal polynomial of $\cos(2\pi/n)$, *Missouri J. Math. Sci.* **15** (2003), 4–14.

[8] W. Watkins and J. Zeitlin, The minimal polynomials of $\cos(2\pi/n)$, *Amer. Math. Monthly* **100** (1993), 471–474.

[9] R. Wituła, D. Słota and A. Warzyński, Quasi-Fibonacci numbers of the seventh order, *J. Integer Seq.* **9** (2006), Article 06.4.3.

[10] R. Wituła and D. Słota, New Ramanujan-type formulas and quasi-Fibonacci numbers of order 7, *J. Integer Seq.* **10** (2007), Article 07.5.6.

2000 *Mathematics Subject Classification*: Primary 11B83, 11A07; Secondary 12D10, 39A10.
Keywords: Fibonacci numbers, primitive roots of unity, recurrence relation.

(Concerned with sequence A062883.)

Received December 9 2006. Revised version received August 3 2007. Published in *Journal of Integer Sequences*, August 5 2007.

Return to Journal of Integer Sequences home page.