

On multiple sums of products of Lucas numbers

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Abstract

This paper studies some sums of products of the Lucas numbers, which are a generalization of the sums of the Lucas numbers from [12]. These sums are related to the denominator of the generating function of the k -th powers of the Fibonacci numbers. We considered [10] a special case for an even positive integer k and we generalize this result to an arbitrary positive integer k in this paper. These sums are expressed as the sum of the binomial and Fibonomial coefficients. We see Theorem 2 and its proof, which is based on special inverse formulas, as the main result of paper.

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1 Introduction

Generating functions are very helpful in finding of relations for sequences of integers. Some authors found miscellaneous identities for the Fibonacci numbers F_n , defined by recurrence relation $F_{n+2} = F_n + F_{n+1}$, with $F_0 = 0$, $F_1 = 1$, and the Lucas numbers L_n , defined by the same recurrence but with the initial conditions $L_0 = 2$, $L_1 = 1$, by manipulation with their generating functions. Our approach is rather different in this paper.

In 1718 DeMoivre found the generating function of the Fibonacci numbers F_n and used it for deriving the closed form $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, with $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \frac{1}{2}(1 - \sqrt{5})$ (similarly the formula $L_n = \alpha^n + \beta^n$ holds for the Lucas numbers). In 1957 S. W. Golomb [2] found the generating function for the square of F_n and this result started the effort to find a recurrence or a closed form for the generating function $f_k(x) = \sum_{n=0}^{\infty} F_n^k x^n$ of the k -th powers

of the Fibonacci numbers. Riordan [7] found a general recurrence for $f_k(x)$. Carlitz [1], Horadam [4] and Mansour [6] presented some generalizations of Riordan's results and found similar recurrences for the generating functions of powers of any second-order recurrence sequences.

Horadam gave some closed forms for the numerator and the denominator of this generating function. From his results follows, for example

$$f_k(x) = \frac{\sum_{i=0}^k \sum_{j=0}^i (-1)^{\frac{j(j+1)}{2}} \begin{bmatrix} k+1 \\ j \end{bmatrix} F_{i-j}^k x^i}{\sum_{i=0}^{k+1} (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix} x^i}, \quad (1)$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ are the Fibonomial coefficients defined for any nonnegative integers n and k by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=0}^{k-1} \frac{F_{n-i}}{F_{i+1}} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < k$.

Using Carlitz' method, Shannon [11] obtained some special results for the numerator and the denominator in the expression of the generating function $f_k(x)$. For example, he used the q -analog of the terminating binomial theorem (firstly published by Rothe [9], but from Gauss's posthumous papers it is known that he had found it around 1808, see [5]) and obtained the relation

$$\prod_{i=0}^k (1 - q^i x) = \sum_{i=0}^{k+1} (-1)^i q^{\frac{i(i-1)}{2}} \left\{ \begin{matrix} k+1 \\ i \end{matrix} \right\} x^i.$$

Q -binomial coefficients are defined $\left\{ \begin{matrix} k+1 \\ i \end{matrix} \right\} = \frac{(q^{k+1}-1)(q^k-1)\cdots(q^{k-i+2}-1)}{(q-1)(q^2-1)\cdots(q^i-1)}$ for $i \geq 1$ and any complex numbers q, x and any positive integer k , where $\left\{ \begin{matrix} k+1 \\ 0 \end{matrix} \right\} = 1$. Replacing q by β/α and x by $\alpha^k x$ he got

$$\prod_{i=0}^k (1 - \alpha^{k-i} \beta^i x) = \sum_{i=0}^{k+1} (-1)^{\frac{i(i+1)}{2}} \left[\begin{matrix} k+1 \\ i \end{matrix} \right] x^i. \quad (2)$$

We paid attention [10] to a generalization of a type of the well-known formulas for the Fibonacci and Lucas numbers, see [12] pp. 179–183, for example

$$\sum_{i=0}^n (-1)^i L_{n-2i} = 2F_{n+1}.$$

In this paper we concentrate on the sums

$$\sum_{i_n=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{i_{n-1}=i_n+1}^{\lfloor \frac{k-1}{2} \rfloor} \cdots \sum_{i_{n-2}=i_{n-1}+1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^n L_{k-2i_j}, \quad (3)$$

where k is an arbitrary positive integer. The special case of (3) for an odd k was solved up in [10]. Here we use analogous method to find formulas for an even integer k .

Throughout the paper we adopt the conventions that the sum and the product over an empty set is 0 and 1, respectively, $\lfloor x \rfloor$ represents the greatest integer less than or equal to x , the relation $f(x) \sim g(x)$ means that $f(x)$ is asymptotic to $g(x)$ and Iverson's notation (see, e. g., [3]) that

$$\lfloor P(k) \rfloor = \begin{cases} 1, & \text{if statement } P(k) \text{ is true;} \\ 0, & \text{if statement } P(k) \text{ is false.} \end{cases}$$

2 The main results

Definition 1. Let k be any positive integer. We define the sequence $\{S_n(k)\}_{n=0}^\infty$ in the following way

$$S_0(k) = 1, \quad S_1(k) = \sum_{i_1=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1} L_{k-2i_1}$$

and

$$S_n(k) = \sum_{i_n=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{i_{n-1}=i_n+1}^{\lfloor \frac{k-1}{2} \rfloor} \cdots \sum_{i_1=i_2+1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^n L_{k-2i_j}, \quad (4)$$

for any integer $n > 1$.

Let us denote

$$\Theta(i, k, n) = \binom{\lfloor \frac{k+1}{2} \rfloor - n + i}{i} + \binom{\lfloor \frac{k+1}{2} \rfloor - n + i - 1}{i - 1}$$

for any positive integers i, k and any nonnegative integer n .

Theorem 2. Let n be any nonnegative integer and let k be any positive integer. Then

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor - i} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \quad (5)$$

if k is odd and

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i+n(\frac{k}{2}+1)+\frac{j}{2}(j+k+1)} \Theta(i, k, n) \begin{bmatrix} k+1 \\ j \end{bmatrix} \quad (6)$$

if k is even.

Corollary 3. *Let n be any nonnegative integer and let k be any positive integer. Then the asymptotic formula*

$$S_n(k) \sim \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor + ik} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \quad (7)$$

holds as $k \rightarrow \infty$.

Theorem 4. *Let m be any integer and let k be any even positive integer. Then*

$$\sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} = (-1)^{\frac{m}{2}(m+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \begin{bmatrix} k+2 \\ m-2i \end{bmatrix} F_{\frac{k+2}{2}-m+2i}.$$

Corollary 5. *Let n be any nonnegative integer and let k be any even positive integer. Then*

$$S_n(k) = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{i+j} \Theta(i, k, n) \begin{bmatrix} k+2 \\ n-j \end{bmatrix} F_{\frac{k+2}{2}-n+2j}. \quad (8)$$

Theorem 6. *Let m be any integer. Then*

$$\begin{aligned} \sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} &= \frac{(-1)^{\frac{m}{2}(m+k+1)}}{F_{\frac{k}{2}+1} F_{k+3} F_{k+4}} \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} \begin{bmatrix} k+4 \\ m-4i \end{bmatrix} \times \\ &\times \left(F_{\frac{k}{2}+1-(m-4i)} L_{\frac{k}{2}+2-(m-4i)} F_{k+3} - F_{m-4i} F_{m-4i-1} \right). \end{aligned}$$

Corollary 7. *Let n be any nonnegative integer. Then*

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left(\binom{\frac{k+1}{2} - n + i}{i} + \binom{\frac{k-1}{2} - n + i}{i-1} \right) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} = 0 \quad (9)$$

if k is an odd positive integer, $k < 2n - 1$, and

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i+\frac{j}{2}(j+k+1)} \left(\binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k-2}{2} - n + i}{i-1} \right) \begin{bmatrix} k+1 \\ j \end{bmatrix} = 0 \quad (10)$$

if k is an even, $k < 2n$.

Corollary 8. *Let k be any even positive integer. Then*

$$\begin{aligned} \sum_{i=0}^{\frac{k-2}{2}} (-1)^i L_{k-2i} &= F_{k+1} - (-1)^{\frac{k}{2}}, \\ \sum_{i_2=0}^{\frac{k-2}{2}} \sum_{i_1=i_2+1}^{\frac{k-2}{2}} (-1)^{i_1+i_2+1} L_{k-2i_1} L_{k-2i_2} &= \frac{k-2}{2} + (-1)^{\frac{k}{2}} F_{k+1} + F_k F_{k+1} \end{aligned}$$

and

$$\begin{aligned} & \sum_{i_3=0}^{\frac{k-2}{2}} \sum_{i_2=i_3+1}^{\frac{k-2}{2}} \sum_{i_1=i_2+1}^{\frac{k-2}{2}} (-1)^{i_1+i_2+i_3} L_{k-2i_1} L_{k-2i_2} L_{k-2i_3} \\ &= \frac{k-4}{2} \left((-1)^{\frac{k}{2}} - F_{k+1} \right) + F_k F_{k+1} \left((-1)^{\frac{k}{2}} - \frac{1}{2} F_{k-1} \right). \end{aligned}$$

3 The preliminary results

Lemma 9. *Let k be any positive integer. Then $S_n(k) = 0$ for each positive integer $n > \lfloor \frac{k+1}{2} \rfloor$.*

Proof. After rewriting relation (4) from Definition 1 into the form

$$S_n(k) = \sum_{\substack{i_1, i_2, \dots, i_n \\ 0 \leq i_n < i_{n-1} < \dots < i_1 \leq \lfloor \frac{k-1}{2} \rfloor}} (-1)^{i_1+i_2+\dots+i_n} \prod_{j=1}^n L_{k-2i_j}$$

the assertion easily follows from the condition

$$0 \leq i_n < i_{n-1} < \dots < i_1 \leq \left\lfloor \frac{k-1}{2} \right\rfloor$$

which does not hold for any values i_1, i_2, \dots, i_n if $\lfloor \frac{k-1}{2} \rfloor < n-1$. \square

Lemma 10. *Let k be any even positive integer and let n be any positive integer. Then*

$$\begin{aligned} (i) \quad & \sum_{i=0}^n \binom{\frac{k}{2} - 2i}{n-i} S_{2i}(k) = 0 \quad \text{for } n \geq \frac{k}{2} + 1 \\ (ii) \quad & \sum_{i=0}^n \binom{\frac{k}{2} - (2i+1)}{n-i} S_{2i+1}(k) = 0 \quad \text{for } n \geq \frac{k}{2}. \end{aligned}$$

Proof. We show the proof of (i). Case (ii) can be proved by an analogous way. Each positive integer $n \geq \frac{k}{2} + 1$ can be written in the form $n = \frac{k}{2} + l$, where l is any positive integer. We will present that just one of factors in the product $\binom{\frac{k}{2}-2i}{n-i} S_{2i}(k)$ is equal to zero. Concretely, the first one equals zero for $i \leq \lfloor \frac{k}{4} \rfloor$ and the second one equals zero for $i > \lfloor \frac{k}{4} \rfloor$. For the sum in (i) the following holds:

$$\sum_{i=0}^{\frac{k}{2}+l} \binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} S_{2i}(k) = Q_1(k, l) + Q_2(k, l),$$

where

$$Q_1(k, l) = \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} \binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} S_{2i}(k)$$

and

$$Q_2(k, l) = \sum_{i=\lfloor \frac{k}{4} \rfloor + 1}^{\frac{k}{2} + l} \binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} S_{2i}(k) = \sum_{p=1}^{\frac{k}{2} - \lfloor \frac{k}{4} \rfloor + l} \binom{\frac{k}{2} - 2\lfloor \frac{k}{4} \rfloor - 2p}{\frac{k}{2} - \lfloor \frac{k}{4} \rfloor + l - p} S_{2\lfloor \frac{k}{4} \rfloor + 2p}(k).$$

It is obvious that $\binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} = 0$ if $i \leq \lfloor \frac{k}{4} \rfloor$ and therefore $Q_1(k, l) = 0$ for any k and l . Since the equality $S_{2\lfloor \frac{k}{4} \rfloor + 2p}(k) = 0$ is implied by Lemma 9 for any nonnegative integer p , it follows that $Q_2(k, l) = 0$. \square

Lemma 11. *Let n be any positive integer and let q be any integer. Then the following inverse formula holds:*

$$a_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{q - n + 2i}{i} b_{n-2i}$$

if and only if

$$b_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i} \left(\binom{q - n + i}{i} + \binom{q - n + i - 1}{i - 1} \right) a_{n-2i}. \quad (11)$$

Proof. Riordan [8] (p. 243) gave the following inverse formula:

$$a_n = \sum_{i=0}^n \binom{q - 2i}{n - i} b_i$$

if and only if

$$b_n = \sum_{i=0}^n (-1)^{n+i} \left(\binom{q - n - i}{n - i} + \binom{q - n - i - 1}{n - i - 1} \right) a_i.$$

To get Lemma 11 from this formula first we substitute $\{a_n\}$ by $\{a_{2n}\}$, $\{b_i\}$ by $\{b_{2i}\}$, n by $\frac{n}{2}$ and i by $\frac{n}{2} - i$ and then $\{a_n\}$ by $\{a_{2n+1}\}$, $\{b_i\}$ by $\{-b_{2i+1}\}$, n by $\frac{n-1}{2}$, i by $\frac{n-1}{2} - i$ and q by $q - 1$. This leads to the proved formula. \square

Lemma 12. *Let n, k, l be any positive integers, $l < n < k$. Let $c_i, i = 1, 2, \dots, n$, be any real numbers, $c_n \neq 0$. Then*

$$(i) \quad \lim_{k \rightarrow \infty} \begin{bmatrix} k \\ l \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}^{-1} = 0, \quad (ii) \quad \sum_{i=l}^n c_i \begin{bmatrix} k \\ i \end{bmatrix} \sim c_n \begin{bmatrix} k \\ n \end{bmatrix} \quad \text{as } k \rightarrow \infty.$$

Proof. Relation (i) follows from the definition of the Fibonomial coefficients and the obvious fact that $\lim_{k \rightarrow \infty} F_k = \infty$. Thus,

$$\begin{aligned} \lim_{k \rightarrow \infty} \begin{bmatrix} k \\ l \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}^{-1} &= \lim_{k \rightarrow \infty} \frac{F_k F_{k-1} \cdots F_{k-l+1}}{F_1 F_2 \cdots F_l} \cdot \frac{F_1 F_2 \cdots F_n}{F_k F_{k-1} \cdots F_{k-n+1}} = \\ &= \frac{F_1 F_2 \cdots F_n}{F_1 F_2 \cdots F_l} \lim_{k \rightarrow \infty} \frac{F_k F_{k-1} \cdots F_{k-l+1}}{F_k F_{k-1} \cdots F_{k-n+1}} = \\ &= \prod_{i=l+1}^n F_i \cdot \lim_{k \rightarrow \infty} \frac{1}{F_{k-l} \cdots F_{k-n+1}} = 0 . \end{aligned}$$

Asymptotic formula (ii) is implied by (i). \square

Lemma 13. *Let $\{a_n\}$, $\{b_n\}$ be any sequences of real numbers, with $b_{-1} = 0$, and let h be any integer. Then for an arbitrary positive integer n*

$$a_n = b_n - (-1)^h b_{n-1} \quad (12)$$

if and only if

$$b_n = \sum_{i=0}^n (-1)^{h(n+i)} a_i . \quad (13)$$

Proof. Let us show that identity (12) implies identity (13). We have

$$\begin{aligned} \sum_{i=0}^n (-1)^{h(n+i)} a_i &= \sum_{i=0}^n (-1)^{h(n+i)} (b_i - (-1)^h b_{i-1}) \\ &= \sum_{i=0}^n (-1)^{h(n+i)} b_i - \sum_{i=1}^n (-1)^{h(n-1+i)} b_{i-1} - (-1)^{h(n-1)} b_{-1} \\ &= \sum_{i=0}^n (-1)^{h(n+i)} b_i - \sum_{j=0}^{n-1} (-1)^{h(n+j)} b_j = b_n . \end{aligned}$$

Thus, this part of the assertion is true and similarly we can prove the reversed implication. \square

Lemma 14. *Let k be any even positive integer and let a be any positive integer. Then*

$$\begin{bmatrix} k+1 \\ a \end{bmatrix} + (-1)^{\frac{k}{2}+a} \begin{bmatrix} k+1 \\ a-1 \end{bmatrix} = \frac{F_{\frac{k}{2}+1-a}}{F_{\frac{k}{2}+1}} \begin{bmatrix} k+2 \\ a \end{bmatrix} .$$

Proof. Using the definition of the Fibonomial coefficients we get the relation

$$F_{\frac{k}{2}-a+1} F_{k+2} = F_{\frac{k}{2}+1} \left(F_{k-a+2} + (-1)^{\frac{k}{2}+a} F_a \right) ,$$

which can be written in the form

$$F_{\frac{k}{2}-a+1} L_{\frac{k}{2}+1} = F_{k-a+2} + (-1)^{\frac{k}{2}+a} F_a$$

as $F_{2n} = F_n L_n$ ([12], p. 176). We get the previous relation by setting $l = \frac{k}{2} - a + 1$ and $n = \frac{k}{2} + 1$ into the identity ([12], p. 177)

$$F_{l+n} = F_l L_n + (-1)^{n+1} F_{l-n}, \quad (14)$$

which holds for any integers l, n . The assertion follows at once. \square

The following form of $\Theta(i, k, n)$ is more effective for the computation of the sums $S_n(k)$:

Lemma 15. *Let i, n be any integers and let k be any even positive integer. Then*

$$\Theta(i, k, n) = \begin{cases} 0, & i < 0; \\ 1, & i = 0; \\ \frac{k-2(n-2i)}{2i} \prod_{j=1}^{i-1} \frac{k-2(n+j-i)}{2(i-j)}, & i > 0. \end{cases}$$

Proof. The cases for $i \leq 0$ are clear. For $i > 0$ we can write:

$$\begin{aligned} \Theta(i, k, n) &= \binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k}{2} - n + i - 1}{i - 1} \\ &= \frac{k - 2(n - 2i)}{2i} \binom{\frac{k}{2} - n + i - 1}{i - 1} = \frac{k - 2(n - 2i)}{2i} \prod_{j=1}^{i-1} \frac{\frac{k}{2} - n + i - j}{i - j} \end{aligned}$$

and the proof is over. \square

4 Additional properties of the inner sum

Now we will investigate properties of the inner sum involved in (6). Let us denote

$$\sigma_k(m) = \sigma(m) := \sum_{j=0}^{k-m} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix}, \quad (15)$$

where k is any even positive integer and m is any integer.

Lemma 16. *Let k be any even positive integer and let m be any integer. Then*

(i)

$$\sigma(m) = 0, \text{ for } m \leq -1 \text{ or } m \geq k+1,$$

(ii)

$$\sigma(k-m) = \sigma(m),$$

(iii)

$$\begin{aligned} \sigma(0) &= 1, & \sigma(1) &= 1 + (-1)^{\frac{k-2}{2}} F_{k+1}, \\ \sigma(2) &= 1 - L_{\frac{k+2}{2}} F_{k+1} F_{\frac{k-2}{2}}, & \sigma(3) &= 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left(2 - F_k F_{\frac{k-4}{2}} L_{\frac{k+2}{2}} \right). \end{aligned}$$

Proof. (i) First we prove the case for $m = -1$:

$$\begin{aligned} \sigma(-1) &= \sum_{j=0}^{k+1} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \\ &= \sum_{j=0}^{\frac{k}{2}} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} + \sum_{j=\frac{k}{2}+1}^{k+1} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \\ &= \sum_{j=0}^{\frac{k}{2}} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} + \sum_{i=0}^{\frac{k}{2}} (-1)^{\frac{k+1-i}{2}(2k+2-i)} \begin{bmatrix} k+1 \\ k+1-i \end{bmatrix} \\ &= \sum_{j=0}^{\frac{k}{2}} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} + \sum_{i=0}^{\frac{k}{2}} (-1)^{-1} (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} = 0. \end{aligned}$$

For $m \geq k+1$ the assertion is obvious, according to defining formula (15). The case for $m < -1$ follows from $\sigma(-1) = 0$ and $\begin{bmatrix} k+1 \\ i \end{bmatrix} = 0$, for $i > k+1$, with respect to the definition of the Fibonomial coefficients.

(ii) We can write successively

$$\begin{aligned} \sigma(k-m) &= \sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} = \sum_{i=k-m+1}^{k+1} (-1)^{\frac{k+1-i}{2}(2k+2-i)} \begin{bmatrix} k+1 \\ k+1-i \end{bmatrix} \\ &= \sum_{i=k-m+1}^{k+1} (-1)^1 (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} \\ &= \sum_{i=0}^{k+1} (-1)^1 (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} - \sum_{i=0}^{k-m} (-1)^1 (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} \\ &= -\sigma(-1) + \sum_{i=0}^{k-m} (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} = \sigma(m). \end{aligned}$$

(iii) Identities for $\sigma(0)$ and $\sigma(1)$ are directly implied by $\sigma(-1) = 0$. Using case (ii) and identity (14) we have

$$\begin{aligned}\sigma(2) &= \sum_{j=0}^{k-2} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} = 1 + (-1)^{\frac{k-2}{2}} F_{k+1} - F_{k+1} F_k \\ &= 1 - F_{k+1} \left(F_k + (-1)^{\frac{k}{2}} \right) = 1 - F_{k+1} L_{\frac{k}{2}+1} F_{\frac{k}{2}-1}, \\ \sigma(3) &= \sigma(2) - \frac{1}{2} (-1)^{\frac{k-2}{2}} F_{k+1} F_k F_{k-1} \\ &= 1 - F_{k+1} F_k - (-1)^{\frac{k}{2}} F_{k+1} + \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} F_k F_{k-1} \\ &= 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left(2 - F_k \left(F_{k-1} - 2(-1)^{\frac{k}{2}} \right) \right) \\ &= 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left(2 - F_k F_{\frac{k}{2}-2} L_{\frac{k}{2}+1} \right).\end{aligned}$$

This finishes the proof. \square

The sum $\sigma(m)$ can be simplified by the following lemma.

Lemma 17. *Let k be any even positive integer and let m be any integer. Then*

$$\sigma(m) - \sigma(m-2) = (-1)^{\frac{m}{2}(m+k+1)} \begin{bmatrix} k+2 \\ m \end{bmatrix} \frac{F_{\frac{k}{2}+1-m}}{F_{\frac{k}{2}+1}}.$$

Proof. For $m < 2$ the assertion follows from the definition of the Fibonomial coefficients and Lemma 16. For $m \geq 2$ we have, with respect to Lemma 16,

$$\begin{aligned}\sigma(m) - \sigma(m-2) &= \sigma(k-m) - \sigma(k-m+2) \\ &= \sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} - \sum_{j=0}^{m-2} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \\ &= (-1)^{\frac{m}{2}(m+k+1)} \begin{bmatrix} k+1 \\ m \end{bmatrix} + (-1)^{\frac{m-1}{2}((m-1)+k+1)} \begin{bmatrix} k+1 \\ m-1 \end{bmatrix} \\ &= (-1)^{\frac{m}{2}(m+k+1)} \left(\begin{bmatrix} k+1 \\ m \end{bmatrix} + (-1)^{\frac{k}{2}+m} \begin{bmatrix} k+1 \\ m-1 \end{bmatrix} \right),\end{aligned}$$

which, by Lemma 14, implies the assertion. \square

Lemma 18. *Let k be any even positive integer and let m be any integer. Then*

$$\sigma(m) - \sigma(m-4) = (-1)^{\frac{m}{2}(m+k+1)} \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{F_{\frac{k}{2}+2-m}}{F_{\frac{k}{2}+1} F_{k+3} F_{k+4}} \omega(m, k), \quad (16)$$

where

$$\omega(m, k) = F_{\frac{k}{2}+1-m} L_{\frac{k}{2}+2-m} F_{k+3} - F_m F_{m-1} .$$

Proof. With respect to Lemma 17 we have for any integer m

$$\begin{aligned} \sigma(m) - \sigma(m-4) &= (\sigma(m) - \sigma(m-2)) + (\sigma(m-2) - \sigma(m-4)) = \\ &= (-1)^{\frac{m}{2}(m+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \left(F_{\frac{k}{2}+1-m} \begin{bmatrix} k+2 \\ m \end{bmatrix} - F_{\frac{k}{2}+3-m} \begin{bmatrix} k+2 \\ m-2 \end{bmatrix} \right) . \end{aligned}$$

The bracket term can be rewritten as

$$\begin{aligned} F_{\frac{k}{2}+1-m} \begin{bmatrix} k+2 \\ m \end{bmatrix} - F_{\frac{k}{2}+3-m} \begin{bmatrix} k+2 \\ m-2 \end{bmatrix} &= \\ &= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{1}{F_{k+3} F_{k+4}} \left(F_{\frac{k}{2}+1-m} F_{k+3-m} F_{k+4-m} - F_{\frac{k}{2}+3-m} F_m F_{m-1} \right) . \end{aligned}$$

The identity

$$F_{k+3-m} F_{k+4-m} = F_{k+4-2m} F_{k+3} + F_m F_{m-1}$$

follows from the identity ([12], p. 177)

$$F_{n+h} F_{n+l} - F_n F_{n+h+l} = (-1)^n F_h F_l ,$$

with any integers h, n, l . Hence, we obtain

$$\begin{aligned} &F_{\frac{k}{2}+1-m} \begin{bmatrix} k+2 \\ m \end{bmatrix} - F_{\frac{k}{2}+3-m} \begin{bmatrix} k+2 \\ m-2 \end{bmatrix} \\ &= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{1}{F_{k+3} F_{k+4}} \left(F_{\frac{k}{2}+1-m} \left(F_{k+4-2m} F_{k+3} - F_m F_{m-1} \right) - F_{\frac{k}{2}+3-m} F_m F_{m-1} \right) \\ &= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{1}{F_{k+3} F_{k+4}} \left(F_{\frac{k}{2}+1-m} F_{k+4-2m} F_{k+3} - \left(F_{\frac{k}{2}+3-m} - F_{\frac{k}{2}+1-m} \right) F_m F_{m-1} \right) \\ &= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{F_{\frac{k}{2}+2-m}}{F_{k+3} F_{k+4}} \left(F_{\frac{k}{2}+1-m} L_{\frac{k}{2}+2-m} F_{k+3} - F_m F_{m-1} \right) \end{aligned}$$

and the assertion follows. \square

Lemma 19. *Let $m \geq 5$ be any integer and let k be any positive even integer in one of the following forms*

$$(i) k = m - 4 + [2 \nmid m] , \quad (ii) k = 2(m - 3) , \quad (iii) k = 2(m - 1) .$$

Then $\omega(m, k)$ can be factored into a product of the Fibonacci or Lucas numbers.

Proof. Condition (i), with respect to the identities ([12], pp. 176–177) $F_{-n} = (-1)^{n+1}F_n$, $L_{-n} = (-1)^n L_n$ and $F_{2n} = F_n L_n$, leads to the relation

$$\begin{aligned}\omega(m, m-3) &= F_{-\frac{m+1}{2}} F_m L_{-\frac{m-1}{2}} - F_m F_{m-1} = F_m L_{\frac{m-1}{2}} (F_{\frac{m+1}{2}} - F_{\frac{m-1}{2}}) \\ &= F_m F_{\frac{m-3}{2}} L_{\frac{m-1}{2}}\end{aligned}$$

if m is odd and to the relation

$$\begin{aligned}\omega(m, m-4) &= F_{\frac{m+2}{2}} F_{m-1} L_{\frac{m}{2}} - F_m F_{m-1} = F_{m-1} L_{\frac{m}{2}} (F_{\frac{m+2}{2}} - F_{\frac{m}{2}}) \\ &= F_{m-1} F_{\frac{m-2}{2}} L_{\frac{m}{2}}\end{aligned}$$

if m is even.

Using the identity $F_{n+1}^2 + F_n^2 = F_{2n+1}$ ([12], p. 177), we have from condition (ii)

$$\begin{aligned}\omega(m, 2(m-3)) &= F_{2m-3} - F_m F_{m-1} = F_{m-2}^2 + F_{m-1}^2 - F_m F_{m-1} \\ &= F_{m-2}^2 - F_{m-1} (F_m - F_{m-1}) = F_{m-2}^2 - F_{m-1} F_{m-2} \\ &= F_{m-2} (F_{m-2} - F_{m-1}) = -F_{m-2} F_{m-3} .\end{aligned}$$

Condition (iii) gives $\omega(m, 2(m-1)) = -F_m F_{m-1}$. □

Remark 20. The right-hand side of (16) can not be factored in a product of the Fibonacci or Lucas numbers for arbitrary values of k and m . The trivial factorization can be done for $m = 0$ and $m = 1$. Table 1 lists the values of m and k , $2 \leq m \leq 10$, $2 \leq k \leq 170$, for which $\omega(m, k)$ can be factored into a product of the Fibonacci or Lucas numbers. These values were found by computer. The computer search for $10 \leq m \leq 100$ showed that $\omega(m, k)$ can be factored into a product of the Fibonacci or Lucas numbers only at values of m, k satisfying conditions from Lemma 20.

Table 1. The values for which $\omega(m, k)$ is factorizable.

m	k				
2	2	6			
3	2	4	6		
4	2	4	6	8	
5	2	4	8	10	
6	2	6	10		
7	4	6	8	12	
8	4	6	8	10	14
9	2	6	10	12	16
10	2	6	14	18	

5 The proofs of the main results

Proof of Theorem 2. First we prove identity (5). We showed [10] that for any positive odd integer k and any positive integer n

$$S_{2n-1}(k) = \sum_{i=1}^n (-1)^{i+1} \left(\binom{\frac{k+3}{2} - n - i}{n - i} + \binom{\frac{k+1}{2} - n - i}{n - i - 1} \right) \begin{bmatrix} k + 1 \\ 2i - 1 \end{bmatrix} \quad (17)$$

and

$$S_{2(n-1)}(k) = \sum_{i=1}^n (-1)^{i+1} \left(\binom{\frac{k+5}{2} - n - i}{n - i} + \binom{\frac{k+3}{2} - n - i}{n - i - 1} \right) \begin{bmatrix} k + 1 \\ 2(i - 1) \end{bmatrix}. \quad (18)$$

Relation (5) can be obtained from (17) and (18). Replacing n by $n + 1$ and i by $n + 1 - i$ we have for any nonnegative integer n

$$S_{2n+1}(k) = \sum_{i=0}^n (-1)^{n-i} \left(\binom{\frac{k+1}{2} - (2n+1) + i}{i} + \binom{\frac{k-1}{2} - (2n+1) + i}{i - 1} \right) \begin{bmatrix} k + 1 \\ 2n+1-2i \end{bmatrix}$$

and

$$S_{2n}(k) = \sum_{i=0}^n (-1)^{n-i} \left(\binom{\frac{k+1}{2} - 2n + i}{i} + \binom{\frac{k-1}{2} - 2n + i}{i - 1} \right) \begin{bmatrix} k + 1 \\ 2n - 2i \end{bmatrix},$$

which can be joined into the proved identity.

We begin the proof of relation (6) by defining the polynomial

$$P_k(x) = \sum_{i=0}^k p_i(k) x^i = \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j L_{k-2j} x + x^2) \quad (19)$$

for an even nonnegative integer k . By direct multiplication of the factors in (19) we get the identities

$$p_{2i+1}(k) = - \sum_{j=0}^i \binom{\frac{k}{2} - (2j+1)}{i-j} S_{2j+1}(k), \quad (20)$$

for $i = 0, 1, 2, \dots, \frac{k-2}{2}$, and

$$p_{2i}(k) = \sum_{j=0}^i \binom{\frac{k}{2} - 2j}{i-j} S_{2j}(k), \quad (21)$$

for $i = 0, 1, 2, \dots, \frac{k}{2}$. By shifting indexes of summation it is possible to join (20) and (21) into the relation

$$p_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{\frac{k}{2} - n + 2i}{i} S_{n-2i}(k), \quad (22)$$

for $n = 0, 1, 2, \dots, k$. This identity can be extended to any positive integer n with respect to Lemma 9, as $p_n(k) = 0$ for $n < 0$ or $n > k$.

If k is an even positive integer, the denominator in (1) is a polynomial of an odd degree $k+1$:

$$D_{k+1}(x) = \sum_{i=0}^{k+1} d_{k+1,i} x^i,$$

where integers $d_{k+1,i} = (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix}$ are terms of the sequence which was named as “signed Fibonomial triangle” in the on-line encyclopedia of integer sequences (maintained by Sloane) with ID Number A055870, see [13]. Identity (2) implies

$$\begin{aligned} D_{k+1}(x) &= \prod_{j=0}^k (1 - \alpha^{k-j} \beta^j x) = (1 - (\alpha\beta)^{\frac{k}{2}} x) \prod_{\substack{j=0 \\ j \neq \frac{k}{2}}}^k (1 - \alpha^{k-j} \beta^j x) \\ &= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j \alpha^{k-2j} x) (1 - (-1)^j \beta^{k-2j} x) \\ &= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j (\alpha^{k-2j} + \beta^{k-2j}) x + (\alpha\beta)^{k-2j} x^2) \\ &= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j L_{k-2j} x + x^2), \end{aligned}$$

according to the relation $\alpha\beta = -1$ and the formula $L_{k-2j} = \alpha^{k-2j} + \beta^{k-2j}$. Thus, with respect to (19), $D_{k+1}(x) = (1 - (-1)^{\frac{k}{2}}x)P_k(x)$. By multiplying on the right-hand side and comparing coefficients of x^i we have the following relations between coefficients $d_{k+1,i}$ of $D_{k+1}(x)$ and coefficients $p_i(k)$ of $P_k(x)$

$$\begin{aligned} d_{k+1,0} &= p_0(k) = 1, \\ d_{k+1,i} &= p_i(k) + (-1)^{\frac{k}{2}+1} p_{i-1}(k), \quad i = 1, 2, \dots, k, \\ d_{k+1,k+1} &= (-1)^{\frac{k}{2}+1} p_k(k) = (-1)^{\frac{k}{2}+1}. \end{aligned}$$

As $p_n(k) = 0$ for $n < 0$ or $n > k$ we can rewrite the previous relations in the recurrence

$$p_n(k) + (-1)^{\frac{k}{2}+1} p_{n-1}(k) = d_{k+1,n},$$

which holds for any integer n . Using Lemma 13 we have

$$p_n(k) = \sum_{j=0}^n (-1)^{\frac{k}{2}(n+j)} d_{k+1,j} \quad (23)$$

for any nonnegative integer n .

To complete the proof of (6) we have to invert identity (22). Setting $a_n = p_{2n}(k)$, $b_n = S_{2n}(k)$ and $q = \frac{k}{2}$ in inverse formula (11) we obtain

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i} \left(\binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k}{2} - n + i - 1}{i-1} \right) p_{n-2i}(k). \quad (24)$$

From (23) and (24) we deduce that

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{n+i} (-1)^{\frac{k}{2}(n+j)} \left(\binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k}{2} - n + i - 1}{i-1} \right) d_{k+1,j}.$$

Putting $d_{k+1,j} = (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix}$ we obtain (6) after simplification. \square

Proof of Corollary 3. The assertion is obviously true with respect to (5) if k is any odd integer. For even values of k identity (6) can be written using (15) as

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} \sigma(n-2i) \Theta(i, k, n).$$

With respect to Lemma 12 for $k \rightarrow \infty$

$$\sigma(n-2i) \sim (-1)^{\frac{n-2i}{2}(n-2i+k+1)} \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} = (-1)^i (-1)^{\frac{n}{2}(n+k+1)} \begin{bmatrix} k+1 \\ n-2i \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} S_n(k) &\sim \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} (-1)^i (-1)^{\frac{n}{2}(n+k+1)} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\frac{n}{2}(n-1)} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \end{aligned}$$

and the assertion follows from the congruence $\frac{n}{2}(n-1) \equiv \lfloor \frac{n}{2} \rfloor \pmod{2}$. \square

Proof of Theorem 4. For any even m we have

$$\sum_{i=0}^{\frac{m}{2}} (\sigma(m-2i) - \sigma(m-2(i+1))) = \sigma(m) - \sigma(-2)$$

and analogously for any odd m

$$\sum_{i=0}^{\frac{m-1}{2}} (\sigma(m-2i) - \sigma(m-2(i+1))) = \sigma(m) - \sigma(-1) .$$

Thus, using Lemma 16 we obtain for any integer m

$$\sigma(m) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (\sigma(m-2i) - \sigma(m-2(i+1)))$$

and with respect to Lemma 17

$$\begin{aligned} \sigma(m) &= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{\frac{m-2i}{2}(m-2i+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \begin{bmatrix} k+2 \\ m-2i \end{bmatrix} F_{\frac{k}{2}+1-(m-2i)} \\ &= (-1)^{\frac{m}{2}(m+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \begin{bmatrix} k+2 \\ m-2i \end{bmatrix} F_{\frac{k+2}{2}-m+2i} . \end{aligned}$$

\square

Proof of Corollary 5. Applying Theorem 2 and Theorem 4, consecutively,

we get

$$\begin{aligned}
S_n(k) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} \sigma(n-2i) \Theta(i, k, n) \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} \Theta(i, k, n) \frac{(-1)^{\frac{n}{2}(n+k+1)}}{F_{\frac{k}{2}+1}} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\frac{n}{2}(n-1)+i} \frac{1}{F_{\frac{k}{2}+1}} \Theta(i, k, n) \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor + i} \Theta(i, k, n) \frac{1}{F_{\frac{k}{2}+1}} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} \\
&= \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{i+j} \Theta(i, k, n) \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} .
\end{aligned}$$

□

Proof of Theorem 6. Similarly as in the proof of Theorem 4 we obtain for any integer m the relation

$$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} (\sigma(m-4i) - \sigma(m-4(i+1))) = \sigma(m) - \sigma\left(m-4\left(\left\lfloor \frac{m}{4} \right\rfloor + 1\right)\right) .$$

Thus, using Lemma 16 we obtain

$$\sigma(m) = \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} (\sigma(m-4i) - \sigma(m-4(i+1))) .$$

With respect to Lemma 18 we have

$$\begin{aligned}
\sigma(m) &= \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} (-1)^{\frac{m-4i}{2}(m-4i+k+1)} \begin{bmatrix} k+4 \\ m-4i \end{bmatrix} \frac{F_{\frac{k}{2}+2-(m-4i)}}{F_{\frac{k}{2}+1} F_{k+3} F_{k+4}} \\
&\quad \cdot \left(F_{\frac{k}{2}+1-(m-4i)} L_{\frac{k}{2}+2-(m-4i)} F_{k+3} - F_{m-4i} F_{m-4i-1} \right) \\
&= \frac{(-1)^{\frac{m}{2}(m+k+1)}}{F_{\frac{k}{2}+1} F_{k+3} F_{k+4}} \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} \begin{bmatrix} k+4 \\ m-4i \end{bmatrix} F_{\frac{k}{2}+2-(m-4i)} \\
&\quad \cdot \left(F_{\frac{k}{2}+1-(m-4i)} L_{\frac{k}{2}+2-(m-4i)} F_{k+3} - F_{m-4i} F_{m-4i-1} \right) .
\end{aligned}$$

□

Proof of Corollary 7. Identities (9) and (10) can be obtained from identities (5) and (6) with respect to $S_n(k) = 0$ for positive integers $k, n > \lfloor \frac{k+1}{2} \rfloor$ (see Lemma 9). \square

Proof of Corollary 8. Each of these three sums follows from identity (6) after some tedious simplification. \square

6 Concluding remark

It is interesting to compare the effectiveness of formulas (6) and (8) in contrast to defining formula (4) for computation of $S_n(k)$. Therefore, we found the CPU time (in seconds) required for computation of sums $S_3(k)$ for some values of k using the system Mathematica on a standard PC. There is the measured time in Table 2.

Table 2. CPU time for $S_3(k)$

	k							
	100	200	300	400	500	600	700	800
(4)	0.297	2.438	8.547	21.296	43.172	77.078	130.125	203.594
(6)	0	0	0.047	0.094	0.172	0.297	0.484	0.719
(8)	0	0	0.015	0.046	0.078	0.156	0.25	0.359

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