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# Some Generalized Fibonacci Polynomials

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**Abstract**

We introduce polynomial generalizations of the $r$-Fibonacci, $r$-Gibonacci, and $r$-Lucas sequences which arise in connection with two statistics defined, respectively, on linear, phased, and circular $r$-mino arrangements.

## 1 Introduction

In what follows, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{P}$ denote, respectively, the integers, the nonnegative integers, and the positive integers. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. If $q$ is an indeterminate, then $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ for $n \in \mathbb{P}$, $0_q^! := 1$, $n_q^! := 1_q 2_q \cdots n_q$ for $n \in \mathbb{P}$, and

$$\binom{n}{k}_q := \begin{cases} \frac{n_q^!}{k_q^!(n-k)_q^!}, & \text{if } 0 \leqslant k \leqslant n; \\ 0, & \text{if } k < 0 \text{ or } 0 \leqslant n < k. \end{cases} \tag{1.1}$$

The $\binom{n}{k}_q$ are also given, equivalently, by the column generating function [12, pp. 201–202]

$$\sum_{n\geqslant 0}\binom{n}{k}_q x^n = \frac{x^k}{(1-x)(1-qx)\cdots(1-q^kx)}, \qquad k\in\mathbb{N}. \tag{1.2}$$

If $r \geqslant 2$, the $r$-Fibonacci numbers $F_n^{(r)}$ are defined by $F_0^{(r)} = F_1^{(r)} = \cdots = F_{r-1}^{(r)} = 1$, with $F_n^{(r)} = F_{n-1}^{(r)} + F_{n-r}^{(r)}$ if $n \geqslant r$. The $r$-Lucas numbers $L_n^{(r)}$ are defined by $L_1^{(r)} = L_2^{(r)} = \cdots = L_{r-1}^{(r)} = 1$ and $L_r^{(r)} = r+1$, with $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geqslant r+1$. If $r = 2$, the $F_n^{(r)}$ and $L_n^{(r)}$ reduce, respectively, to the classical Fibonacci and Lucas numbers (parametrized, as in Wilf [13] by $F_0 = F_1 = 1$, etc., and $L_1 = 1$, $L_2 = 3$, etc.).

Polynomial generalizations of $F_n$ and/or $L_n$ have arisen as distribution polynomials for statistics on binary words [3], lattice paths [8], Morse code sequences [7], and linear and circular domino arrangements [9]. Generalizations of $F_n^{(r)}$ and/or $L_n^{(r)}$ have arisen similarly in connection with statistics on Morse code sequences [7] as well as on linear and circular $r$-mino arrangements [10, 11].

In the next section, we consider the $q$-generalization

$$F_n^{(r)}(q,t) := \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} q^{k+r\binom{k}{2}}\binom{n-(r-1)k}{k}_q t^k \tag{1.3}$$

of $F_n^{(r)}$. The $r = 2$ case of (1.3) or close variants thereof have appeared several times in the literature starting with Carlitz (see, e.g., [3, 4, 5, 8, 9]. The $F_n^{(r)}(q,t)$ arise as joint distribution polynomials for two statistics on linear $r$-mino arrangements which naturally extend well known statistics on domino arrangements. When defined, more broadly, on phased $r$-mino arrangements, these statistics lead to a further generalization of the $F_n^{(r)}(q,t)$ which we denote by $G_n^{(r)}(q,t)$. In the third section, we consider the $q$-generalization

$$L_n^{(r)}(q,t) := \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} q^{k+r\binom{k}{2}}\left[\frac{n_q}{(n-(r-1)k)_q}\right]\binom{n-(r-1)k}{k}_q t^k \tag{1.4}$$

of $L_n^{(r)}$, which arises as the joint distribution polynomial for the same two statistics, now defined on circular $r$-mino arrangements. The $r = 2$ case of (1.4) was introduced by Carlitz [3] and has been subsequently studied (see, e.g., [9]).

## 2 Linear and Phased $r$-Mino Arrangements

Let $\mathcal{R}_{n,k}^{(r)}$ denote the set of coverings of the numbers $1, 2, \ldots, n$ arranged in a row by $k$ indistinguishable $r$-minos and $n - rk$ indistinguishable squares, where pieces do not overlap, an *$r$-mino*, $r \geqslant 2$, is a rectangular piece covering $r$ numbers, and a square is a piece covering a single number. Each such covering corresponds uniquely to a word in the alphabet $\{r, s\}$ comprising $k$ $r$'s and $n - rk$ $s$'s so that

$$|\mathcal{R}_{n,k}^{(r)}| = \binom{n-(r-1)k}{k}, \qquad 0\leqslant k\leqslant \lfloor n/r\rfloor, \tag{2.1}$$

for all $n \in \mathbb{P}$. (If we set $\mathcal{R}_{0,0}^{(r)} = \{\emptyset\}$, the "empty covering," then (2.1) holds for $n = 0$ as well.) In what follows, we will identify coverings $c$ with such words $c_1c_2\cdots$ in $\{r, s\}$. With

$$\mathcal{R}_n^{(r)} := \bigcup_{0 \leqslant k \leqslant \lfloor n/r \rfloor} \mathcal{R}_{n,k}^{(r)}, \qquad n \in \mathbb{N}, \tag{2.2}$$

it follows that

$$|\mathcal{R}_n^{(r)}| = \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} \binom{n-(r-1)k}{k} = F_n^{(r)}, \tag{2.3}$$

where $F_0^{(r)} = F_1^{(r)} = \cdots = F_{r-1}^{(r)} = 1$, with $F_n^{(r)} = F_{n-1}^{(r)} + F_{n-r}^{(r)}$ if $n \geqslant r$. Note that

$$\sum_{n \geqslant 0} F_n^{(r)} x^n = \frac{1}{1 - x - x^r}. \tag{2.4}$$

Given $c \in \mathcal{R}_n^{(r)}$, let $v(c) :=$ the number of $r$-minos in the covering $c$, let $\sigma(c) :=$ the sum of the numbers covered by the leftmost segments of each of these $r$-minos, and let

$$F_n^{(r)}(q,t) := \sum_{c \in \mathcal{R}_n^{(r)}} q^{\sigma(c)} t^{v(c)}, \qquad n \in \mathbb{N}. \tag{2.5}$$

Categorizing linear covers of $1, 2, \ldots, n$ according to the final and initial pieces, respectively, yields the recurrences

$$F_n^{(r)}(q,t) = F_{n-1}^{(r)}(q,t) + q^{n-r+1} t F_{n-r}^{(r)}(q,t), \qquad n \geqslant r, \tag{2.6}$$

and

$$F_n^{(r)}(q,t) = F_{n-1}^{(r)}(q,qt) + qtF_{n-r}^{(r)}(q,q^r t), \qquad n \geqslant r, \tag{2.7}$$

where $F_0^{(r)}(q,t) = F_1^{(r)}(q,t) = \cdots = F_{r-1}^{(r)}(q,t) = 1$. Iterating (2.6) or (2.7) gives $F_{-i}^{(r)}(q,t) = 0$ if $1 \leqslant i \leqslant r-1$ with $F_{-r}^{(r)}(q,t) = q^{r-1}t^{-1}$, which we'll take as a convention.

With the ordinary generating function

$$\Phi^{(r)}(x,q,t) := \sum_{n \geqslant 0} F_n^{(r)}(q,t) x^n, \tag{2.8}$$

recurrence (2.6) is equivalent to the identity

$$\Phi^{(r)}(x,q,t) = 1 + x\Phi^{(r)}(x,q,t) + qtx^r \Phi^{(r)}(qx,q,t), \tag{2.9}$$

which may be rewritten, with the operator $\varepsilon f(x) := f(qx)$, as

$$(1 - x - qtx^r \varepsilon)\Phi^{(r)}(x,q,t) = 1,$$

or

$$\left(1 - \frac{qtx^r}{1-x}\varepsilon\right)\Phi^{(r)}(x,q,t) = \frac{1}{1-x}. \tag{2.10}$$

From (2.10), we immediately get

$$\Phi^{(r)}(x,q,t)=\sum_{k\geqslant 0}\left(\frac{qtx^r}{1-x}\varepsilon\right)^k\frac{1}{1-x},$$

which implies

**Theorem 2.1.**

$$\Phi^{(r)}(x,q,t)=\sum_{k\geqslant 0}\frac{q^{k+r\binom{k}{2}}t^k x^{rk}}{(1-x)(1-qx)\cdots(1-q^k x)}. \tag{2.11}$$

By (2.11) and (1.2),

$$\begin{aligned}\Phi^{(r)}(x,q,t) &= \sum_{k\geqslant 0} q^{k+r\binom{k}{2}}t^k x^{(r-1)k}\cdot\frac{x^k}{(1-x)(1-qx)\cdots(1-q^k x)}\\ &= \sum_{k\geqslant 0} q^{k+r\binom{k}{2}}t^k x^{(r-1)k}\sum_{n\geqslant rk}\binom{n-(r-1)k}{k}_q x^{n-(r-1)k}\\ &= \sum_{n\geqslant 0}\left(\sum_{0\leqslant k\leqslant\lfloor n/r\rfloor} q^{k+r\binom{k}{2}}\binom{n-(r-1)k}{k}_q t^k\right)x^n,\end{aligned}$$

which establishes the explicit formula:

**Theorem 2.2.** *For all $n\in\mathbb{N}$,*

$$F_n^{(r)}(q,t)=\sum_{0\leqslant k\leqslant\lfloor n/r\rfloor} q^{k+r\binom{k}{2}}\binom{n-(r-1)k}{k}_q t^k. \tag{2.12}$$

*Remark:* Cigler [7] has studied algebraically the polynomials

$$F_n(j,x,s,q):=\sum_{0\leqslant jk\leqslant n-j+1} q^{j\binom{k}{2}}\binom{n-(j-1)(k+1)}{k}_q s^k x^{n-j(k+1)+1},\quad n\geqslant 0,$$

which, by (2.12), are related to the $F_n^{(r)}(q,t)$ by

$$F_n(j,x,s,q)=x^{n-j+1}F_{n-j+1}^{(j)}\left(q,\frac{s}{qx^j}\right),\quad n\geqslant 0. \tag{2.13}$$

From (2.5) and (2.13), one gets a combinatorial interpretation for the $F_n(j,x,s,q)$ in terms of $j$-mino arrangements; viz., $F_n(j,x,s,q)$ is the joint distribution polynomial for the statistics on $\mathcal{R}_{n-j+1}^{(j)}$ recording the number of squares, the number of $j$-minos, and the sum of the numbers directly preceding leftmost segments of $j$-minos.

Note that (2.11) and (2.12) reduce, respectively, to (2.4) and (2.3) when $q=t=1$. Setting $q=1$ and $q=-1$ in (2.11) gives

**Corollary 2.3.**

$$\Phi^{(r)}(x,1,t)=\frac{1}{1-x-tx^r}. \tag{2.14}$$

and

**Corollary 2.4.**

$$\Phi^{(r)}(x,-1,t)=\frac{1+x-tx^r}{1-x^2+(-1)^{r+1}t^2x^{2r}}. \tag{2.15}$$

Taking the even and odd parts of both sides of (2.15), replacing $x$ with $x^{1/2}$, and applying (2.14) yields

**Theorem 2.5.** *Let $m\in\mathbb{N}$. If $m$ and $r$ have the same parity, then*

$$F_m^{(r)}(-1,t)=F_{\lfloor m/2\rfloor}^{(r)}(1,(-1)^rt^2)-tF_{(m-r)/2}^{(r)}(1,(-1)^rt^2), \tag{2.16}$$

*and if $m$ and $r$ have different parity, then*

$$F_m^{(r)}(-1,t)=F_{\lfloor m/2\rfloor}^{(r)}(1,(-1)^rt^2). \tag{2.17}$$

One can provide combinatorial proofs of (2.16) and (2.17) similar to those in [10, 11] given for comparable formulas involving other $q$-Fibonacci polynomials.

The $F_n^{(r)}(q,t)$ may be generalized as follows:

If $r\geqslant 2$ and $a,b\in\mathbb{P}$, then define the sequence $(G_n^{(r)})_{n\in\mathbb{Z}}$ by the recurrence $G_n^{(r)}=G_{n-1}^{(r)}+G_{n-r}^{(r)}$ for all $n\in\mathbb{Z}$ with the initial conditions $G_{-(r-2)}^{(r)}=\cdots=G_{-1}^{(r)}=0$, $G_0^{(r)}=a$, and $G_1^{(r)}=b$. When $r=2$, these are the *Gibonacci numbers* $G_n$ (shorthand for *generalized Fibonacci numbers*) occurring in Benjamin and Quinn [2, p. 17]. When $a=b=1$ and $a=r$, $b=1$, the $G_n^{(r)}$ reduce to the $r$-Fibonacci and $r$-Lucas numbers, respectively. We'll call the $G_n^{(r)}$ *$r$-Gibonacci numbers.*

From the initial conditions and recurrence, one sees that the $G_n^{(r)}$, when $n\geqslant 1$, count linear $r$-mino coverings of length $n$ in which an initial $r$-mino is assigned one of $a$ phases and an initial square is assigned one of $b$ phases. We'll call such coverings *phased $r$-mino tilings* (*of length* $n$), in accordance with Benjamin and Quinn [1, 2] in the case $r=2$. Let $\widehat{\mathcal{R}}_n^{(r)}$ be the set consisting of these phased tilings and let

$$G_n^{(r)}(q,t):=\sum_{c\in\widehat{\mathcal{R}}_n^{(r)}}q^{\sigma(c)}t^{v(c)},\quad n\geqslant 1, \tag{2.18}$$

where the $\sigma$ and $v$ statistics on $\widehat{\mathcal{R}}_n^{(r)}$ are defined as above. When $a=b=1$, the $G_n^{(r)}(q,t)$ reduce to the $F_n^{(r)}(q,t)$.

Conditioning on the final and initial pieces of a phased $r$-mino tiling yields the respective recurrences

$$G_n^{(r)}(q,t)=G_{n-1}^{(r)}(q,t)+q^{n-r+1}tG_{n-r}^{(r)}(q,t),\quad n\geqslant r+1, \tag{2.19}$$

and

$$G_n^{(r)}(q,t) = bF_{n-1}^{(r)}(q,qt) + aqtF_{n-r}^{(r)}(q,q^r t), \quad n \geqslant r+1, \tag{2.20}$$

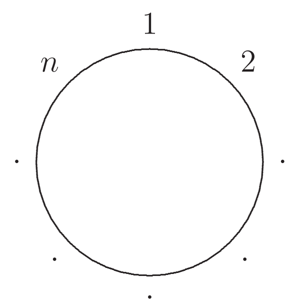
with $G_1^{(r)}(q,t) = \cdots = G_{r-1}^{(r)}(q,t) = b$ and $G_r^{(r)}(q,t) = b + aqt$. From (2.20), one gets formulas for $G_n^{(r)}(q,t)$ similar to those for $F_n^{(r)}(q,t)$. For example, taking $a = r$, $b = 1$ in (2.20), and applying (2.12), yields

$$\widehat{L}_n^{(r)}(q,t) := \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{k+r\binom{k}{2}} \left[ \frac{(r-1)k_q + (n-(r-1)k)_q}{(n-(r-1)k)_q} \right] \binom{n-(r-1)k}{k}_q t^k, \tag{2.21}$$

a $q$-generalization of the $r$-Lucas numbers.

# 3 Circular $r$-Mino Arrangements

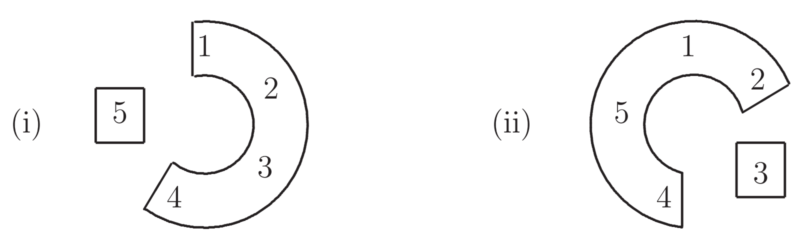
If $n \in \mathbb{P}$ and $0 \leqslant k \leqslant \lfloor n/r \rfloor$, let $\mathcal{C}_{n,k}^{(r)}$ denote the set of coverings by $k$ $r$-minos and $n - rk$ squares of the numbers $1, 2, \ldots, n$ arranged clockwise around a circle:




By the *initial segment* of an $r$-mino occurring in such a cover, we mean the segment first encountered as the circle is traversed clockwise. Classifying members of $\mathcal{C}_{n,k}^{(r)}$ according as (i) 1 is covered by one of $r$ segments of an $r$-mino or (ii) 1 is covered by a square, and applying (2.1), yields

$$\begin{aligned} \left|\mathcal{C}_{n,k}^{(r)}\right| &= r\binom{n-(r-1)k-1}{k-1} + \binom{n-(r-1)k-1}{k} \\ &= \frac{n}{n-(r-1)k}\binom{n-(r-1)k}{k}, \qquad 0 \leqslant k \leqslant \lfloor n/r \rfloor. \end{aligned} \tag{3.1}$$

Below we illustrate two members of $\mathcal{C}_{5,1}^{(4)}$:



In covering (i), the initial segment of the 4-mino covers 1, and in covering (ii), the initial segment covers 4.

With

$$\mathcal{C}_n^{(r)} := \bigcup_{0\leqslant k\leqslant \lfloor n/r\rfloor} \mathcal{C}_{n,k}^{(r)}, \qquad n \in \mathbb{P}, \tag{3.2}$$

it follows that

$$\left|\mathcal{C}_n^{(r)}\right| = \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} \frac{n}{n-(r-1)k}\binom{n-(r-1)k}{k} = L_n^{(r)}, \tag{3.3}$$

where $L_1^{(r)} = L_2^{(r)} = \cdots = L_{r-1}^{(r)} = 1$, $L_r^{(r)} = r+1$, and $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geqslant r+1$. Note that

$$\sum_{n\geqslant 1} L_n^{(r)} x^n = \frac{x + rx^r}{1 - x - x^r} \tag{3.4}$$

and that

$$L_n^{(r)} = F_n^{(r)} + (r-1)F_{n-r}^{(r)}, \quad n \geqslant 1. \tag{3.5}$$

Given $c \in \mathcal{C}_n^{(r)}$, let $v(c) :=$ the number of $r$-minos in the covering $c$, let $\sigma(c) :=$ the sum of the numbers covered by the initial segments of each of these $r$-minos, and let

$$L_n^{(r)}(q,t) := \sum_{c\in\mathcal{C}_n^{(r)}} q^{\sigma(c)} t^{v(c)}. \tag{3.6}$$

Conditioning on whether the number 1 is covered by a square or by an initial segment of an $r$-mino or by an $r$-mino with initial segment $n-(r-1-i)$ for some $i$, $1 \leqslant i \leqslant r-1$, yields the formula

$$L_n^{(r)}(q,t) = F_n^{(r)}(q,t) + q^{n-r+1} t \sum_{i=1}^{r-1} q^i F_{n-r}^{(r)}(q, q^i t), \quad n \geqslant 1, \tag{3.7}$$

which reduces to the well known formula (see, e.g., [10])

$$L_n^{(r)}(1,t) = F_n^{(r)}(1,t) + (r-1)tF_{n-r}^{(r)}(1,t), \quad n \geqslant 1, \tag{3.8}$$

when $q = 1$. The $L_n^{(r)}(q,t)$, though, do not appear to satisfy a simple recurrence like (2.6) or (2.7).

With the ordinary generating function

$$\lambda^{(r)}(x,q,t) := \sum_{n\geqslant 1} L_n^{(r)}(q,t) x^n, \tag{3.9}$$

one sees that (3.7) is equivalent to

$$\lambda^{(r)}(x,q,t) = -1 + \Phi^{(r)}(x,q,t) + qtx^r \sum_{i=1}^{r-1} q^i \Phi^{(r)}(qx, q, q^i t). \tag{3.10}$$

By (2.11), identity (3.10) is equivalent to

**Theorem 3.1.**

$$\lambda^{(r)}(x,q,t) = \frac{x}{1-x} + \sum_{k\geqslant 1} \frac{q^{k+r\binom{k}{2}} t^k x^{rk} \left[1+(1-x)\sum_{i=1}^{r-1} q^{ki}\right]}{(1-x)(1-qx)\cdots(1-q^k x)}. \tag{3.11}$$

The following theorem gives an explicit formula for the $L_n^{(r)}(q,t)$:

**Theorem 3.2.** *For all* $n \in \mathbb{P}$,

$$L_n^{(r)}(q,t) = \sum_{0\leqslant k\leqslant \lfloor n/r \rfloor} q^{k+r\binom{k}{2}} \left[\frac{n_q}{(n-(r-1)k)_q}\right] \binom{n-(r-1)k}{k}_q t^k. \tag{3.12}$$

*Proof.* It suffices to show

$$\sum_{c\in\mathcal{C}_{n,k}^{(r)}} q^{\sigma(c)} = q^{k+r\binom{k}{2}} \left[\frac{n_q}{(n-(r-1)k)_q}\right] \binom{n-(r-1)k}{k}_q.$$

Partitioning $\mathcal{C}_{n,k}^{(r)}$ into three classes according to whether (i) 1 is covered by an initial segment of an $r$-mino, (ii) 1 is covered by an $r$-mino with initial segment $n-(r-1-i)$ for some $i$, $1 \leqslant i \leqslant r-1$, or (iii) 1 is covered by a square, and applying (2.12) to each class, yields

$$\begin{aligned}
\sum_{c\in\mathcal{C}_{n,k}^{(r)}} q^{\sigma(c)} &= q^{(k-1)+r\binom{k-1}{2}} \binom{n-(r-1)k-1}{k-1}_q \left(q^{r(k-1)+1} + \sum_{i=1}^{r-1} q^{(k-1)i+(n-r+1+i)}\right) \\
&\quad + q^{k+r\binom{k}{2}} \binom{n-(r-1)k-1}{k}_q \cdot q^k \\
&= q^{k+r\binom{k}{2}} \binom{n-(r-1)k-1}{k-1}_q \left(1+\sum_{i=1}^{r-1} q^{n-(r-i)k}\right) + q^{2k+r\binom{k}{2}} \binom{n-(r-1)k-1}{k}_q \\
&= q^{k+r\binom{k}{2}} \left[\binom{n-(r-1)k-1}{k-1}_q \left(1+\sum_{i=1}^{r-1} q^{n-ki}\right) + q^k \binom{n-(r-1)k-1}{k}_q\right] \\
&= \frac{q^{k+r\binom{k}{2}}}{(n-(r-1)k)_q} \binom{n-(r-1)k}{k}_q \left[k_q \left(1+\sum_{i=1}^{r-1} q^{n-ki}\right) + q^k (n-rk)_q\right],
\end{aligned}$$

from which (3.12) now follows from the easily verified identity

$$n_q = k_q \left(1+\sum_{i=1}^{r-1} q^{n-ki}\right) + q^k (n-rk)_q.$$

□

Note that (3.11) and (3.12) reduce, respectively, to (3.4) and (3.3) when $q = t = 1$. Setting $q = 1$ and $q = -1$ in (3.11) gives

**Corollary 3.3.**

$$\lambda^{(r)}(x,1,t) = \frac{x + rtx^r}{1 - x - tx^r}. \tag{3.13}$$

and

**Corollary 3.4.**

$$\lambda^{(r)}(x,-1,t) = \frac{x + x^2 - tx^{2\lfloor \frac{r}{2} \rfloor + 1} + r(-1)^r t^2 x^{2r}}{1 - x^2 + (-1)^{r+1} t^2 x^{2r}}. \tag{3.14}$$

Either setting $q = -1$ in (3.7) and applying (2.16), (2.17), and (3.8) or taking the even and odd parts of both sides of (3.14), replacing $x$ with $x^{1/2}$, and applying (3.13) and (2.14) yields

**Theorem 3.5.** *If $m \in \mathbb{P}$, then*

$$L_{2m}^{(r)}(-1,t) = L_m^{(r)}(1,(-1)^r t^2) \tag{3.15}$$

*and*

$$L_{2m-1}^{(r)}(-1,t) = F_{m-1}^{(r)}(1,(-1)^r t^2) - tF_{m-\lfloor \frac{r}{2} \rfloor - 1}^{(r)}(1,(-1)^r t^2). \tag{3.16}$$

For a combinatorial proof of (3.15) and (3.16), we first associate to each $c \in \mathcal{C}_n^{(r)}$ a word $u_c = u_1 u_2 \cdots$ in the alphabet $\{r, s\}$, where

$$u_i := \begin{cases} r, & \text{if the } i^{th} \text{ piece of } c \text{ is an } r\text{-mino;} \\ s, & \text{if the } i^{th} \text{ piece of } c \text{ is a square,} \end{cases}$$

and one determines the $i^{th}$ piece of $c$ by starting with the piece covering 1 and proceeding clockwise from that piece. Note that for each word starting with $r$, there are exactly $r$ associated members of $\mathcal{C}_n^{(r)}$, while for each word starting with $s$, there is only one associated member.

Assign to each covering $c \in \mathcal{C}_n^{(r)}$ the weight $w_c := (-1)^{\sigma(c)} t^{v(c)}$, where $t$ is an indeterminate. Let $\mathcal{C}_n^{(r)'}$ consist of those $c$ in $\mathcal{C}_n^{(r)}$ whose associated words $u_c = u_1 u_2 \cdots$ satisfy the conditions $u_{2i} = u_{2i+1}$, $i \geqslant 1$. Suppose $c \in \mathcal{C}_n^{(r)} - \mathcal{C}_n^{(r)'}$, with $i_0$ being the smallest value of $i$ for which $u_{2i} \neq u_{2i+1}$. Exchanging the positions of the $(2i_0)^{th}$ and $(2i_0+1)^{st}$ pieces within $c$ produces a $\sigma$-parity changing, $v$-preserving involution of $\mathcal{C}_n^{(r)} - \mathcal{C}_n^{(r)'}$.

First assume $n = 2m$ and let $\mathcal{C}_{2m}^{(r)*} \subseteq \mathcal{C}_{2m}^{(r)'}$ comprise those $c$ whose first and last pieces are the same and containing an even number of pieces in all. We extend the involution of $\mathcal{C}_{2m}^{(r)} - \mathcal{C}_{2m}^{(r)'}$ above to $\mathcal{C}_{2m}^{(r)'} - \mathcal{C}_{2m}^{(r)*}$ as follows. Let $c \in \mathcal{C}_{2m}^{(r)'} - \mathcal{C}_{2m}^{(r)*}$, first assuming $r$ is even. If the initial segment of the $r$-mino covering 1 in $c$ lies on an odd (resp., even) number, then rotate the entire arrangement counterclockwise (resp., clockwise) one position, moving the pieces but keeping the numbered positions fixed.

Now assume $r$ is odd. If 1 is covered by a segment of an $r$-mino which isn't initial, the rotate the entire arrangement clockwise or counterclockwise depending on whether the initial segment of this $r$-mino covers an odd or an even number. If 1 is covered by a square or by an initial segment of an $r$-mino, then pair $c$ with the covering obtained by reading $u_c = u_1 u_2 \cdots$ backwards. Thus,

$$\begin{aligned}L_{2m}^{(r)}(-1,t) &= \sum_{c\in\mathcal{C}_{2m}^{(r)}} w_c = \sum_{c\in\mathcal{C}_{2m}^{(r)*}} w_c = \sum_{c\in\mathcal{C}_{2m}^{(r)*}} (-1)^{rv(c)/2}t^{v(c)}\\ &= \sum_{c\in\mathcal{C}_{m}^{(r)}} (-1)^{rv(c)}t^{2v(c)} = L_m^{(r)}(1,(-1)^r t^2),\end{aligned}$$

which gives (3.15).

Next, assume $n = 2m-1$ and let $\mathcal{C}_{2m-1}^{(r)*} \subseteq \mathcal{C}_{2m-1}^{(r)'}$ comprise those $c$ in which 1 is covered by a square or by an initial segment of an $r$-mino and containing an odd number of pieces in all if 1 is covered by a square. Define an involution of $\mathcal{C}_{2m-1}^{(r)'} - \mathcal{C}_{2m-1}^{(r)*}$ as follows. If $r$ is odd, then use the mapping defined above for $\mathcal{C}_{2m}^{(r)'} - \mathcal{C}_{2m}^{(r)*}$ when $r$ was even. If $r$ is even, then slightly modify the mapping defined above for $\mathcal{C}_{2m}^{(r)'} - \mathcal{C}_{2m}^{(r)*}$ when $r$ was odd (i.e., replace the word "initial" with "second" in a couple of places). Thus,

$$\begin{aligned}L_{2m-1}^{(r)}(-1,t) &= \sum_{c\in\mathcal{C}_{2m-1}^{(r)*}} w_c = \sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)*}\\ u_1=s \text{ in } u_c}} w_c + \sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)*}\\ u_1=r \text{ in } u_c}} w_c\\ &= \sum_{\substack{c\in\mathcal{R}_{2m-2}^{(r)'}\\ v(c) \text{ even}}} w_c - t\sum_{c\in\mathcal{R}_{2m-r-1}^{(r)'}} w_c\\ &= F_{m-1}^{(r)}(1,(-1)^r t^2) - tF_{m-\lfloor\frac{r}{2}\rfloor-1}^{(r)}(1,(-1)^r t^2),\end{aligned}$$

which gives (3.16), where $\mathcal{R}_n^{(r)'} \subseteq \mathcal{R}_n^{(r)}$ consists of those $c = c_1c_2\cdots$ such that $c_{2i-1} = c_{2i}$, $i \geqslant 1$.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