On Multiple Sums of Products of Lucas Numbers

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Abstract

This paper studies some sums of products of the Lucas numbers. They are a generalization of the sums of the Lucas numbers, which were studied another authors. These sums are related to the denominator of the generating function of the \( k \)-th powers of the Fibonacci numbers. We considered a special case for an even positive integer \( k \) in the previous paper and now we generalize this result to an arbitrary positive integer \( k \). These sums are expressed as the sum of the binomial and Fibonomial coefficients. The proofs of the main theorems are based on special inverse formulas.

1 Introduction

Generating functions are very helpful in finding of relations for sequences of integers. Some authors found miscellaneous identities for the Fibonacci numbers \( F_n \), defined by recurrence relation \( F_{n+2} = F_n + F_{n+1} \), with \( F_0 = 0, F_1 = 1 \), and the Lucas numbers \( L_n \), defined by the same recurrence but with the initial conditions \( L_0 = 2, L_1 = 1 \), by manipulation with their generating functions. Our approach is rather different in this paper.

In 1718 DeMoivre found the generating function of the Fibonacci numbers \( F_n \) and used it for deriving the closed form \( F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \), with \( \alpha = \frac{1}{2}(1 + \sqrt{5}) \) and \( \beta = \frac{1}{2}(1 - \sqrt{5}) \) (similarly the formula \( L_n = \alpha^n + \beta^n \) holds for the Lucas numbers). In 1957 S. W. Golomb [2] found the generating function for the square of \( F_n \) and this result started the effort to find a recurrence or a closed form for the generating function \( f_k(x) = \sum_{n=0}^{\infty} F_n^k x^n \) of the

Horadam gave some closed forms for the numerator and the denominator of this generating function. From his results follows, for example

\[ f_k(x) = \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{j(i+1) \left[ k+1 \right] - j} F_{i-j}^{k} \frac{[k+1]_{i} x^i}{\sum_{i=0}^{k+1} (-1)^{j(i+1) \left[ k+1 \right] - j} x^i}, \]  

where $\left[ n \right]_k$ are the Fibonomial coefficients defined for any nonnegative integers $n$ and $k$ by

\[ \left[ n \right]_k = \prod_{i=0}^{k-1} \frac{F_{n-i}}{F_{i+1}} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}, \]

with $\left[ n \right]_0 = 1$ and $\left[ n \right]_k = 0$ for $n < k$.

Using Carlitz’ method, Shannon [11] obtained some special results for the numerator and the denominator in the expression of the generating function $f_k(x)$. For example, he used the $q$–analog of the terminating binomial theorem (firstly published by Rothe [9], but from Gauss’s posthumous papers it is known that he had found it around 1808, see [5]) and obtained the relation

\[ \prod_{i=0}^{k} (1 - q^i x) = \sum_{i=0}^{k+1} (-1)^i q^{i(i+1)} \left\{ k+1 \atop i \right\} x^i. \]

Q–binomial coefficients are defined $\left\{ k+1 \atop i \right\} = \frac{(q^{k+1}-1)(q^{k-1}) \cdots (q^{i+2}-1)}{(q-1)(q^2-1) \cdots (q^i-1)}$ for $i \geq 1$ and any complex numbers $q$, $x$ and any positive integer $k$, where $\left\{ k+1 \atop 0 \right\} = 1$. Replacing $q$ by $\beta/\alpha$ and $x$ by $\alpha^k x$ he got

\[ \prod_{i=0}^{k} (1 - \alpha^{k-i} \beta^i x) = \sum_{i=0}^{k+1} (-1)^{i} \beta^{i(i+1)} \left\{ k+1 \atop i \right\} x^i. \]  

(2)

We paid attention [10] to a generalization of a type of the well–known formulas for the Fibonacci and Lucas numbers, see [12, pp. 179–183], for example

\[ \sum_{i=0}^{n} (-1)^i L_{n-2i} = 2F_{n+1}. \]

In this paper we concentrate on the sums

\[ \sum_{i_n=0}^{\left\lfloor k+1 \atop i_n \right\rfloor} \sum_{i_{n-1}=i_n+1}^{\left\lfloor k+1 \atop i_{n-1} \right\rfloor} \cdots \sum_{i_{n-2}=i_{n-1}+1}^{\left\lfloor k+1 \atop i_{n-2} \right\rfloor} (-1)^{i_1+\cdots+i_n} \prod_{j=1}^{n} L_{k-2i_j}, \]

where $k$ is an arbitrary positive integer. The special case of (3) for an odd $k$ was solved up in [10]. Here we use analogous method to find formulas for an even integer $k$. 

2
Throughout the paper we adopt the conventions that the sum and the product over an empty set is 0 and 1, respectively, \(\lfloor x \rfloor\) represents the greatest integer less than or equal to \(x\), the relation \(f(x) \sim g(x)\) means that \(f(x)\) is asymptotic to \(g(x)\) and Iverson’s notation (see, e. g., [3]) that
\[
[P(k)] = \begin{cases} 
1, & \text{if statement } P(k) \text{ is true;} \\
0, & \text{if statement } P(k) \text{ is false.}
\end{cases}
\]

2 The main results

Definition 1. Let \(k\) be any positive integer. We define the sequence \(\{S_n(k)\}_{n=0}^\infty\) in the following way
\[
S_0(k) = 1, \quad S_1(k) = \sum_{i_1=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1} L_{k-2i_1}
\]
and
\[
S_n(k) = \sum_{i_n=0}^{\lfloor \frac{k-1}{2} \rfloor} \cdots \sum_{i_2=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{i_1=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{k-2i_j},
\]
for any integer \(n > 1\).

Let us denote
\[
\Theta(i, k, n) = \left(\left\lfloor \frac{k+1}{2} \right\rfloor - n + i\right) + \left(\left\lfloor \frac{k+1}{2} \right\rfloor - n + i - 1\right)
\]
for any positive integers \(i, k\) and any nonnegative integer \(n\).

Theorem 2. Let \(n\) be any nonnegative integer and let \(k\) be any positive integer. Then
\[
S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor - i} \Theta(i, k, n) \left[\frac{k+1}{n-2i}\right]
\]
if \(k\) is odd and
\[
S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i+n(\frac{k}{2}+1)+\frac{k}{2}(j+k+1)} \Theta(i, k, n) \left[\frac{k+1}{j}\right]
\]
if \(k\) is even.

Corollary 3. Let \(n\) be any nonnegative integer and let \(k\) be any positive integer. Then the asymptotic formula
\[
S_n(k) \sim \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor + i} k \Theta(i, k, n) \left[\frac{k+1}{n-2i}\right]
\]
holds as \(k \to \infty\).
Theorem 4. Let $m$ be any integer and let $k$ be any even positive integer. Then

$$\sum_{j=0}^{m} (-1)^{j} \binom{k+1}{j} = (-1)^{\frac{m}{2}} (m+1) \frac{1}{F_{k+1}} \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^{i} \binom{k+2}{m-2i} F_{k+2-m+2i}.$$ 

Corollary 5. Let $n$ be any nonnegative integer and let $k$ be any even positive integer. Then

$$S_n(k) = \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}}{F_{k+1}} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{i=j}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{i+j} \Theta(i, k, n) \binom{k+2}{n-j} F_{k+2-n+2j}. \quad (8)$$

Theorem 6. Let $m$ be any integer. Then

$$\sum_{j=0}^{m} (-1)^{j} \binom{k+1}{j} = (-1)^{\frac{m}{2}} (m+k+1) \frac{1}{F_{k+1}} \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left\lfloor \frac{k+4}{m-4i} \right\rfloor \times \left( F_{k+1} - (m-4i) F_{k+3} - F_{m-4i} F_{m-4i-1} \right).$$

Corollary 7. Let $n$ be any nonnegative integer. Then

$$\sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{i} \left( \binom{k+1}{i} - \binom{k-1}{i} \right) \binom{k+1}{n-2i} = 0 \quad (9)$$

if $k$ is an odd positive integer, $k < 2n-1$, and

$$\sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{i+j+\left\lfloor \frac{n}{2} \right\rfloor} \left( \binom{k}{i} - \binom{k-2}{i} \right) \binom{k+1}{j} = 0 \quad (10)$$

if $k$ is an even integer, $k < 2n$.

Corollary 8. Let $k$ be any even positive integer. Then

$$\sum_{i=0}^{k-2} (-1)^{i} L_{k-2i} = F_{k+1} - (-1)^{\frac{k}{2}};$$

$$\sum_{i_2=0}^{k-2} \sum_{i_1=i_2+1}^{k-2} (-1)^{i_1+i_2+1} L_{k-2i_1} L_{k-2i_2} = \frac{k-2}{2} + (-1)^{\frac{k}{2}} F_{k+1} + F_{k} F_{k+1}$$

and

$$\sum_{i_3=0}^{k-2} \sum_{i_2=i_3+1}^{k-2} \sum_{i_1=i_2+1}^{k-2} (-1)^{i_1+i_2+i_3} L_{k-2i_1} L_{k-2i_2} L_{k-2i_3}$$

$$= \frac{k-4}{2} \left( (-1)^{\frac{k}{2}} - F_{k+1} \right) + F_{k} F_{k+1} \left( (-1)^{\frac{k}{2}} - \frac{1}{2} F_{k-1} \right).$$
3 The preliminary results

Lemma 9. Let \( k \) be any positive integer. Then \( S_n(k) = 0 \) for each positive integer \( n > \left\lfloor \frac{k+1}{2} \right\rfloor \).

Proof. After rewriting relation (4) from Definition 1 into the form

\[
S_n(k) = \sum_{i_1,i_2, \ldots, i_n \geq 0 \atop 0 \leq i_n < i_{n-1} < \cdots < i_1 \leq \left\lfloor \frac{k-1}{2} \right\rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{k-2i_j}
\]

the assertion easily follows from the condition

\[
0 \leq i_n < i_{n-1} < \cdots < i_1 \leq \left\lfloor \frac{k-1}{2} \right\rfloor
\]

which does not hold for any values \( i_1, i_2, \ldots, i_n \) if \( \left\lfloor \frac{k-1}{2} \right\rfloor < n-1 \). \( \square \)

Lemma 10. Let \( k \) be any even positive integer and let \( n \) be any positive integer. Then

\[
(i) \quad \sum_{i=0}^{n} \left( \frac{k}{2} - 2i \right) \frac{k}{n-i} S_{2i}(k) = 0 \quad \text{for} \quad n \geq \frac{k}{2} + 1
\]

\[
(ii) \quad \sum_{i=0}^{n} \left( \frac{k}{2} - (2i + 1) \right) \frac{k}{n-i} S_{2i+1}(k) = 0 \quad \text{for} \quad n \geq \frac{k}{2}.
\]

Proof. We show the proof of (i). Case (ii) can be proved analogously. Each positive integer \( n \geq \frac{k}{2} + 1 \) can be written in the form \( n = \frac{k}{2} + l \), where \( l \) is any positive integer. We will show that just one of factors in the product \( \left( \frac{k}{2} - 2i \right) \frac{k}{n-i} S_{2i}(k) \) is equal to zero. Concretely, the first one equals zero for \( i \leq \left\lfloor \frac{k}{4} \right\rfloor \) and the second one equals zero for \( i > \left\lfloor \frac{k}{4} \right\rfloor \). For the sum in (i) the following holds:

\[
\sum_{i=0}^{\frac{k}{2}+l} \left( \frac{k}{2} - 2i \right) \frac{k}{\frac{k}{2} + l - i} S_{2i}(k) = Q_1(k, l) + Q_2(k, l),
\]

where

\[
Q_1(k, l) = \sum_{i=0}^{\left\lfloor \frac{k}{4} \right\rfloor} \left( \frac{k}{2} - 2i \right) \frac{k}{\frac{k}{2} + l - i} S_{2i}(k)
\]

and

\[
Q_2(k, l) = \sum_{i=\left\lfloor \frac{k}{4} \right\rfloor+1}^{\frac{k}{2}+l} \left( \frac{k}{2} - 2i \right) \frac{k}{\frac{k}{2} + l - i} S_{2i}(k) = \sum_{p=1}^{\left\lfloor \frac{k}{4} \right\rfloor+2} \left( \frac{k}{2} - \left\lfloor \frac{k}{4} \right\rfloor - 2p \right) S_{2\left\lfloor \frac{k}{4} \right\rfloor+2p}(k).\]

It is obvious that \( \left( \frac{k}{2} - 2i \right) \frac{k}{\frac{k}{2} + l - i} = 0 \) if \( i \leq \left\lfloor \frac{k}{4} \right\rfloor \) and therefore \( Q_1(k, l) = 0 \) for any \( k \) and \( l \). Since the equality \( S_{2\left\lfloor \frac{k}{4} \right\rfloor+2p}(k) = 0 \) is implied by Lemma 9 for any nonnegative integer \( p \), it follows that \( Q_2(k, l) = 0 \). \( \square \)
Lemma 11. Let \( n \) be any positive integer and let \( q \) be any integer. Then the following inverse formula holds:

\[
a_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{q - n + 2i}{i} b_{n-2i}
\]

if and only if

\[
b_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i} \left( \binom{q - n + i}{i} + \binom{q - n + i - 1}{i-1} \right) a_{n-2i}.
\] (11)

Proof. Riordan [8, p. 243] gave the following inverse formula:

\[
a_n = \sum_{i=0}^{n} \binom{q - 2i}{n - i} b_i
\]

if and only if

\[
b_n = \sum_{i=0}^{n} (-1)^{n+i} \left( \binom{q - n - i}{n - i} + \binom{q - n - i - 1}{n - i - 1} \right) a_i.
\]

To get Lemma 11 from this formula first we substitute \( \{a_n\} \) by \( \{a_{2n}\} \), \( \{b_i\} \) by \( \{b_{2i}\} \), \( n \) by \( \frac{n}{2} \) and \( i \) by \( \frac{n}{2} - i \) and then \( \{a_n\} \) by \( \{a_{2n+1}\} \), \( \{b_i\} \) by \( \{-b_{2i+1}\} \), \( n \) by \( \frac{n-1}{2} \), \( i \) by \( \frac{n-1}{2} - i \) and \( q \) by \( q - 1 \). This leads to the proved formula. \( \square \)

Lemma 12. Let \( n, k, l \) be any positive integers, \( l < n < k \). Let \( c_i, i = 1, 2, \ldots, n \), be any real numbers, \( c_n \neq 0 \). Then

\[
(i) \lim_{k \to \infty} \binom{k}{l}^{-1} = 0, \quad (ii) \sum_{i=l}^{n} c_i \binom{k}{i} \sim c_n \binom{k}{n} \text{ as } k \to \infty.
\]

Proof. Relation (i) follows from the definition of the Fibonomial coefficients and the obvious fact that \( \lim_{k \to \infty} F_k = \infty \). Thus,

\[
\lim_{k \to \infty} \binom{k}{l}^{-1} = \lim_{k \to \infty} \frac{F_k F_{k-1} \cdots F_{k-l+1}}{F_k F_{k-1} \cdots F_{l-1}} \cdot \frac{F_1 F_2 \cdots F_n}{F_k F_{k-1} \cdots F_{k-n+1}} =
\]

\[
= \frac{F_1 F_2 \cdots F_n}{F_k F_{k-1} \cdots F_l} \lim_{k \to \infty} \frac{F_k F_{k-1} \cdots F_{k-l+1}}{F_k F_{k-1} \cdots F_{k-n+1}} =
\]

\[
= \prod_{i=l+1}^{n} F_i \cdot \lim_{k \to \infty} \frac{1}{F_{k-l} \cdots F_{k-n+1}} = 0.
\]

Asymptotic formula (ii) is implied by (i). \( \square \)

Lemma 13. Let \( \{a_n\} \), \( \{b_n\} \) be any sequences of real numbers, with \( b_{-1} = 0 \), and let \( h \) be any integer. Then for an arbitrary positive integer \( n \)

\[
a_n = b_n - (-1)^h b_{n-1}
\] (12)

if and only if

\[
b_n = \sum_{i=0}^{n} (-1)^{h(n+i)} a_i.
\] (13)
Proof. Let us show that identity (12) implies identity (13). We have
\[
\sum_{i=0}^{n} (-1)^{h(n+i)} a_i = \sum_{i=0}^{n} (-1)^{h(n+i)} (b_i - (-1)^h b_{i-1})
\]
\[
= \sum_{i=0}^{n} (-1)^{h(n+i)} b_i - \sum_{i=1}^{n} (-1)^{h(n-1+i)} b_{i-1} - (-1)^{h(n-1)} b_{-1}
\]
\[
= \sum_{i=0}^{n} (-1)^{h(n+i)} b_i - \sum_{j=0}^{n-1} (-1)^{h(n+j)} b_j = b_n.
\]
Thus, this part of the assertion is true and similarly we can prove the reversed implication.

Lemma 14. Let \( k \) be any even positive integer and let \( a \) be any positive integer. Then
\[
\left[ \frac{k+1}{a} \right] + (-1)^{\frac{k}{2} + a} \left[ \frac{k+1}{a-1} \right] = \frac{F_{k+1-a}}{F_{k+1}} \left[ \frac{k+2}{a} \right].
\]

Proof. Using the definition of the Fibonomial coefficients we get the relation
\[
F_{k-a+1} F_{k+2} = F_{k+1} \left( F_{k-a+2} + (-1)^{\frac{k}{2} + a} F_a \right),
\]
which can be written in the form
\[
F_{k-a+1} L_{k+1} = F_{k-a+2} + (-1)^{\frac{k}{2} + a} F_a
\]
as \( F_{2n} = F_n L_n \) ([12, p. 176]). We get the previous relation by setting \( l = \frac{k}{2} - a + 1 \) and \( n = \frac{k}{2} + 1 \) into the identity ([12, p. 177])
\[
F_{l+n} = F_l L_n + (-1)^{n+1} F_{l-n},
\]
which holds for any integers \( l, n \). The assertion follows at once.

The following form of \( \Theta(i, k, n) \) is more effective for the computation of the sums \( S_n(k) \):

Lemma 15. Let \( i, n \) be any integers and let \( k \) be any even positive integer. Then
\[
\Theta(i, k, n) = \begin{cases} 
0, & i < 0; \\
1, & i = 0; \\
\frac{k-2(n-i)}{2i} \prod_{j=1}^{i-1} \frac{k-2(n-j)}{2(i-j)}, & i > 0.
\end{cases}
\]

Proof. The cases for \( i \leq 0 \) are clear. For \( i > 0 \) we can write:
\[
\Theta(i, k, n) = \left( \frac{k}{2} - n + i \right) + \left( \frac{k}{2} - n + i - 1 \right)
\]
\[
= \frac{k-2(n-2i)}{2i} \left( \frac{k}{2} - n + i - 1 \right) = \frac{k-2(n-2i)}{2i} \prod_{j=1}^{i-1} \frac{k}{2} - n + i - j
\]
and the proof is over.
4 Additional properties of the inner sum

Now we will investigate properties of the inner sum involved in (6). Let us denote

\[ \sigma_k(m) = \sigma(m) := \sum_{j=0}^{k-m} (-1)^{\frac{j}{2}(j+k+1)} \left[ \begin{array}{c} k+1 \\ j \end{array} \right], \] (15)

where \( k \) is any even positive integer and \( m \) is any integer.

Lemma 16. Let \( k \) be any even positive integer and let \( m \) be any integer. Then

(i) \[ \sigma(m) = 0 \quad \text{for} \quad m \leq -1 \quad \text{or} \quad m \geq k+1, \]

(ii) \[ \sigma(k-m) = \sigma(m), \]

(iii) \[ \sigma(0) = 1, \quad \sigma(1) = 1 + (-1)^{\frac{k}{2}} F_{k+1}, \quad \sigma(2) = 1 - L_{\frac{k}{2}} F_{k+1} F_{k+\frac{1}{2}}, \quad \sigma(3) = 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left( 2 - F_k F_{k+\frac{1}{2}} L_{k+\frac{1}{2}} \right). \]

Proof. (i) First we prove the case for \( m = -1 \):

\[ \sigma(-1) = \sum_{j=0}^{k+1} (-1)^{\frac{j}{2}(j+k+1)} \left[ \begin{array}{c} k+1 \\ j \end{array} \right] \]

\[ = \sum_{j=0}^{k} (-1)^{\frac{j}{2}(j+k+1)} \left[ \begin{array}{c} k+1 \\ j \end{array} \right] + \sum_{j=\frac{k}{2}+1}^{k+1} (-1)^{\frac{j}{2}(j+k+1)} \left[ \begin{array}{c} k+1 \\ j \end{array} \right] \]

\[ = \sum_{j=0}^{k} (-1)^{\frac{j}{2}(j+k+1)} \left[ \begin{array}{c} k+1 \\ j \end{array} \right] + \sum_{i=0}^{k} (-1)^{\frac{k+1-i}{2}(2k+2-i)} \left[ \begin{array}{c} k+1 \\ k+1-i \end{array} \right] \]

\[ = \sum_{j=0}^{k} (-1)^{\frac{j}{2}(j+k+1)} \left[ \begin{array}{c} k+1 \\ j \end{array} \right] + \sum_{i=0}^{k} (-1)^{-i} (-1)^{\frac{k}{2}+1+i} \left[ \begin{array}{c} k+1 \\ i \end{array} \right] = 0. \]

For \( m \geq k+1 \) the assertion is obvious, according to defining formula (15). The case for \( m \leq -1 \) follows from \( \sigma(-1) = 0 \) and \( \left[ \begin{array}{c} k+1 \\ i \end{array} \right] = 0 \), for \( i > k+1 \), with respect to the definition of the Fibonomial coefficients.
(ii) We can write successively

\[ \sigma(k - m) = \sum_{j=0}^{m} (-1)^j \binom{j+k+1}{j} = \sum_{i=k-m+1}^{k+1} (-1)^{k+1-i} \binom{k+1}{k+1-i} \]

\[ = \sum_{i=k-m+1}^{k+1} (-1)^i (-1)^j \binom{j+k+1}{i} \]

\[ = \sum_{i=0}^{k+1} (-1)^i (-1)^j \binom{j+k+1}{i} - \sum_{i=0}^{k-m} (-1)^i (-1)^j \binom{j+k+1}{i} \]

\[ = -\sigma(-1) + \sum_{i=0}^{k-m} (-1)^i \binom{j+k+1}{i} = \sigma(m). \]

(iii) Identities for \( \sigma(0) \) and \( \sigma(1) \) are directly implied by \( \sigma(-1) = 0 \). Using case (ii) and identity (14) we have

\[ \sigma(2) = \sum_{j=0}^{k-2} (-1)^j \binom{j+k+1}{j} = 1 + (-1) \frac{k-2}{k+1} F_{k+1} - F_{k+1} F_k \]

\[ = 1 - F_{k+1} \left( F_k + (-1)^{\frac{k}{2}} \right) = 1 - F_{k+1} L_{\frac{k}{2}+1} F_{\frac{k}{2}-1}, \]

\[ \sigma(3) = \sigma(2) - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} F_{k-1} \]

\[ = 1 - F_{k+1} F_k - (-1)^{\frac{k}{2}} F_{k+1} + \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} F_k F_{k-1} \]

\[ = 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left( 2 - F_k \left( F_{k-1} - 2(-1)^{\frac{k}{2}} \right) \right) \]

\[ = 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left( 2 - F_k F_{\frac{k}{2}-2} L_{\frac{k}{2}+1} \right). \]

This finishes the proof. \( \square \)

The sum \( \sigma(m) \) can be simplified by the following lemma.

**Lemma 17.** Let \( k \) be any even positive integer and let \( m \) be any integer. Then

\[ \sigma(m) - \sigma(m - 2) = (-1)^{\frac{m}{2}(m+k+1)} \binom{k+2}{m} \frac{F_{k+1}}{F_{\frac{k}{2}+1}}. \]

**Proof.** For \( m < 2 \) the assertion follows from the definition of the Fibonomial coefficients
and Lemma 16. For \( m \geq 2 \) we have, with respect to Lemma 16,
\[
\sigma(m) - \sigma(m - 2) = \sigma(k - m) - \sigma(k - m + 2)
\]
\[
= \sum_{j=0}^{m} (-1) \frac{j}{2} (j + k + 1) \left[ \binom{k + 1}{j} \right] - \sum_{j=0}^{m-2} (-1) \frac{j}{2} (j + k + 1) \left[ \binom{k + 1}{j} \right]
\]
\[
= (-1)^m \left( \frac{k + 1}{m} \right) + (-1)^{m-1} \left( \frac{(m-1)+k+1}{m-1} \right)
\]
which, by Lemma 14, implies the assertion. \( \square \)

**Lemma 18.** Let \( k \) be any even positive integer and let \( m \) be any integer. Then
\[
\sigma(m) - \sigma(m - 4) = (-1)^m \frac{F_{k+2-m}}{F_{k+1} F_{k+3} F_{k+4}} \omega(m, k),
\] (16)

where
\[
\omega(m, k) = F_{k+1-m} L_{k+2-m} F_{k+3} - F_m F_{m-1}.
\]

**Proof.** With respect to Lemma 17 we have for any integer \( m \)
\[
\sigma(m) - \sigma(m - 4) = (\sigma(m) - \sigma(m - 2)) + (\sigma(m - 2) - \sigma(m - 4)) =
\]
\[
= (-1)^m \frac{1}{F_{k+1} F_{k+3} F_{k+4}} \left( F_{k+1-m} \left[ \binom{k + 2}{m} \right] - F_{k+3-m} \left[ \binom{k + 2}{m-2} \right] \right).
\]
The bracket term can be rewritten as
\[
F_{k+1-m} \left[ \binom{k + 2}{m} \right] - F_{k+3-m} \left[ \binom{k + 2}{m-2} \right] =
\]
\[
= \left[ \binom{k + 4}{m} \right] \frac{1}{F_{k+3} F_{k+4}} \left( F_{k+1-m} F_{k+3-m} F_{k+4-m} - F_{k+3-m} F_m F_{m-1} \right)
\]
The identity
\[
F_{k+3-m} F_{k+4-m} = F_{k+4-2m} F_{k+3} + F_m F_{m-1}
\]
follows from the identity ([12, p. 177])
\[
F_{n+h} F_{n+l} - F_n F_{n+h+l} = (-1)^n F_h F_l,
\]
with any integers \( h, n, l \). Hence, we obtain
\[
F_{k+1-m} \left[ \binom{k + 2}{m} \right] - F_{k+3-m} \left[ \binom{k + 2}{m-2} \right] =
\]
\[
= \left[ \binom{k + 4}{m} \right] \frac{1}{F_{k+3} F_{k+4}} \left( F_{k+2-m} \left( F_{k+4-2m} F_{k+3} - F_m F_{m-1} \right) - F_{k+6-m} F_m F_{m-1} \right)
\]
\[
= \left[ \binom{k + 4}{m} \right] \frac{1}{F_{k+3} F_{k+4}} \left( F_{k+1-m} F_{k+4-2m} F_{k+3} - \left( F_{k+3-m} - F_{k+3-m} F_{k+3} \right) F_m F_{m-1} \right)
\]
\[
= \left[ \binom{k + 4}{m} \right] \frac{F_{k+2-m}}{F_{k+3} F_{k+4}} \left( F_{k+1-m} L_{k+2-m} F_{k+3} - F_m F_{m-1} \right).
\]

10
Lemma 19. Let \( m \geq 5 \) be any integer and let \( k \) be any positive even integer in one of the following forms

\[
(i) \ k = m - 4 + [2 \mid m], \quad (ii) \ k = 2(m - 3), \quad (iii) \ k = 2(m - 1).
\]

Then \( \omega(m,k) \) can be factored into a product of the Fibonacci or Lucas numbers.

Proof. Condition (i), with respect to the identities ([12, pp. 176–177]) \( F_{-n} = (-1)^{n+1}F_n \), \( L_{-n} = (-1)^nL_n \) and \( F_{2n} = F_n L_n \), leads to the relation

\[
\omega(m, m - 3) = F_{\frac{m+1}{2}} F_m L_{\frac{m-1}{2}} - F_m F_{m-1} = F_m L_{\frac{m-1}{2}} (F_{\frac{m+1}{2}} - F_{\frac{m-1}{2}})
\]

if \( m \) is odd and to the relation

\[
\omega(m, m - 4) = F_{\frac{m+2}{2}} F_{m-1} L_{\frac{m}{2}} - F_m F_{m-1} = F_{m-1} L_{\frac{m}{2}} (F_{\frac{m+2}{2}} - F_{\frac{m}{2}})
\]

if \( m \) is even.

Using the identity \( F_{n+1}^2 + F_n^2 = F_{2n+1} \) ([12, p. 177]), we have from condition (ii)

\[
\omega(m, 2(m - 3)) = F_{2m-3} - F_m F_{m-1} = F_{m-2}^2 + F_{m-1}^2 - F_m F_{m-1}
\]

\[
= F_{m-2}^2 - F_{m-1} (F_m - F_{m-1}) = F_{m-2}^2 - F_{m-1} F_{m-2}
\]

\[
= F_{m-2}(F_{m-2} - F_{m-1}) = -F_{m-2} F_{m-3}.
\]

Condition (iii) gives \( \omega(m, 2(m - 1)) = -F_m F_{m-1} \).

Remark 20. The right–hand side of (16) can not be factored in a product of the Fibonacci or Lucas numbers for arbitrary values of \( k \) and \( m \). The trivial factorization can be done for \( m = 0 \) and \( m = 1 \). Table 1 lists the values of \( m \) and \( k \), \( 2 \leq m \leq 10 \), \( 2 \leq k \leq 170 \), for which \( \omega(m,k) \) can be factored into a product of the Fibonacci or Lucas numbers. These values were found by computer. The computer search for \( 10 \leq m \leq 100 \) showed that \( \omega(m,k) \) can be factored into a product of the Fibonacci or Lucas numbers only at values of \( m, k \) satisfying conditions from Lemma 20.
Table 1. The values for which \( \omega(m, k) \) is factorizable.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( k )</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>2 6</td>
</tr>
<tr>
<td>3</td>
<td>2 4 6</td>
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<td>2 4 6 8</td>
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<td>2 4 8 10</td>
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<td>4 6 8 12</td>
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<td>9</td>
<td>2 6 10 12 16</td>
</tr>
<tr>
<td>10</td>
<td>2 6 14 18</td>
</tr>
</tbody>
</table>

5 The proofs of the main results

Proof of Theorem 2. First we prove identity (5). We showed \cite{10} that for any positive odd integer \( k \) and any positive integer \( n \)

\[
S_{2n-1}(k) = \sum_{i=1}^{n} (-1)^{i+1} \left( \binom{k+1}{2} - n - i \right) \left[ k + 1 \right] \left[ 2i - 1 \right] \tag{17}
\]

and

\[
S_{2(n-1)}(k) = \sum_{i=1}^{n} (-1)^{i+1} \left( \binom{k+3}{2} - n - i \right) \left[ k + 1 \right] \left[ 2(i-1) \right]. \tag{18}
\]

Relation (5) can be obtained from (17) and (18). Replacing \( n \) by \( n + 1 \) and \( i \) by \( n + 1 - i \) we have for any nonnegative integer \( n \)

\[
S_{2n+1}(k) = \sum_{i=0}^{n} (-1)^{n-i} \left( \binom{k+1}{2} - (2n+1) + i \right) \left[ k + 1 \right] \left[ 2n+1-2i \right] \tag{19}
\]

and

\[
S_{2n}(k) = \sum_{i=0}^{n} (-1)^{n-i} \left( \binom{k+1}{2} - 2n + i \right) \left[ k + 1 \right] \left[ 2n-2i \right] \tag{19}
\]

which can be joined into the proved identity.

We begin the proof of relation (6) by defining the polynomial

\[
P_k(x) = \sum_{i=0}^{k} p_i(k) x^i = \prod_{j=0}^{k-1} \left( 1 - (-1)^j L_{k-2j} x + x^2 \right) \tag{19}
\]
for an even nonnegative integer $k$. By direct multiplication of the factors in (19) we get the identities

$$p_{2i+1}(k) = -\sum_{j=0}^{i} \binom{k/2 - (2j + 1)}{i-j} S_{2j+1}(k),$$

for $i = 0, 1, 2, \ldots, \frac{k-2}{2}$, and

$$p_{2i}(k) = \sum_{j=0}^{i} \binom{k/2 - 2j}{i-j} S_{2j}(k),$$

for $i = 0, 1, 2, \ldots, \frac{k}{2}$. By shifting indexes of summation it is possible to join (20) and (21) into the relation

$$p_n(k) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^n \binom{k/2 - n + 2i}{i} S_{n-2i}(k),$$

for $n = 0, 1, 2, \ldots, k$. This identity can be extended to any positive integer $n$ with respect to Lemma 9, as $p_n(k) = 0$ for $n < 0$ or $n > k$.

If $k$ is an even positive integer, the denominator in (1) is a polynomial of an odd degree $k + 1$:

$$D_{k+1}(x) = \sum_{i=0}^{k+1} d_{k+1,i} x^i,$$

where integers $d_{k+1,i} = (-1)^{\frac{i(i+1)}{2} \left\lfloor \frac{k+1}{i} \right\rfloor}$ are terms of sequence A055870, called the “signed Fibonomial triangle” in Sloane’s On-Line Encyclopedia of Integer Sequences [13]. Identity (2) implies

$$D_{k+1}(x) = \prod_{j=0}^{k} (1 - \alpha^{k-j} \beta^j x) (1 - (\alpha \beta)^{\frac{k}{2}}) \prod_{j=0}^{k} (1 - \alpha^{k-j} \beta^j x)$$

$$= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k-1}{2}} (1 - (-1)^j \alpha^{k-2j} x) (1 - (-1)^j \beta^{k-2j} x)$$

$$= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k-1}{2}} (1 - (-1)^j (\alpha^{k-2j} + \beta^{k-2j}) x + (\alpha \beta)^{k-2j} x^2)$$

$$= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k-1}{2}} (1 - (-1)^j L_{k-2j} x + x^2),$$

according to the relation $\alpha \beta = -1$ and the formula $L_{k-2j} = \alpha^{k-2j} + \beta^{k-2j}$. Thus, with respect to (19), $D_{k+1}(x) = (1 - (-1)^{\frac{k}{2}} x) P_k(x)$. By multiplying on the right–hand side and comparing coefficients of $x^i$ we have the following relations between coefficients $d_{k+1,i}$ of $D_{k+1}(x)$ and
coefficients $p_i(k)$ of $P_k(x)$
\[
d_{k+1,0} = p_0(k) = 1, \\
d_{k+1,i} = p_i(k) + (-1)^{\frac{i}{2}+1} p_{i-1}(k), \quad i = 1, 2, \ldots, k, \\
d_{k+1,k+1} = (-1)^{\frac{k}{2}+1} p_k(k) = (-1)^{\frac{k}{2}+1}.
\]

As $p_n(k) = 0$ for $n < 0$ or $n > k$ we can rewrite the previous relations in the recurrence
\[
p_n(k) + (-1)^{\frac{k}{2}+1} p_{n-1}(k) = d_{k+1,n},
\]
which holds for any integer $n$. Using Lemma 13 we have
\[
p_n(k) = \sum_{j=0}^{n} (-1)^{\frac{j}{2}(n+j)} d_{k+1,j}
\]
for any nonnegative integer $n$.

To complete the proof of (6) we have to invert identity (22). Setting $a_n = p_{2n}(k)$, $b_n = S_{2n}(k)$ and $q = \frac{k}{2}$ in inverse formula (11) we obtain
\[
S_n(k) = \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} (-1)^{n+i} \left( \left(\frac{k}{2} - n + i \right) + \left(\frac{k}{2} - n + i - 1 \right) \right) p_{n-2i}(k).
\]

From (23) and (24) we deduce that
\[
S_n(k) = \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} \sum_{j=0}^{n-2i} (-1)^{n+i} (-1)^{\frac{j}{2}(n+j)} \left( \left(\frac{k}{2} - n + i \right) + \left(\frac{k}{2} - n + i - 1 \right) \right) d_{k+1,j}.
\]

Putting $d_{k+1,j} = (-1)^{\frac{j}{2}(j+1)} {k+1 \choose j}$ we obtain (6) after simplification. \hfill \Box

**Proof of Corollary 3.** The assertion is obviously true with respect to (5) if $k$ is any odd integer. For even values of $k$ identity (6) can be written using (15) as
\[
S_n(k) = \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} (-1)^{n+i+n/2} \sigma(n-2i) \Theta(i,k,n).
\]

With respect to Lemma 12 for $k \to \infty$
\[
\sigma(n-2i) \sim (-1)^{n-2i} (n-2i+k+1) \left[ \frac{k+1}{n-2i} \right] = (-1)^{i} (-1)^{\frac{n+k+1}{2}} \left[ \frac{k+1}{n-2i} \right].
\]

Hence, we obtain
\[
S_n(k) \sim \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} (-1)^{n+i+n/2} (-1)^i (-1)^{\frac{n+k+1}{2}} \Theta(i,k,n) \left[ \frac{k+1}{n-2i} \right]
\]
\[
= \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} (-1)^{\frac{n}{2}(n-1)} \Theta(i,k,n) \left[ \frac{k+1}{n-2i} \right]
\]
and the assertion follows from the congruence $\frac{n}{2}(n-1) \equiv \left\lfloor \frac{n}{2} \right\rfloor \pmod{2}$. \hfill \Box
Proof of Theorem 4. For any even \( m \) we have
\[
\sum_{i=0}^{\frac{m}{2}} (\sigma(m - 2i) - \sigma(m - 2(i + 1))) = \sigma(m) - \sigma(-2)
\]
and analogously for any odd \( m \)
\[
\sum_{i=0}^{\frac{m-1}{2}} (\sigma(m - 2i) - \sigma(m - 2(i + 1))) = \sigma(m) - \sigma(-1) .
\]
Thus, using Lemma 16 we obtain for any integer \( m \)
\[
\sigma(m) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (\sigma(m - 2i) - \sigma(m - 2(i + 1)))
\]
and with respect to Lemma 17
\[
\sigma(m) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{\frac{m-2i}{2}(m-2i+1)} \frac{1}{F_{k}^{2}+1} \left[ \frac{k+2}{m-2i} \right] F_{k+1-2(m-2i)}
\]
\[
= (-1)^{\frac{m}{2}(m+k+1)} \frac{1}{F_{k}^{2}+1} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{i} \left[ \frac{k+2}{m-2i} \right] F_{k+2-m+2i} .
\]

Proof of Corollary 5. Applying Theorem 2 and Theorem 4, consecutively, we get
\[
S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n-k} \sigma(n - 2i) \Theta(i, k, n)
\]
\[
= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n} \Theta(i, k, n) \frac{1}{F_{k}^{2}+1} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{j} \left[ \frac{k+2}{n-2j} \right] F_{k+2-n+2j}
\]
\[
= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n/2(n-1)+i} \Theta(i, k, n) \frac{1}{F_{k}^{2}+1} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{j} \left[ \frac{k+2}{n-2j} \right] F_{k+2-n+2j}
\]
\[
= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n/2+1} \Theta(i, k, n) \frac{1}{F_{k}^{2}+1} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{j} \left[ \frac{k+2}{n-2j} \right] F_{k+2-n+2j}
\]
\[
= \frac{(-1)^{n/2}}{F_{k}^{2}+1} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{i+j} \Theta(i, k, n) \left[ \frac{k+2}{n-2j} \right] F_{k+2-n+2j} .
\]
Proof of Theorem 6. Similarly as in the proof of Theorem 4 we obtain for any integer $m$
the relation
\[
\sum_{i=0}^{\lfloor m/4 \rfloor} (\sigma(m - 4i) - \sigma(m - 4(i + 1))) = \sigma(m) - \sigma\left(m - 4\left(\lfloor \frac{m}{4} \rfloor + 1\right)\right).
\]
Thus, using Lemma 16 we obtain
\[
\sigma(m) = \sum_{i=0}^{\lfloor m/4 \rfloor} (\sigma(m - 4i) - \sigma(m - 4(i + 1))).
\]
With respect to Lemma 18 we have
\[
\sigma(m) = \sum_{i=0}^{\lfloor m/4 \rfloor} (-1)^{m/2(m-4i+k+1)} \begin{bmatrix} k + 4 \\ m - 4i \end{bmatrix} \frac{F_{\frac{k+2-(m-4i)}}{k+1}F_k}{F_{\frac{k+3}{k+4}}F_{k+4}} \\
\cdot \left( F_{\frac{k+1-(m-4i)}}{k+1}L_{\frac{k+2}{k+4}} F_k - F_{m-4i} F_{m-4i-1} \right)
\]
\[
= (-1)^{m/2(m+k+1)} \sum_{i=0}^{\lfloor m/4 \rfloor} \begin{bmatrix} k + 4 \\ m - 4i \end{bmatrix} \frac{F_{\frac{k+2}{k+4}}}{F_{\frac{k+3}{k+4}}} \\
\cdot \left( F_{\frac{k+1-(m-4i)}}{k+1}L_{\frac{k+2}{k+4}} F_k - F_{m-4i} F_{m-4i-1} \right).
\]

Proof of Corollary 7. Identities (9) and (10) can be obtained from identities (5) and (6)
with respect to $S_n(k) = 0$ for positive integers $k$, $n > \lfloor \frac{k+1}{2} \rfloor$ (see Lemma 9).

Proof of Corollary 8. Each of these three sums follows from identity (6) after some tedious
simplification.

6 Concluding remark

It is interesting to compare the effectiveness of formulas (6) and (8) in contrast to defining
formula (4) for computation of $S_n(k)$. Therefore, we found the CPU time (in seconds)
required for computation of sums $S_3(k)$ for some values of $k$ using the system Mathematica
on a standard PC. There is the measured time in Table 2.

Table 2. CPU time for $S_3(k)$

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<th>$k$</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
<th>700</th>
<th>800</th>
</tr>
</thead>
<tbody>
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<td>(4)</td>
<td>0.297</td>
<td>2.438</td>
<td>8.547</td>
<td>21.296</td>
<td>43.172</td>
<td>77.078</td>
<td>130.125</td>
<td>203.594</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
<td>0</td>
<td>0.047</td>
<td>0.094</td>
<td>0.172</td>
<td>0.297</td>
<td>0.484</td>
<td>0.719</td>
</tr>
<tr>
<td>(8)</td>
<td>0</td>
<td>0</td>
<td>0.015</td>
<td>0.046</td>
<td>0.078</td>
<td>0.156</td>
<td>0.25</td>
<td>0.359</td>
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References


2000 *Mathematics Subject Classification*: Primary 11B39; Secondary 05A15, 05A10.

*Keywords*: generating function, Riordan’s theorem, generalized Fibonacci numbers, Fibonomial coefficients.

(Concerned with sequence [A055870](http://www.research.att.com/~njas/sequences/index.html).)

Received January 19 2006; revised version received May 2 2007. Published in *Journal of Integer Sequences*, May 2 2007.

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