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Wild Partitions and Number Theory

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Abstract

We introduce the notion of wild partition to describe in combinatorial language an important situation in the theory of p-adic fields. For Q a power of p, we get a sequence of numbers $\lambda_{Q,n}$ counting the number of certain wild partitions of n. We give an explicit formula for the corresponding generating function $\Lambda_Q(x) = \sum \lambda_{Q,n} x^n$ and use it to show that $\lambda_{Q,n}^{1/n}$ tends to $Q^{1/(p-1)}$. We apply this asymptotic result to support a finiteness conjecture about number fields. Our finiteness conjecture contrasts sharply with known results for function fields, and our arguments explain this contrast.

1 Introduction

The sequence A000041 of integers λ_n giving the number of partitions of n is important throughout mathematics. Its generating function is

$$\Lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n = \prod_{e=1}^{\infty} \frac{1}{1-x^e} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots . \tag{1}$$

In this paper, we consider for any prime power $Q = p^{n_0}$, an analogous integer sequence $\lambda_{Q,n}$ arising in a fundamental way in the number theory of p-adic fields. We evaluate the associated generating functions $\Lambda_Q(x) = \sum \lambda_{Q,n} x^n$, obtain corresponding asymptotics, and apply our results to support a finiteness conjecture about number fields. Generally speaking, our goal is to describe how a combinatorial viewpoint clarifies an important number-theoretic situation.

The following two displays, incorporated in the sequence database [24] as A131139 and A131140, give a first sense of the functions $\Lambda_Q(x)$ and the corresponding sequences $\lambda_{Q,n}$:

$$\begin{aligned}
\Lambda_2(x) &= \frac{1}{1-x}\cdot\frac{1-x^2}{(1-2x^2)^2}\cdot\frac{1}{1-x^3}\cdot\frac{(1-x^4)(1-4x^4)^2}{(1-8x^4)^4}\cdot\frac{1}{1-x^5}\cdot\frac{1-x^6}{(1-8x^6)^2}\cdot\,\cdots\\
&= 1+x+4x^2+5x^3+36x^4+40x^5+145x^6+180x^7+1572x^8+1712x^9+\cdots,
\end{aligned}$$

$$\begin{aligned}
\Lambda_3(x) &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{(1-x^3)^2}{(1-3x^3)^3}\cdot\frac{1}{1-x^4}\cdot\frac{1}{1-x^5}\cdot\frac{(1-x^6)^2}{(1-9x^6)^3}\cdot\,\cdots\\
&= 1+x+2x^2+9x^3+11x^4+19x^5+83x^6+99x^7+172x^8+1100x^9+\cdots.
\end{aligned}$$

In general, $\Lambda_Q(x)$, like its model $\Lambda(x)$, is given by a product over positive integers e. For p not dividing e, the corresponding factor is $1/(1-x^e)$ again. However for p dividing e, this factor is more complicated. In the number-theoretic context, the former factors reflect *tame* ramification and the latter reflect *wild* ramification.

The paper is organized so its starts in combinatorics and ends in number theory. The main combinatorial objects, wild partitions, are defined so that they correspond bijectively to the main number theoretic objects, geometric classes of p-adic algebras. We do not pursue it here, but a future goal is to specify one bijection as the conventional one, so that the very simple objects, wild partitions, index the more complicated objects, geometric classes of p-adic algebras. Such a labeling of p-adic algebras could be incorporated into the database of local fields [12] and would considerably facilitate the p-adic analysis of number fields.

Sections 2-4 are our combinatoric sections. For a factorization $n_0 = e_0 f_0$, we define (p, e_0, f_0)-*wild partitions* and a corresponding complicated three-variable generating function $\Lambda_{p,e_0,f_0}(x, y, z)$. Our definitions are not particularly motivated from a purely combinatoric point of view. Rather, as indicated above, they are chosen to mimic the structure of p-adic fields. For the sake of comparison, we consider first the specialization $(y, z) = (1, p^{-f_0})$ and get the remarkable simplification

$$\Lambda_{p,e_0,f_0}(x, 1, p^{-f_0}) = \Lambda(x). \tag{2}$$

As we'll indicate, this identity is related to the Serre mass formula [23]. Our main interest is in the new specialization $(y, z) = (1, 1)$. We define $\Lambda_Q(x)$ by an explicit formula and find

$$\Lambda_{p,e_0,f_0}(x, 1, 1) = \Lambda_Q(x), \tag{3}$$

independently of the factorization $n_0 = e_0 f_0$. Our explicit formula allows us to consider arbitrary real powers $Q = p^\nu \geq 1$ so that Q no longer determines p and we accordingly write $\Lambda_{p,Q}(x)$. One has $\Lambda_{p,1}(x) = \Lambda(x)$, independently of p. Thus another point of view is that for each prime p we have a Q-analog of $\Lambda(x)$.

Sections 5 and 6 consider analytic number theory associated to $\Lambda_{p,Q}(x)$. We express $\Lambda_{p,Q}(x)$ directly in terms of $\Lambda(x)$ and observe that as a consequence

$$\lim_{n\to\infty} \lambda_{p,Q,n}^{1/n} = Q^{1/(p-1)}. \tag{4}$$

Thus $\lambda_{p,Q,n}$ grows exponentially with growth factor $Q^{1/(p-1)}$. Equation (4) contrasts with the famous Hardy-Ramanujan statement [8] of subexponential growth, $\lambda_n \sim e^{\pi\sqrt{2n/3}}/(4n\sqrt{3})$. It quantifies the extent to which wild ramification predominates over tame ramification in number theory. Another contrast between ordinary partitions and our Q-analogs is that λ_n/λ_{n-1} tends to 1 while $\lambda_{p,Q,n}/\lambda_{p,Q,n-1}$ has oscillatory behavior which becomes more pronounced as Q increases. We conjecture an asymptotic of the form

$$\lambda_{p,Q,n} \sim c_{p,Q}(n) C_p(Q) n^{B_p(Q)} e^{A_p(Q)\sqrt{n}} Q^{n/(p-1)}, \tag{5}$$

with an explicit factor $c_{p,Q}(n)$ capturing the oscillatory behavior of $\lambda_{p,Q,n}/\lambda_{p,Q,n-1}$.

Sections 7-10 are set in the framework of local algebraic number theory. The material here is somewhat more technical, but we have arranged our presentation so that the only prerequisite is familiarity with basic facts about p-adic fields. Section 7 sets up the general situation and illustrates it for the fields $\mathbb{R}$ and $\mathbb{C}$, getting simple functions $\Lambda_{\mathbb{R}}(x) = e^x$ and $\Lambda_{\mathbb{C}}(x) = e^{x+x^2/2}$ which serve as analogs of our $\Lambda_Q(x)$. For Sections 8-10, we let F be an extension field of the p-adic field $\mathbb{Q}_p$, of ramification index e_0, inertial degree f_0, and thus degree $n_0 = e_0 f_0$ and residual cardinality $q = p^{f_0}$. Section 9 explains how the coefficient λ_{n,c_t,c_w} of $x^n y^{c_t} z^{c_w}$ gives the "total mass" of algebras K over F having relative degree n, tame conductor c_t, and wild conductor c_w. Section 10 works over the maximal unramified extension F^{un} of F. It explains how λ_{n,c_t,c_w} also counts extension algebras of F^{un} with the corresponding invariants. The perspective of Section 9 is more directly connected with the literature, while the perspective of Section 10 explains why the coefficients of $\Lambda_{p,e_0,f_0}(x,y,z)$ are integers. Summing over the possible c_t and c_w, one gets that the total mass $\lambda_{F,n}$ of degree n extension algebras of F is $\lambda_{Q,n}$, where $Q = p^{n_0}$.

Section 11 shifts to global algebraic number theory, working over an arbitrary number field F. It addresses a question raised in [16] on the size of sets $\mathrm{Fields}^{\mathrm{big}}_{F,n,S}$ of relative degree n number fields K/F. To be in $\mathrm{Fields}^{\mathrm{big}}_{F,n,S}$, the extension K/F must have associated Galois group A_n or S_n and ramification contained within the prescribed finite set of places S of F including the Archimedean places. A recent heuristic of Bhargava [4] yields

$$\frac{1}{2}\prod_{v\in S} \lambda_{F_v,n} \tag{6}$$

as a first guess (after slight modifications in degrees ≤ 3) for the size of $\mathrm{Fields}^{\mathrm{big}}_{F,n,S}$. The Archimedean factors $\lambda_{\mathbb{R},n}$, $\lambda_{\mathbb{C},n}$ decay superexponentially and we have proved that the remaining $\lambda_{F_v,n}$ grow only exponentially. Thus (6) leads to the prediction that for fixed (F,S), the set $\mathrm{Fields}^{\mathrm{big}}_{F,n,S}$ is empty for sufficiently large n. In other words $\mathrm{Fields}^{\mathrm{big}}_{F,S} = \coprod_n \mathrm{Fields}^{\mathrm{big}}_{F,n,S}$ is finite. This finiteness statement has a certain irony to it: normally one considers A_n and especially S_n to be the "generic expectation" for Galois groups of number fields; the statement says that in the setting of prescribed ramification, these groups are in fact the exceptions.

Finally, Section 12 considers positive characteristic analogs of all the previous considerations. In positive characteristic, our finiteness statement fails very badly. The theory we present here explains this failure as due to two sources, either one of which suffices to void our argument for the finiteness statement. One source is that Q must be considered ∞ and this forces all the $\lambda_{F_v,n}$ appearing in (6) to be infinite for $n \geq p$. Another source is that there are no Archimedean places of F, and thus no superexponentially decaying factors in (6).

Readers who want to quickly see the main ideas in a streamlined setting are invited to first focus on the special case $e_0 = f_0 = 1$. Then $n_0 = 1$ too, $p = q = Q$, and later $E = e$. The ground fields F of Sections 7-10 are then only the completions of $\mathbb{Q}$, i.e. $\mathbb{R}$ and the $\mathbb{Q}_p$. The ground fields F of Section 11 are then limited to simply $\mathbb{Q}$ itself. However by explicit examples with $n_0 = 2$ in Sections 2, 3, 6, and 11, we try to assist readers in appreciating the case of general (e_0, f_0). Sections 7-11 make clear that general (e_0, f_0) is the natural setting from a number-theoretic point of view. The naturality of this setting is emphasized by Sections 5 and 6 which interpolate $n_0 = e_0 f_0 \in \mathbb{Z}_{\geq 1}$ with general reals $\nu \geq 0$. The naturality of the general setting is further underscored by Section 12, which is based on the limiting case $e_0 = \infty$.

2 Wild partitions

Basic notation. An ordinary partition is an element of the free abelian monoid generated by the set of allowed parts $P = \{1, 2, 3, \dots\}$, for example

$$\mu_{\text{ordinary}} = 9 + 7 + 3 + 3 + 2 + 2 + 2 + 1. \tag{7}$$

Wild partitions are more complicated in two ways. First, the set P is replaced by a set $P(p, e_0)$ mapping surjectively to P, with infinite fibers above multiples of p. Second, necessary for obtaining finiteness, an invariance condition with respect to an operator $\sigma = \sigma_p^{f_0}$ enters.

As just indicated, our notion of wild partition depends not only on a prime p, but also on two positive integers e_0 and f_0. Let $q = p^{f_0}$ and $n_0 = e_0 f_0$. Our notations e_0, f_0, n_0 and q come from standard notations in the number-theoretic situation of Sections 8-10 inspiring our definitions. The quantity $Q = p^{n_0}$ is important for us, but does not have a standard number-theoretic notation.

Strictly speaking, our sets of wild partitions depend also on a choice of algebraic closure $\overline{\mathbb{F}}_p$ of the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. However all algebraic closures are isomorphic and so our final formulas counting certain wild partitions are independent of this choice. As usual, for a power p^u of p we denote by $\mathbb{F}_{p^u}$ the unique subfield of $\overline{\mathbb{F}}_p$ of cardinality p^u. We denote by σ_p the Frobenius element $k \mapsto k^p$ in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Similarly, we denote by σ_{p^u} the element σ_p^u; it is a topological generator of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^u})$. Most important for us is the operator σ_q, which we abbreviate by simply σ.

We reserve e for our main variable running over P. As a standing convention, we systematically write $e = p^w t$ with p^w the largest power of p dividing e. We think of w as the *wildness* of e, p^w as the *wild part* of e, and t as the *tame part* of e. As another standing convention, we abbreviate $e_0 e$ by E.

Ore numbers and their associated dimensions and spaces. An important notion in number theory is the set of *Ore numbers*

$$\mathrm{Ore}(p, e_0, e) \subseteq \{0, 1, \dots, wE - 1, wE\}. \tag{8}$$

To understand the set $\mathrm{Ore}(p, e_0, e)$, it is convenient present it as an array, as for the case $(p, e_0, e) = (3, 2, 9)$ for which $w = 2$:

$$\begin{array}{ccccccccc}
. & & & & & & & & \\
\hline
. & 8 & 7 & . & 5 & 4 & . & 2 & 1 \\
. & 17 & 16 & . & 14 & 13 & . & 11 & 10 \\
\hline
. & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 \\
36 & 35 & 34 & 33 & 32 & 31 & 30 & 29 & 28
\end{array} \tag{9}$$

In general, the array $\mathrm{Ore}(p, e_0, e)$ consists of a degenerate zeroth block, followed by w full blocks. The zeroth block has only a single spot, filled by 0 if $w = 0$ and empty otherwise. The full blocks each have e_0 rows and e columns. For $1 \leq j \leq w-1$, the j^{th} block consists of the integers in the interval $[(j-1)E + 1, jE]$ which are not multiples of p^j. The w^{th} block consists of these entries together with wE. Considering the table as a whole, we refer to all the entries as *non-maximal*, except for wE which is *maximal*. Our array format, including the right-to-left order, is intended to facilitate the discussion in Section 8, where the number-theoretic origin of $\mathrm{Ore}(p, e_0, e)$ is explained.

An important quantity in our situation is the *dimension* $d(p, e_0, e, s)$ associated to an Ore number $s \in \mathrm{Ore}(p, e_0, e)$. It is the number of integers in $[0, s-1]$ which are not in $\mathrm{Ore}(p, e_0, e)$. Thus $d(3, 2, 9, 20) = 7$, as there are seven omitted numbers less than 20 on the displayed Ore table (9). The way dimensions arise in number theory is explained in Section 9.

An Ore number $s \in \mathrm{Ore}(p, e_0, e)$ determines a subset $W(p, e_0, e, s)$ of the vector space $\overline{\mathbb{F}}_p^{d(p,e_0,e,s)}$ as follows. For non-maximal s, the set $W(p, e_0, e, s)$ consists of the subset of vectors with non-zero first coordinate. In the maximal case $s = wE$, the subset $W(p, e_0, e, s)$ is defined to be all of $\overline{\mathbb{F}}_p^{d(p,e_0,e,s)}$. The Frobenius element $\sigma = \sigma_p^{f_0}$ acts coordinate-wise on each $W(p, e_0, e, s)$, as indeed σ_p itself acts. The number of fixed points of σ is clearly $q^{d(p,e_0,e,s)}(1 - 1/q)$ for non-maximal s and $q^{d(p,e_0,e,s)}$ for the maximal $s = wE$. The explicit σ-sets $W(p, e_0, e, s)$ just introduced are isomorphic to less explicit σ-sets arising naturally in number theory, as explained in Section 10. We use s as our variable running over Ore numbers, because Ore numbers are also called Swan conductors.

Wild partitions and associated invariants. We are now in a position to make the main definition of the combinatorial part of this paper.

Definition 2.1. *A (p, e_0, f_0)-wild partition is an element of the free abelian monoid on the set*

$$P(p, e_0) = \coprod_{e \in \mathbb{Z}_{\geq 0}} \coprod_{s \in \mathrm{Ore}(p,e_0,e)} W(p, e_0, e, s) \tag{10}$$

which is fixed by $\sigma = \sigma_p^{f_0}$.

Usually (p, e_0, f_0) is fixed and clear from the context. Then we just say "wild partition" rather than (p, e_0, f_0)-wild partition.

We denote elements of $P(p, e_0)$ as doubly-subscripted integers $e_{s;\omega}$, with $s \in \mathrm{Ore}(p, e_0, e)$ and $\omega \in W(p, e_0, e, s)$. If p does not divide e, then the only possible subscript is "0;0" and so we allow ourselves to omit it. As an example of our notation, let i be one of the two square roots of -1 in $\mathbb{F}_9$. Then

$$\mu_{\mathrm{wild}} = 9_{20;1,1,0,2,2,0,1} + 7 + 3_{1;i} + 3_{1;-i} + 2 + 2 + 2 + 1 \tag{11}$$

is a wild partition for $(p, e_0, f_0) = (3, 2, 1)$. To check that μ_{wild} is indeed formed according to our rules, note that $d(3, 2, 9, 20) = |\{0, 3, 6, 9, 12, 15, 18\}| = 7$ from (9), and so it is proper that first subscripted ω has length 7. Also the first coordinate of this ω is non-zero, as required. The Ore table for $(p, e_0, e) = (3, 2, 3)$ omits 0 and has first row "$\cdot$ 2 1", so that $d(3, 2, 3, 1) = |\{0\}| = 1$; thus $3_{1;i}$ and $3_{1;-i}$ are properly constructed wild parts. Finally $q = p^{f_0} = 3^1 = 3$ and so $\sigma(i) = i^3 = -i$; thus μ_{wild} satisfies the σ-invariance condition.

By definition, one can add wild partitions just as one can add ordinary partitions. Wild partitions have three obvious additive integer invariants, all important in the underlying number-theoretic situation. First, as for ordinary partitions, one has the *degree* n, defined as usual as the sum of the parts e_i. Second, defined but often not important for ordinary partitions, one has the *tame conductor* c_t, the sum of the $e_i - 1$. Third, particular to our wild situation, one has the *wild conductor* c_w, the sum of the first subscripts s_i. Thus for the wild partition (11), one has $(n, c_t, c_w) = (29, 21, 22)$.

3 The generating functions $\Phi_{\mathcal{F}}(x, y, z)$ and $\Lambda_{\mathcal{F}}(x, y, z)$

Fix for this section a triple (p, e_0, f_0) as in the previous section. These three quantities figure rather passively into our current considerations. We will have other more active quantities as well. Accordingly, we abbreviate via $\mathcal{F} = (p, e_0, f_0)$. When we are continuing the example started in (7), (11), we will take $\mathcal{F} = (3, 2, 1)$.

Irreducible and isotypical partitions. We say a wild partition is *irreducible* if it is non-zero and cannot be written as the sum of two non-zero wild partitions. Every wild partition is uniquely the sum of its irreducible constituents. For example, μ_{wild} has seven irreducible constituents,

$$\mu_{\mathrm{wild}} = 9_{20;1,1,0,2,2,0,1} + 7 + (3_{1;i} + 3_{1;-i}) + 2 + 2 + 2 + 1. \tag{12}$$

We similarly say that a wild partition is *isotypical* if it has the form $m\mu$ for μ an irreducible wild partition and m a positive integer. Every wild partition is uniquely the sum of its isotypical constituents. For example, μ_{wild} has five isotypical constituents,

$$\mu_{\mathrm{wild}} = 9_{20;1,1,0,2,2,0,1} + 7 + (3_{1;i} + 3_{1;-i}) + (2 + 2 + 2) + 1. \tag{13}$$

If the wild partition μ is irreducible, we say that the isotypical wild partition $m\mu$ has mass $1/m$. In the number-theoretic settings of Sections 8-10, wild partitions correspond to geometric packets of algebras of total mass one. A packet contains fields if and only if the corresponding wild partition is isotypical; in this case, the fields in the packets have total mass $1/m$ and the non-fields total mass $1 - 1/m$.

Definition of $\Phi_{\mathcal{F}}(x,y,z)$ and $\Lambda_{\mathcal{F}}(x,y,z)$. Let $\phi_{\mathcal{F},n,c_t,c_w}$ be the total mass of $\mathcal{F}$-wild isotypical partitions of degree n, tame conductor c_t, and wild conductor c_w. Then the corresponding generating function is expressible as a sum over irreducibles,

$$\begin{aligned}\Phi_{\mathcal{F}}(x,y,z) &= \sum_{n=0}^{\infty}\sum_{c_t=0}^{\infty}\sum_{c_w=0}^{\infty}\phi_{\mathcal{F},n,c_t,c_w}x^n y^{c_t} z^{c_w} &(14)\\ &= \sum_{\mu}\sum_{m=1}^{\infty}\frac{1}{m}x^{mn(\mu)}y^{mc_t(\mu)}z^{mc_w(\mu)} &(15)\\ &= \sum_{\mu}\log\left(\frac{1}{1-x^{n(\mu)}y^{c_t(\mu)}z^{c_w(\mu)}}\right). &(16)\end{aligned}$$

Similarly, let $\lambda_{\mathcal{F},n,c_t,c_w}$ be the total number of $\mathcal{F}$-wild partitions of degree n, tame conductor c_t, and wild conductor c_w. Then its generating function is expressible as a product over irreducibles,

$$\begin{aligned}\Lambda_{\mathcal{F}}(x,y,z) &= \sum_{n=0}^{\infty}\sum_{c_t=0}^{\infty}\sum_{c_w=0}^{\infty}\lambda_{\mathcal{F},n,c_t,c_w}x^n y^{c_t} z^{c_w} &(17)\\ &= \prod_{\mu}\sum_{m=0}^{\infty}x^{mn(\mu)}y^{mc_t(\mu)}z^{mc_w(\mu)} &(18)\\ &= \prod_{\mu}\left(\frac{1}{1-x^{n(\mu)}y^{c_t(\mu)}z^{c_w(\mu)}}\right). &(19)\end{aligned}$$

The presence of $1/m$ in (15) and the absence of a corresponding factor in (18) reflects that (14)-(16) are in the setting of total mass while (17)-(19) are in the setting of total number.

Comparison of (14)-(16) with (17)-(19) shows that one has an exponential formula

$$\Lambda_{\mathcal{F}}(x,y,z) = \exp\left(\Phi_{\mathcal{F}}(x,y,z)\right). \tag{20}$$

We will have analogous exponential formulas on the level of fields and algebras in the sequel.

Computation of $\Phi_{\mathcal{F}}(x,y,z)$ and $\Lambda_{\mathcal{F}}(x,y,z)$. The degree n of an isotypical partition factors into ef, where e is the degree of any constituent part and f is the number of such parts. Following the terminology of the number-theoretic situation from which we are abstracting, we call e the *ramification index* and f the *inertial degree*. In our continuing example, both $(3_{1,i}+3_{1,-i})$ and $(2+2+2)$ have degree $n=6$. The corresponding (e,f) are $(3,2)$ in the first case and $(2,3)$ in the second. The tame conductor is always $c_t=(e-1)f$, thus 4 in the first case and 3 in the second. The Swan conductor s of an isotypical partition is the first subscript on any of the parts, these first subscripts being all equal. One has $c_w=fs$, this equation being $2=2\cdot 1$ in the first case and $0=3\cdot 0$ in the second.

Let $\phi_{\mathcal{F}}(e,f,s)$ be the total mass of isotypical $\mathcal{F}$-wild partitions with ramification index e, inertial degree f, and Swan conductor s. One has

$$\phi_{\mathcal{F}}(e,f,s) = \frac{1}{f}|W(p,e_0,e,s)^{\sigma^f}| = \frac{1}{f}q^{f\,d(p,e_0,e,s)}\left(1-\delta_s^{wE}q^{-f}\right). \tag{21}$$

Here, as usual, X^g is the set of fixed points of an operator g on a set X. Also, to unify cases, δ_s^{wE} is 1 in the non-maximal case $s < wE$ and 0 in the maximal case $s = wE$.

Since (21) plays a particularly important role in this paper, we explain how it looks in our continuing example. Take $(p, e_0, e, s) = (3, 2, 3, 1)$ so that $\sigma = \sigma_3$ and take also $f = 2$. The fixed point set of σ^2 on $W(p, e_0, e, s) \cong \overline{\mathbb{F}}_3 - \{0\}$ is $\mathbb{F}_9 - \{0\}$. The Frobenius operator σ on $\mathbb{F}_9 - \{0\}$ has three orbits of size two, corresponding to the irreducible wild partitions $3_{1,\omega} + 3_{1,\omega^3}$ for $\omega \in \{i, 1+i, 2+i\}$. Similarly, σ has two orbits of size one, corresponding to the isotypical but non-irreducible $\mathcal{F}$-wild partitions $3_{1,\omega}+3_{1,\omega}$ for $\omega \in \{1, 2\}$; the multiplicity of these isotypical partitions is 2, so each contributes only 1/2 towards the total mass. So in this case the three quantities equated in (21) are all 4.

The concepts of ramification index and inertial degree together with the key formula (21) let us evaluate our first generating function explicitly:

$$\Phi_{\mathcal{F}}(x, y, z) = \sum_{e=1}^{\infty} \sum_{s \in \mathrm{Ore}(p, e_0, e)} \sum_{f=1}^{\infty} \phi_{\mathcal{F}}(e, f, s) x^{ef} y^{(e-1)f} z^{sf} \tag{22}$$

$$= \sum_{e=1}^{\infty} \sum_{s \in \mathrm{Ore}(p, e_0, e)} \sum_{f=1}^{\infty} \frac{1}{f} q^{d(p,e_0,e,s)f} \left(1 - \delta_s^{wE} q^{-f}\right) x^{ef} y^{(e-1)f} z^{sf} \tag{23}$$

$$= \sum_{e=1}^{\infty} \sum_{s \in \mathrm{Ore}(p, e_0, e)} \log \left(\frac{1 - \delta_s^{wE} q^{d(p,e_0,e,s)-1} x^e y^{e-1} z^s}{1 - q^{d(p,e_0,e,s)} x^e y^{e-1} z^s} \right). \tag{24}$$

The logarithm appearing in (24) is very welcome because it cancels the exponential in (20) to yield the main result of this section:

$$\Lambda_{\mathcal{F}}(x, y, z) = \prod_{e=1}^{\infty} \prod_{s \in \mathrm{Ore}(p, e_0, e)} \left(\frac{1 - \delta_s^{wE} q^{d(p,e_0,e,s)-1} x^e y^{e-1} z^s}{1 - q^{d(p,e_0,e,s)} x^e y^{e-1} z^s} \right). \tag{25}$$

4 The generating function $\Lambda_Q(x)$

Formula (25) is somewhat complicated, as it involves a product over the set $\mathrm{Ore}(p, e_0, e)$ and makes reference to the numbers $d(p, e_0, e, s)$. In this section, we evaluate the two specializations mentioned in the introduction. In both cases, the set $\mathrm{Ore}(p, e_0, e)$ and the numbers $d(p, e_0, e, s)$ each disappear from the final formula.

The specialization $(y,z)=(1,1/q)$**.** Equality (2), namely $\Lambda_{p,e_0,f_0}(x,1,1/q)=\Lambda(x)$, is established by reducing the e-factor of $\Lambda_{p,e_0,f_0}(x,1,1/q)$ to the e-factor of $\Lambda(x)$:

$$\begin{aligned}
\Lambda_{p,e_0,f_0;e}(x,1,\frac{1}{q}) &= \left(\prod_{s\in \mathrm{Ore}(p,e_0,e)-\{wE\}} \frac{1-q^{d(p,e_0,e,s)-1-s}x^e}{1-q^{d(p,e_0,e,s)-s}x^e}\right)\frac{1}{1-q^{d(p,e_0,e,wE)-wE}x^e} \\
&= \left(\prod_{k=0}^{wE-d(p,e_0,e,wE)-1} \frac{1-q^{-k-1}x^e}{1-q^{-k}x^e}\right)\frac{1}{1-q^{d(p,e_0,e,wE)-wE}x^e} \\
&= \frac{1}{1-x^e}.
\end{aligned}$$

Here the first equality is simply a specialization of (25). The second equality holds because as s increases through $\mathrm{Ore}(p,e_0,e)$, the new index $k=s-d(p,e_0,e,s)$ increases by uniform steps of 1 from 0 to $wE-d(p,e_0,e,wE)$, as at each step $(s,d(p,e_0,e,s))$ increases by either $(1,0)$ or $(2,1)$. The third equality holds because each numerator cancels the next denominator, leaving only the denominator of the initial $k=0$ factor.

The specialization $(y,z)=(1,1/q)$ in the context of p-adic fields and algebras corresponds to counting fields and algebras, but weighting wildly ramified algebras less, in accordance with their wild conductor. In the context of fields, this method of weighting was first introduced by Serre [23]. Serre's mass formula was translated into the context of algebras by Bhargava [4]. The technique of generating functions was first used in this context by Kedlaya [13]. The specialization $(y,z)=(1,1/q)$ is the relevant one for the application to number fields made by Bhargava [4], [3]. However to support our Conjecture 11.1 below, we need another specialization.

Our specialization. Our specialization is $(y,z)=(1,1)$. In fact, if we did not take the detour to evaluate Serre's specialization, it would have sufficed to take $z=y$ throughout this paper and work with two-variable generating functions. Following the model provided by Serre's case, we evaluate $\Lambda_{p,e_0,f_0}(x,1,1)$ by evaluating each e-factor separately. As for Serre's case, the main point is that $\Lambda_{p,e_0,f_0;e}(x,1,1)$ is given by a telescoping product. In our case, however, the cancellation is not as complete.

Theorem 4.1. *The quantity* $\Lambda_{p,e_0,f_0;e}(x,1,1)$ *depends only on* $Q=p^{e_0f_0}$ *and is equal to*

$$\Lambda_{Q;e}(x)=\frac{\prod_{j=0}^{w-1}(1-Q^{(p^w-p^{w-j})t/(p-1)}x^e)^{(p-1)p^j}}{(1-Q^{(p^w-1)t/(p-1)}x^e)^{p^w}}. \tag{26}$$

Proof. Let $D(k)=d(p,e_0,e,kE-1)$ be the number of integers in $\{0,\ldots,kE-1\}$ which are not in $\mathrm{Ore}(p,e_0,e)$. This quantity is best understood by thinking in terms of the j^{th} shifted block $[(j-1)E,jE-1]$ rather that the j^{th} block $[(j-1)E+1,jE]$. In these terms, $D(k)$ is the number of omitted entries in the first k shifted blocks of the corresponding Ore table. The j^{th} block contains exactly those integers in $[(j-1)E,jE-1]$ which are not multiples of p^j. So

$$D(k)=E\sum_{j=1}^{k}p^{-j}=E\frac{p^{-1}-p^{-k-1}}{1-p^{-1}}=e_0tp^w\frac{1-p^{-k}}{p-1}=e_0t\frac{p^w-p^{w-k}}{p-1}. \tag{27}$$

The e-factor $\Lambda_{p,e_0,f_0;e}(x,1,1)$ then simplifies to the e-factor of $\Lambda_Q(x)$ as follows:

$$\begin{aligned}
\Lambda_{p,e_0,f_0;e}(x,1,1) &= \left(\prod_{s\in \mathrm{Ore}(p,e_0,e)-\{wE\}} \frac{1-q^{d(p,e_0,e,s)-1}x^e}{1-q^{d(p,e_0,e,s)}x^e}\right)\frac{1}{1-q^{d(p,e_0,e,wE)}x^e}\\
&= \left(\prod_{j=1}^{w}\prod_{d=D(j-1)+1}^{D(j)}\left(\frac{1-q^{d-1}x^e}{1-q^dx^e}\right)^{p^j-1}\right)\frac{1}{1-q^{D(w)}x^e}\\
&= \left(\prod_{j=1}^{w}\left(\frac{1-q^{D(j-1)}x^e}{1-q^{D(j)}x^e}\right)^{p^j-1}\right)\frac{1}{1-q^{D(w)}x^e}\\
&= \frac{\prod_{j=1}^{w}(1-q^{D(j-1)}x^e)^{(p-1)p^{j-1}}}{(1-q^{D(w)}x^e)^{p^w}}\\
&= \frac{\prod_{j=1}^{w}(1-Q^{(p^w-p^{w-j+1})t/(p-1)}x^e)^{(p-1)p^{j-1}}}{(1-Q^{(p^w-1)t/(p-1)}x^e)^{p^w}}.
\end{aligned}$$

Here the first equality is simply a specialization of (25). The second equality holds because as s increases through $\mathrm{Ore}(p,e_0,e)$, the quantity $d=d(p,e_0,e,s)$ increases from 1 to $D(w)$ by steps of 0 or 1. If d occurs in the k^{th} shifted block, then it occurs exactly p^k-1 times, yielding the same factor each time. The third equality makes cancellations within a shifted block of Ore numbers. The fourth equality makes cancellations between adjacent shifted blocks. The fifth equality uses (27) and the fact that $q^{e_0}=Q$. The statement of the theorem is then obtained by the index shift $j\mapsto j-1$. (The indexing in the statement is preferable in the sequel; otherwise, e.g., the three p^j in (29) would all have to be p^{j-1}.) □

For wildness w in $\{0,1,2,3\}$, Formula (26) simplifies to

$$\begin{aligned}
\Lambda_{Q;t}(x) &= \frac{1}{1-x},\\
\Lambda_{Q;pt}(x) &= \frac{(1-x^e)^{p-1}}{(1-Q^tx^e)^p},\\
\Lambda_{Q;p^2t}(x) &= \frac{(1-x^e)^{p-1}(1-Q^{pt}x^e)^{p^2-p}}{(1-Q^{(p+1)t}x^e)^{p^2}},\\
\Lambda_{Q;p^3t}(x) &= \frac{(1-x^e)^{p-1}(1-Q^{p^2t}x^e)^{p^2-p}(1-Q^{(p^2+p)t}x^e)^{p^3-p^2}}{(1-Q^{(p^2+p+1)t}x^e)^{p^3}}.
\end{aligned}$$

These cases more than suffice for the factors of $\Lambda_2(x)$ and $\Lambda_3(x)$ displayed in the introduction. In general, to pass from $\Lambda_{p;e}(x)$ to $\Lambda_{p^{n_0};e}(x)$, each written in this form, one simply replaces each factor of the form $1-cx^e$ by a new factor $1-c^{n_0}x^e$.

For Q regarded as a formal variable, the right side of (26) defines an element of $\mathbb{Z}[Q][[x]]$, i.e. a formal power series in x with coefficients in the polynomial ring $\mathbb{Z}[Q]$. Accordingly, we could allow Q to be any complex number. To not stray too far from our main focus, we allow Q only to be real number ≥ 1, so that $Q=p^\nu$ for some $\nu\geq 0$. As remarked already in the introduction, Q no longer determines p in our enlarged context. Accordingly, we write $\Lambda_{p,Q}(x)$ rather than $\Lambda_p(x)$.

The new flexibility gained by allowing Q to vary offers new insights into our situation. To start, note that if $Q = 1$ then all factors on the right side of (26) reduce to $1 - x^e$, with $p^w - 1$ such factors on the top and p^w on the bottom. Thus (26) reduces to $1/(1 - x^e)$ independent of the wildness w, and so $\Lambda_{p,1}(x) = \Lambda(x)$ for all p. This observation further justifies our terminology "wild partitions," because wild partitions are now continuously related to ordinary partitions.

5 Relation with p-cores

In this section and the next, we continue with the set-up just introduced, so that $Q \geq 1$ is a real number independent from p. Our results in these sections go through even when p is allowed to be any integer larger than one, as long as one still defines wildness by the equation $e = tp^w$ and the condition that p does not divide t. However to not superfluously complicate our exposition, we will stick with our requirement that p is prime.

p-cores. Define

$$\Theta_p(x) = \frac{\Lambda(x)}{\Lambda(x^p)^p} = \left(\prod_{\mathrm{ord}_p(e) \geq 1} (1 - x^e)^{p-1} \right) \left(\prod_{\mathrm{ord}_p(e) = 0} \frac{1}{1 - x^e} \right). \tag{28}$$

The series $\Theta_p(x)$ is the generating function for the sequence of p-cores, i.e. partitions without hook-lengths a multiple of p, as studied in [6]. In the cases $p = 2$ and 3, respectively, $\Theta_p(x)$ is the generating function for A010054 and A033687:

$$\begin{aligned}
\Theta_2(x) &= \sum_{j=-\infty}^{\infty} x^{2j^2 - j} &&= 1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + x^{28} + \cdots \\
\Theta_3(x) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x^{3(j^2+jk+k^2)-j-2k} &&= 1 + x + 2x^2 + 2x^4 + x^5 + 2x^6 + x^8 + 2x^9 + \cdots
\end{aligned}$$

The reference [6] similarly identifies $\Theta_p(x)$ as a theta-series of a quadratic form on $\mathbb{Z}^{p-1}$. By either the combinatoric interpretation or the theta-series interpretation, one gets that $\Theta_p(x)$ has non-negative coefficients, something not obvious from (28).

The analytic properties of $\Theta_p(x)$ are relevant for the asymptotics of the next section. Figure 1 provides a guide. For all p, the radius of convergence is 1. In fact, the sum of the first n coefficients of $\Theta_p(x)^k$ is approximately the volume of a ball of radius $\rho = \sqrt{2n}p^{1/2+1/(2p-2)}$ in Euclidean space of dimension $m = (p-1)k$, thus $\pi^{m/2}\rho^m/(m/2)!$. Because all coefficients are non-negative, for fixed $0 < r < 1$ the function $|\Theta_p(re^{i\theta})|$ takes on its unique maximum at $\theta = 0$.

$\Lambda_{p,Q}(x)$ as a product over p-cores.

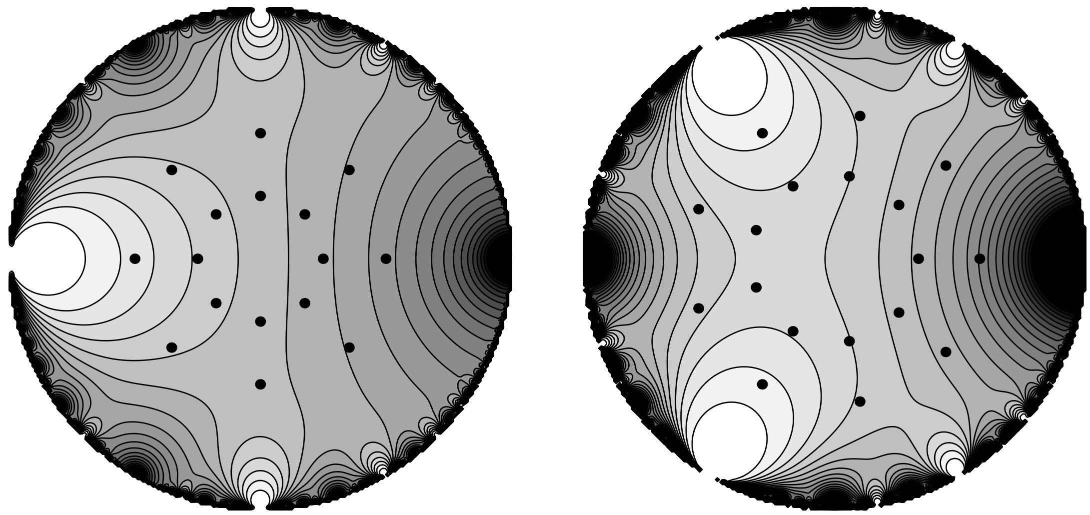


Figure 1: Contour plots of $|\Theta_2(x)|$ on the left and $|\Theta_3(x)|$ on the right, both on the open unit disk $|x| < 1$, with dark shading indicating large values. The arguments indicated by dots are the ones relevant for Table 1. They give the main contribution to the numbers in Table 2.

Corollary 5.1. *One has the identity*

$$\Lambda_{p,Q}(x) \quad = \quad \prod_{j=0}^{\infty} \Theta_p(Q^{(p^j-1)/(p-1)} x^{p^j})^{p^j}. \tag{29}$$

Before beginning the general proof, it is worth noting that the right side of (29) in the case of $Q = 1$ is a telescoping product, namely

$$\Theta_p(x)\Theta_p(x^p)^p\Theta_p(x^{p^2})^{p^2}\cdots = \frac{\Lambda(x)}{\Lambda(x^p)^p}\frac{\Lambda(x^p)^p}{\Lambda(x^{p^2})^{p^2}}\frac{\Lambda(x^{p^2})^{p^2}}{\Lambda(x^{p^3})^{p^3}}\cdots = \Lambda(x). \tag{30}$$

This remark establishes (29) in the case $Q = 1$, as we have already remarked at the end of the previous section that $\Lambda_{p,1}(x) = \Lambda(x)$.

Proof. Substituting $Q^{(p^j-1)/(p-1)}x^{p^j}$ in for x in (28), we get that the factor on the right of (29) with index j is

$$\begin{aligned}
&\Theta_p(Q^{(p^j-1)/(p-1)}x^{p^j})^{p^j} \\
&\quad = \left(\prod_{\mathrm{ord}_p(e)>0} (1 - Q^{(p^j-1)e/(p-1)}x^{p^j e})\right)^{p^{j+1}-p^j} \left(\prod_{\mathrm{ord}_p(e)=0} \frac{1}{1 - Q^{(p^j-1)e/(p-1)}x^{p^j e}}\right)^{p^j} \\
&\quad = \left(\prod_{\mathrm{ord}_p(e)>j} (1 - Q^{(1-p^{-j})e/(p-1)}x^{e})\right)^{p^{j+1}-p^j} \left(\prod_{\mathrm{ord}_p(e)=j} \frac{1}{1 - Q^{(1-p^{-j})e/(p-1)}x^{e}}\right)^{p^j}
\end{aligned} \tag{31}$$

Let $e = p^w t$ as usual. Then the e-factor of the left side of (29) is given by (26) and the e-factor of the right side of (29) is the product of (31) for $j = 0, \ldots, w$. The w numerator factors of (26) match the numerator factors in (31) for $j = 0, \ldots, w-1$. The denominator factor of (26) matches the denominator factor in (31) for $j = w$. $\square$

6 Asymptotics

In this section we study the asymptotic behavior of the coefficients $\lambda_{p,Q,n}$ of the power series $\Lambda_{p,Q}(x)$.

A change of variables. Abbreviate $Q^{-1/(p-1)}$ by $r \in (0,1]$ and change variables via $x = ry$. Define $\underline{\Lambda}_{p,Q}(y) = \Lambda_{p,Q}(ry) = \sum_{n=0}^{\infty} \underline{\lambda}_{p,Q,n} y^n$ so that $\underline{\lambda}_{p,Q,n} = r^n \lambda_{p,Q,n}$. Then (29) takes on the simpler form

$$\underline{\Lambda}_{p,Q}(y) = \prod_{j=0}^{\infty} \Theta_p(ry^{p^j})^{p^j}. \tag{32}$$

As the radius of convergence of $\Theta_p(x)$ is 1, the radius of convergence of the j^{th} factor is r^{-1/p^j}. So as j increases from 1, these radii decrease monotonically from $1/r$ with limit 1.

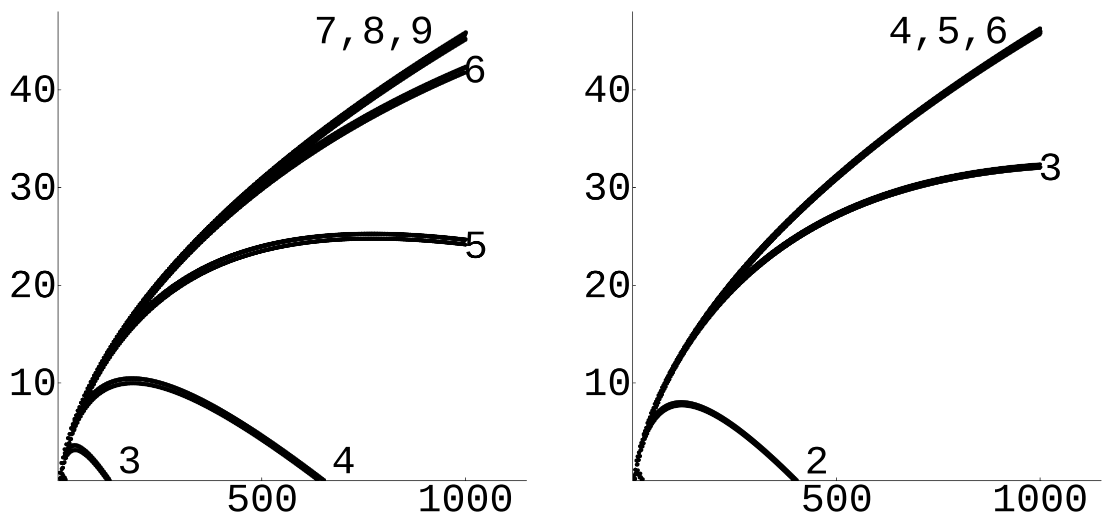


Figure 2: Points $(n, \log(\underline{\lambda}_{2,2,n;k}))$ on the left and $(n, \log(\underline{\lambda}_{3,3,n;k}))$ on the right, with k indicated by text.

We indicate with a subscript k the corresponding objects where the product in (32) is taken from 0 to k. For any given k, the coefficients $\underline{\lambda}_{p,Q,n;k}$ eventually decay exponentially, by the above radius of convergence remarks. Figure 2 illustrates the decay.

Radius of convergence and upper bounds for $\lambda_{p,Q,n}$. We are interested more in the behavior of the coefficients $\underline{\lambda}_{p,Q,n}$ themselves, rather than the cutoff versions $\underline{\lambda}_{p,Q,n;k}$. In general, let $\Theta(z)$ be any power series convergent at least on the closed disk of radius r with $\Theta(0) = 1$ and $|\Theta(r)| > 1$. Then it is elementary that $\prod_{j=0}^{\infty} \Theta(ry^{p^j})^{p^j}$ converges for $|y| < 1$. The product does not converge for $y = 1$ because it formally has the form $\Theta(r)^\infty$. Thus our $\underline{\Lambda}_{p,Q}(y)$ have radius of convergence exactly one. In terms of Figure 2, the upper envelope of the points plotted grows at most sub-linearly. In terms of the original $\lambda_{p,Q,n}$, one eventually has $\lambda_{p,Q,n} < Q^{n/(p-1)+\epsilon}$ for any positive ϵ.

Root growth. Sharpening the statement $\limsup \lambda_{p,Q,n}^{1/n} = Q^{1/(p-1)}$ just observed is the following.

Proposition 6.1. $\lim_{n\to\infty} \lambda_{p,Q,n}^{1/n} = Q^{1/(p-1)}$.

Proof. We need only show that the sequence $\lambda_{p,Q,n}^{1/n}$ has no limit points smaller than $Q^{1/(p-1)}$. In other words, we need only show that the sequence $\underline{\lambda}_{p,Q,n}^{1/n}$ has no limit points smaller than 1. From (28), we know that the expansion of $\Theta_p(x)$ begins with $\sum_{n=0}^{p-1} \lambda_n x^n$ which is term-by-term at least $\sum_{n=0}^{p-1} x^n$. The j^{th} factor $\Theta_p(ry^p)^{p^j}$ of (29) is coefficient-wise bounded from below by $\Theta_p(ry^p)$ which is in turn coefficient-wise bounded from below by $\sum_{a=0}^{p-1} r^a y^{ap^j}$. So $\Lambda_{p,Q}(y)$ is coefficient-wise bounded from below by $\prod_{j=0}^{\infty} \sum_{a=0}^{p-1} r^a y^{ap^j}$. If $n = \sum_{i=0}^{\log_p(n)} a_i p^i$ with $0 \leq a_i \leq p-1$, then

$$\underline{\lambda}_{p,Q,n} \geq \prod_{i=0}^{\log_p(n)} r^{a_i} \geq r^{(1+\log_p(n))(p-1)}$$

so that

$$\underline{\lambda}_{p,Q,n}^{1/n} \geq r^{(1+\log_p(n))(p-1)/n}. \tag{33}$$

The right side of (33) tends to 1 with n, proving that indeed $\underline{\lambda}_{p,Q,n}^{1/n}$ has no limit points smaller than 1. ☐

Expected refined asymptotics. Proposition 6.1 is more than we need to support Conjecture 11.1 below. However the extreme crudeness of the bounds in its proof suggests that stronger statements are provable. Rather than proceed incrementally, in the rest of this section we present numerical evidence and heuristic argument leading up to the very strong statement (37), conjecturally extending the Hardy-Ramanujan asymptotic for λ_n.

Another model of the type of statement sought is given in the next section for analogous quantities $\lambda_{F,n}$ for $F = \mathbb{R}$ and $F = \mathbb{C}$. There, (42) gives an asymptotic equivalent to the root decay factor $\lambda_{F,n}^{1/n}$, analogous to our Proposition 6.1. More subtly, (42) also says that the ratio decay factor $\lambda_{F,n}/\lambda_{F,n-1}$ has the same asymptotic equivalent. Finally (41) is the sharpest statement, giving asymptotic equivalents to $\lambda_{F,n}$ itself.

Ratio oscillation. In contrast to both ordinary partitions and the Archimedean cases, evidence strongly suggests that $\lambda_{p,Q,n}/\lambda_{p,Q,n-1}$ does not have a limiting value for $Q > 1$. Figure 3 graphs $\log(\underline{\lambda}_{p,Q,n})$ for $p = 2, 3$ and $Q = p^j$ with $j = 1, 2$. One sees smooth growth

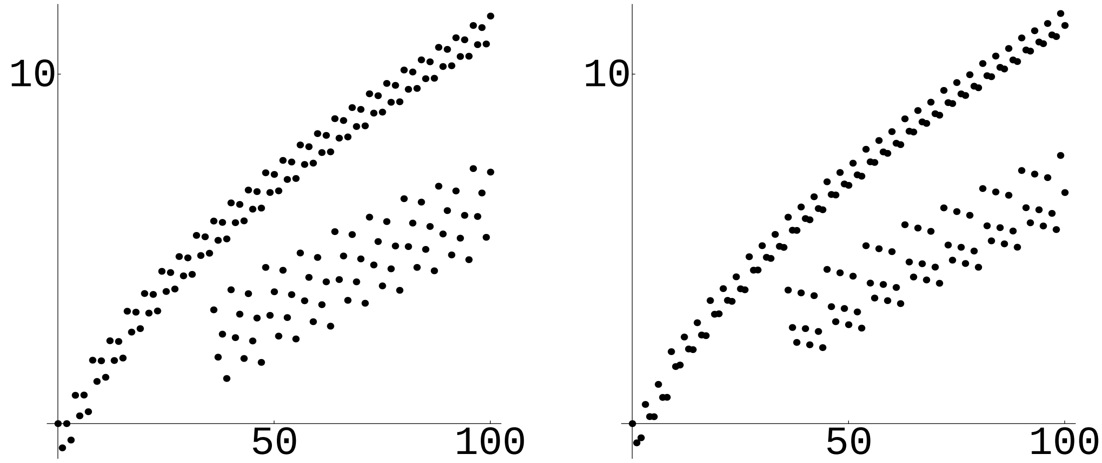


Figure 3: Points $(n, \log(\underline{\lambda}_{2,2,n}))$ on the left and $(n, \log(\underline{\lambda}_{3,3,n}))$ on the right, for $0 \leq n \leq 100$. Also points $(n, \log(\underline{\lambda}_{2,4,n}))$ on the left and $(n, \log(\underline{\lambda}_{3,9,n}))$ on the right, for $36 \leq n \leq 100$. The oscillatory behavior modulo eight on the left and modulo nine on the right matches Table 2 well.

with oscillatory behavior superimposed. The corresponding picture for n out through 4000 shows no damping. In general, oscillatory behavior is barely visible for Q near 1 and increases in amplitude with Q. There seem to be dominant oscillations with period p, secondary oscillations with period p^2, tertiary oscillations with period p^3, and so on. The situation clearly calls for a Fourier analysis.

Figure 4 illustrates with two examples the magnitude of $\underline{\Lambda}_{p,Q}(y)$ as a function of the complex variable y. The needed Fourier analysis is connected with the limiting behavior of $\underline{\Lambda}_{p,Q}(y)$ as $|y|$ increases to 1. For $r < 1$, consider the function

$$\hat{c}_{p,Q}(y) = \prod_{j=0}^{\infty} \left(\frac{\Theta_p(ry^{p^j})}{\Theta_p(r)} \right)^{p^j} \tag{34}$$

defined on the unit circle. One has $\hat{c}_{p,Q}(1) = 1$ as indeed all factors are 1. For y satisfying $y^{p^k} = 1$ but not $y^{p^{k-1}} = 1$ the infinite product reduces to the product of its first k factors, all of which are non-zero with absolute value less than one; thus $0 < |\hat{c}_{p,Q}(y)| < 1$ for these y. Finally if y is otherwise, the infinite product converges to 0.

We view $\hat{c}_{p,Q}(y)$ as giving the normalized boundary values of $\underline{\Lambda}_{p,Q}(y)$. Intuitively, we can view $\underline{\Lambda}_{p,Q}(y)$ as having its most important singularity at 1. This singularity is echoed in quantitatively smaller singularities at primitive p^{th} roots of unity. It is echoed in still smaller singularities at primitive roots of unity of order p^2, and so on, as illustrated by Figure 4. Table 1 gives a more numerical illustration, and includes also the cases $(p, Q) = (2, 4)$ and $(p, Q) = (3, 9)$. For given p, the echoes decay more slowly with larger Q.

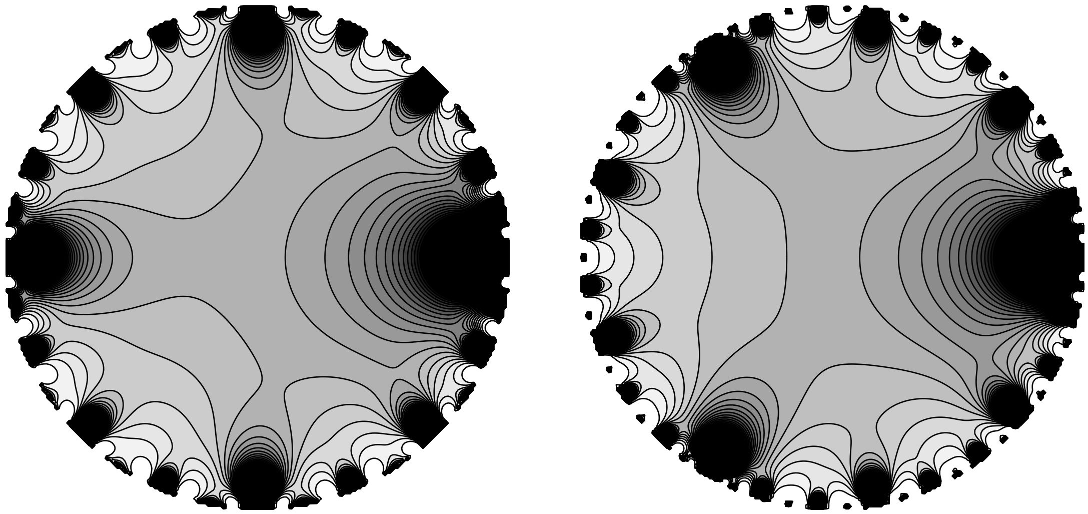


Figure 4: Contour plots of $|\underline{\Lambda}_{2,2}(y)|$ on the left and $|\Lambda_{3,3}(y)|$ on the right, both on the open unit disk $|y| < 1$, with dark shading indicating large values.

The function $\hat{c}_{p,Q}(y)$ enters into our Fourier analysis as follows. Taking an inverse Fourier transform, define

$$c_{p,Q}(n) = \sum_y y^{-n} \hat{c}_{p,Q}(y), \tag{35}$$

the sum being over all p^{th} power roots of unity. One can check that the sum in (35) indeed converges. Moreover, let $\mathbb{Z}_p$ be the p-adic integers, i.e. the completion of $\mathbb{Z}$ with respect to the sequence of finite quotients $\mathbb{Z}/p^j\mathbb{Z}$. Then $c_{p,Q}$, thought of as a function from $\mathbb{Z}$ to $\mathbb{R}$, extends continuously to a function from $\mathbb{Z}_p$ to $\mathbb{R}$. Table 2 gives some values. We expect that

α	$\hat{c}_{2,2}(e^{2\pi i\alpha})$	$\hat{c}_{2,4}(e^{2\pi i\alpha})$
$0/1$	$1.$	$1.$
$1/8$	$0.0005 + 0.0009i$	$0.0567 + 0.0405i$
$1/4$	$0.0341 + 0.0130i$	$0.2660 + 0.0624i$
$3/8$	$0.0007 - 0.0001i$	$0.0490 - 0.0119i$
$1/2$	0.2385	0.5803

α	$\hat{c}_{3,3}(e^{2\pi i\alpha})$	$\hat{c}_{3,9}(e^{2\pi i\alpha})$
$0/1$	$1.$	$1.$
$1/9$	$0.0004 + 0.0010i$	$0.0542 + 0.0569i$
$2/9$	$0.0006 + 0.0002i$	$0.0560 - 0.0029i$
$1/3$	$0.1191 + 0.0210i$	$0.4476 + 0.0718i$
$4/9$	$0.0007 + 0.0001i$	$0.0463 + 0.0205i$

Table 1: Some normalized boundary values $\hat{c}_{p,Q}(y)$ of $\underline{\Lambda}_{p,Q}(y)$, rounded to the nearest ten-thousandth. Only values in the upper half plane are given, because $\hat{c}_{p,Q}(\bar{y}) = \overline{\hat{c}_{p,Q}(y)}$.

n	$c_{2,2}(n)$	$c_{2,4}(n)$
0	1.309	2.324
1	0.788	0.596
2	1.172	1.153
3	0.737	0.324
4	1.304	1.901
5	0.787	0.493
6	1.168	0.944
7	0.734	0.265

n	$c_{3,3}(n)$	$c_{3,9}(n)$
0	1.242	2.208
1	0.918	0.774
2	0.847	0.496
3	1.238	1.878
4	0.918	0.679
5	0.845	0.426
6	1.235	1.600
7	0.915	0.577
8	0.842	0.362

Table 2: Some values of $c_{p,Q}(n)$, rounded to the nearest thousandth. To the nearest thousandth, $c_{2,2}(n)$ depends only on n modulo 8 while $c_{3,3}(n)$ depends only on n modulo 9. Similarly, to the nearest hundredth, $c_{2,4}(n)$ and $c_{3,9}(n)$ depend only on n to the respective moduli 8 and 9.

the function $c_{p,Q}(n)$ fully captures the oscillatory behavior in the sense that

$$\frac{\underline{\lambda}_{p,Q,n}/c_{p,Q}(n)}{\underline{\lambda}_{p,Q,n-1}/c_{p,Q}(n-1)} \sim Q^{1/(p-1)}. \tag{36}$$

Computations such as those illustrated by Figure 5 support this expectation.

Towards an asymptotic equivalent to $\lambda_{p,Q,n}$. The smooth part of $\lambda_{p,Q,n}$ presents more of a mystery. Computations are consistent with the conjecture given as (5) in the introduction, namely

$$\lambda_{p,Q,n} \sim c_{p,Q}(n) C_p(Q) n^{B_p(Q)} e^{A_p(Q)\sqrt{n}} Q^{n/(p-1)} \tag{37}$$

for quantities $A_p(Q)$, $B_p(Q)$, and $C_p(Q)$ to be thought of as functions on the Q-interval $[1,\infty)$.

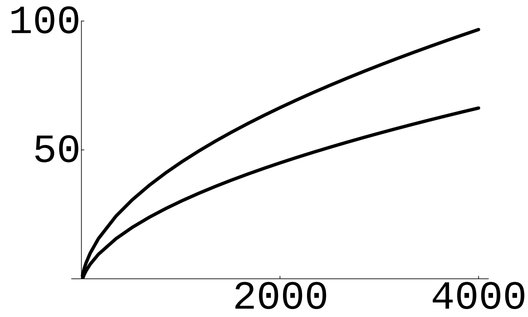
For $Q=1$, the oscillatory factor $c_{p,Q}(n)$ reduces to 1. One has

$$(A_p(1), B_p(1), C_p(1)) = (\pi\sqrt{\frac{2}{3}}, -1, \frac{1}{4\sqrt{3}}) \approx (2.56, -1.00, 0.144),$$

independently of p, by the Hardy-Ramanujan asymptotic for partitions [8]. Figure 5 graphs functions $A_p(Q)\sqrt{n} + B_p(Q)\log n + \log C_p(Q)$ with $(A_p(Q), B_p(Q), C_p(Q))$ deduced from a least squares fit to $\log(\underline{\lambda}_{p,Q,n}/c_{p,Q}(n))$ over $[20, 4000]$. The drawn lines are thick enough so that they contain all the actual points $(n, \log(\underline{\lambda}_{p,Q,n}/c_{p,Q}(n)))$. For the most important quantity $A_p(Q)$, the fit yields

$$A_2(2) \approx 1.66\,, \qquad A_3(3) \approx 1.68\,,$$
$$A_2(4) \approx 1.18\,, \qquad A_3(9) \approx 1.21\,.$$

Numeric computations are not accurate enough to suggest an analytic form for $A_p(Q)$, $B_p(Q)$, and $C_p(Q)$; a more theoretical approach is needed.



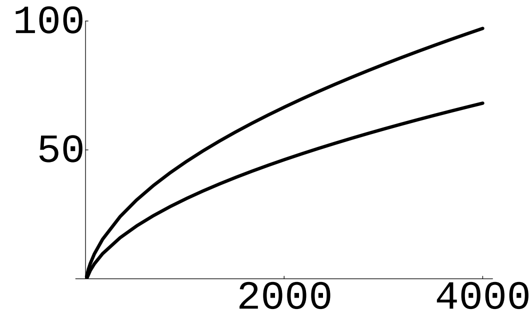


Figure 5: Least square fits to points $(n, \log(\underline{\lambda}_{2,2^j,n}/c_{2,2^j}(n)))$ on the left and $(n, \log(\underline{\lambda}_{3,3^j,n}/c_{3,3^j}(n)))$ on the right, for $j \in \{1, 2\}$ and $20 \leq n \leq 4000$.

7 Fields, algebras, and the cases $F = \mathbb{R}$ and $F = \mathbb{C}$

In this section, we introduce some concepts associated to field and algebra extensions of a given ground field F. These concepts will play a major role in the rest of the paper. Also we illustrate these concepts with the ground fields $F = \mathbb{R}$ and $F = \mathbb{C}$. These ground fields are particularly simple and familiar. Moreover, they play an essential role in the global considerations of Section 11.

Fields and algebras. For F a field, let $\mathrm{Fields}_{F,n}$ be the set of isomorphism classes of separable degree n field extensions of F. Our main interest is in characteristic zero, where all fields are separable; accordingly we drop the adjective "separable." Similarly, let $\mathrm{Algebras}_{F,n}$ be the set of isomorphism classes of degree n algebra extensions which are products of field extensions. For both fields and algebras, we allow ourselves also to drop the phrase "of isomorphism classes" since it always understood. Similarly we write just K instead of say $[K]$ to indicate the isomorphism class of an algebra K.

An algebra K has an automorphism group $\mathrm{Aut}(K/F)$. We define, as is standard, its mass to be $1/|\mathrm{Aut}(K/F)|$. For some fields F, all the sets $\mathrm{Fields}_{F,n}$ are finite. Exactly in this case, all the larger sets $\mathrm{Algebras}_{F,n}$ are finite too. For these F, we define $\phi_{F,n}$ and $\lambda_{F,n}$ to be the total masses of $\mathrm{Fields}_{F,n}$ and $\mathrm{Algebras}_{F,n}$ respectively. Let

$$\Phi_F(x) = \sum_{n=1}^{\infty} \phi_{F,n} x^n = x + \cdots, \qquad \Lambda_F(x) = \sum_{n=0}^{\infty} \lambda_{F,n} x^n = 1 + x + \cdots$$

be the corresponding generating functions. Then one has the exponential formula [25, Chapter 5]

$$\Lambda_F(x) = \exp(\Phi_F(x)), \tag{38}$$

from the definition of mass and the way algebras are built from fields.

The cases $F = \mathbb{R}$ and $F = \mathbb{C}$. With the above definitions, $\mathrm{Fields}_{\mathbb{R},1} = \{\mathbb{R}\}$, $\mathrm{Fields}_{\mathbb{R},2} = \{\mathbb{C}\}$, and otherwise $\mathrm{Fields}_{\mathbb{R},n} = \emptyset$. Also $\mathrm{Algebras}_{\mathbb{R},n} = \{\mathbb{R}^r\mathbb{C}^s : r + 2s = n\}$, with the mass of

$\mathbb{R}^r\mathbb{C}^s$ being $1/(r!s!2^s)$. Even more simply, the only non-empty $\text{Fields}_{\mathbb{C},n}$ is $\text{Fields}_{\mathbb{C},1} = \{\mathbb{C}\}$. One has $\text{Algebras}_{\mathbb{C},n} = \{\mathbb{C}^n\}$, with the mass of $\mathbb{C}^n$ being $1/n!$. Thus

$$\Lambda_{\mathbb{R}}(x) = \sum_{n=0}^{\infty} \lambda_{\mathbb{R},n} x^n \qquad = e^{x+x^2/2} \qquad = 1 + x + \frac{2}{2}x^2 + \frac{4}{6}x^3 + \frac{10}{24}x^4 + \frac{26}{120}x^5 + \cdots, \tag{39}$$

$$\Lambda_{\mathbb{C}}(x) = \sum_{n=0}^{\infty} \lambda_{\mathbb{C},n} x^n \qquad = e^{x} \qquad = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots. \tag{40}$$

The numbers $n!\lambda_{\mathbb{R},n}$ form the sequence A000085 giving, among other interpretations, the number of involutions in the symmetric group S_n. One has the asymptotic formulas

$$\lambda_{\mathbb{R},n} \sim \frac{e^{\frac{n}{2}+\sqrt{n}-\frac{1}{4}} n^{-\frac{n}{2}-\frac{1}{2}}}{2\sqrt{\pi}}, \qquad\qquad \lambda_{\mathbb{C},n} \sim \frac{e^n}{n^{n+\frac{1}{2}}\sqrt{2\pi}}, \tag{41}$$

the first due to Moser and Wyman [17] and the second being Stirling's approximation. On the level of ratio and root behavior, one has

$$\frac{\lambda_{\mathbb{R},n}}{\lambda_{\mathbb{R},n-1}} \sim \lambda_{\mathbb{R},n}^{1/n} \sim \sqrt{\frac{e}{n}}, \qquad\qquad \frac{\lambda_{\mathbb{C},n}}{\lambda_{\mathbb{C},n-1}} \sim \lambda_{\mathbb{C},n}^{1/n} \sim \frac{e}{n}, \tag{42}$$

thus superexponential decay.

8 Eisenstein polynomials

For this and the next two sections, fix a prime number p. Let $\mathbb{Q}_p$ be the field of p-adic numbers. Its ring of integers $\mathbb{Z}_p$ already arose naturally in Section 6. The maximal ideal of $\mathbb{Z}_p$ is generated by the prime number p, and the corresponding residue field is $\mathbb{F}_p = \mathbb{Z}_p/p$. For background on p-adic numbers, see e.g. [7]. We need mainly the algebraic theory of finite degree field extensions of $\mathbb{Q}_p$, i.e. Chapter 5 of [7].

The ground field F. For this and the next two sections, fix also an extension field F of degree n_0 over $\mathbb{Q}_p$. So F can be presented as $\mathbb{Q}_p[x]/g_0(x)$ for some irreducible polynomial $g_0(x)$ in $\mathbb{Q}_p[x]$ of degree n_0. We have no need to consider $g_0(x)$ again, as we will simply regard F as given. Let $\mathcal{O}$ be the ring of integers of F, let Π be its maximal ideal, and let $\kappa = \mathcal{O}/\Pi$. The ramification index of $F/\mathbb{Q}_p$ is the positive integer e_0 such that $\Pi^{e_0} = (p)$. The inertial degree of $F/\mathbb{Q}_p$ is the positive integer f_0 such that $q := |\kappa| = p^{f_0}$. One has $e_0 f_0 = n_0$.

It is often clearer to avoid the language of ideals. To do this we fix a uniformizer π of F, i.e. a generator of Π. For $a \in \mathcal{O} - \{0\}$, we write $\text{ord}_\pi(a) = b$ to mean that a generates the ideal (π^b). We define ord_π on all of $\mathcal{O}$ by writing $\text{ord}_\pi(0) = \infty$.

Extensions of F and their numerical invariants. Likewise, one can consider field extensions $K = F[x]/g(x)$ of F. The degree n of such an extension factors into its ramification index e and its inertial degree f. Another important invariant of a field extension K/F is its discriminant $d(K/F)$, which is an ideal Π^c in $\mathcal{O}$. We focus on the discriminant-exponent c, which we call the conductor. If $K = F[x]/g(x)$, then the ideal generated by the polynomial discriminant

$$D(g) = (-1)^{n(n-1)/2}\mathrm{Res}_x(g(x), g'(x)) \tag{43}$$

has the form Π^{c+2d} for d a non-negative integer, called the defect of $g(x)$.

The conductor of K/F is naturally written as $c = c_t + c_w$, where c_t is the tame conductor and c_w is the wild conductor. Very simply, $c_t = f(e-1)$. The wild conductor c_w is more complicated, but has the form fs, where $s \in \mathrm{Ore}(p, e_0, e)$ is a non-negative integer called the Swan conductor, as detailed below.

We seek to understand the sets $\mathrm{Fields}_{F,n}$ introduced in the previous section. The decomposition

$$\mathrm{Fields}_{F,n} = \coprod_{ef=n} \coprod_{s \in \mathrm{Ore}(p,e_0,e)} \mathrm{Fields}_F(e, f, s) \tag{44}$$

is a natural starting point. In the rest of this section, we explain how Eisenstein polynomials give an explicit understanding of the totally ramified part $\mathrm{Fields}_F(e, 1, s)$. The cases $f > 1$ are easily reduced to the case $f = 1$, as explained in the next section.

Eisenstein polynomials. Consider monic polynomials of degree e with coefficients in $\mathcal{O}$. Such a polynomial

$$g(x) = x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0 \tag{45}$$

is called an Eisenstein polynomial if and only if π divides all the coefficients a_i and moreover π^2 does not divide a_0. Let $\mathrm{Eis}(\mathcal{O}, e)$ be the space of degree e Eisenstein polynomials over $\mathcal{O}$.

If $g(x)$ is an Eisenstein polynomial then $K = F[x]/g(x)$ is a totally ramified field extension of F. Moreover $\mathcal{O}[x]/g(x)$ is its ring of integers which means that the defect d of $g(x)$ is zero. The element $x \in \mathcal{O}[x]/g(x)$ is a uniformizer, meaning that it generates the maximal ideal of $\mathcal{O}[x]/g(x)$.

Conversely, suppose a totally ramified K is given. Then one can consider for each of its uniformizers ω the characteristic polynomial $g_\omega(x)$ of ω acting by multiplication on K, where K is considered as an e-dimensional vector space over F. The resulting map

$$\omega \mapsto g_\omega(x) \tag{46}$$

is $|\mathrm{Aut}(K/F)|$-to-1 over its image $\mathrm{Eis}(\mathcal{O}, e)_K \subseteq \mathrm{Eis}(\mathcal{O}, e)$.

Conductors of Eisenstein polynomials. The conductor $c = c_t + c_w$ of the Eisenstein polynomial (45) is $\mathrm{ord}_x(g'(x))$, where x here is understood as the given uniformizer of $\mathcal{O}[x]/g(x)$ [22, III.6]. The tame conductor c_t is $e - 1$. Thus

$$c_w = \mathrm{ord}_x(g'(x)) - (e-1). \tag{47}$$

For $i = 1, \ldots, e$, define the i^{th} index of an Eisenstein polynomial (45) to be

$$\mathrm{ind}_i(g(x)) = \mathrm{ord}_x(ia_i x^{i-1}) - (e-1) = e\,\mathrm{ord}_\pi(ia_i) + i - e = \begin{cases} e\,\mathrm{ord}_\pi(i\dfrac{a_i}{\pi}) + i, & \text{if } i < e, \\[2ex] ee_0 w = wE, & \text{if } i = e. \end{cases}$$

One has $\mathrm{ind}_i(g(x)) \equiv i$ modulo e, and so the $\mathrm{ind}_i(g(x))$ are all different. The conclusion of these considerations is that the wild conductor c_w of $g(x)$ is the smallest of the indices $\mathrm{ind}_i(g(x))$.

In words, if $i < e$ then $\mathrm{ind}_i(g(x))$ is either greater than wE, and hence irrelevant, or the $\mathrm{ord}_\pi(a_i)^{\mathrm{th}}$ number in column i of the corresponding Ore array $\mathrm{Ore}(p, e_0, e)$. Display (48), a copy of (9) except that Column i has been headed by the corresponding coefficient a_i, illustrates this viewpoint.

$$\begin{array}{ccccccccc} a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ . & & & & & & & & \\ \hline . & 8 & 7 & . & 5 & 4 & . & 2 & 1 \\ . & 17 & 16 & . & 14 & 13 & . & 11 & 10 \\ \hline . & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 \\ 36 & 35 & 34 & 33 & 32 & 31 & 30 & 29 & 28 \end{array} \tag{48}$$

For this displayed case $(p, e_0, e) = (3, 2, 9)$, the wild conductor is the number to the right of the first condition that holds:

$$\begin{array}{rclr} \mathrm{ord}_\pi(a_1) & = & 1 & 1 \\ \mathrm{ord}_\pi(a_2) & = & 1 & 2 \\ \mathrm{ord}_\pi(a_4) & = & 1 & 4 \\ \mathrm{ord}_\pi(a_5) & = & 1 & 5 \\ & \vdots & & \vdots \\ \mathrm{ord}_\pi(a_6) & = & 2 & 33 \\ \mathrm{ord}_\pi(a_7) & = & 4 & 34 \\ \mathrm{ord}_\pi(a_8) & = & 4 & 35 \\ \mathrm{ord}_\pi(a_9) & = & 0 & 36 \end{array} \tag{49}$$

One has a natural decomposition

$$\mathrm{Eis}(\mathcal{O}, e) = \coprod_{s \in \mathrm{Ore}(p,e_0,e)} \mathrm{Eis}(\mathcal{O}, e, s) \tag{50}$$

$$= \coprod_{s \in \mathrm{Ore}(p,e_0,e)} \coprod_{K \in \mathrm{Fields}_F(e,1,s)} \mathrm{Eis}(\mathcal{O}, e, s)_K \tag{51}$$

where $\mathrm{Eis}(\mathcal{O}, e, s)$ consists of all Eisenstein polynomials over $\mathcal{O}$ of degree e and wild conductor s and $\mathrm{Eis}(\mathcal{O}, e, s)_K$ is the subset consisting of polynomials which define K.

9 p-adic algebras: masses

Let $\mathrm{Fields}_{F,n,c_t,c_w}$ be the set of fields over F with degree n, tame conductor c_t, and wild conductor c_w. Let ϕ_{F,n,c_t,c_w} be the total mass of $\mathrm{Fields}_{F,n,c_t,c_w}$. Let $\mathrm{Algebras}_{F,n,c_t,c_w}$ be the corresponding set of algebras and let λ_{F,n,c_t,c_w} be its total mass. One has the corresponding generating functions, satisfying $\Lambda_F(x, y, z) = \exp(\Phi_F(x, y, z))$.

Let $\mathcal{F} = (p, e_0, f_0)$ be the invariants of F. This section explains how the equality

$$\Phi_F(x, y, z) = \Phi_{\mathcal{F}}(x, y, z) \tag{52}$$

follows from the Krasner mass formula. It is this equality which renders the directly defined power series $\Phi_{\mathcal{F}}(x, y, z)$ of interest in algebraic number theory. By exponentiating both sides of (52) one immediately gets $\Lambda_F(x, y, z) = \Lambda_{\mathcal{F}}(x, y, z)$. Besides our given $F = F_1$, there may be non-isomorphic $F_2, \ldots, F_m$ with the same invariants (p, e_0, f_0). Our notation encourages one to also think of $\mathcal{F}$ as representing the numerical equivalence class $\{F_1, \ldots, F_m\}$. Note that if $p|e_0$ then the different F_i have Swan conductors s_0 varying over $\mathrm{Ore}(p, 1, e_0)$, but that not even s_0 enters our considerations.

Volumes. The space of all monic polynomials of degree e with coefficients in $\mathcal{O}$ is naturally identified with $\mathcal{O}^e$ via the coefficients. The quotient space $(\mathcal{O}/\Pi^i)^e$ is a discrete set of size q^{ie}. We view $\mathcal{O}^e$ as a measure space of mass one by requiring that for all i, each fiber of $\mathcal{O}^e \to (\mathcal{O}/\Pi^i)^e$, has mass $1/q^{ie}$. In this measure, $\mathrm{Eis}(\mathcal{O}, e)$ clearly has volume $q^{-e}(1 - q^{-1})$.

As in Section 2, for $s \in \mathrm{Ore}(p, e_0, e)$ let $d(p, e_0, e, s)$ be the number of integers in $\{0, \ldots, s\}$ which are not in $\mathrm{Ore}(p, e_0, e)$. As illustrated by (49), for a random polynomial to be in $\mathrm{Eis}(\mathcal{O}, e, s)$ it has to fail $s - d(p, e_0, e, s)$ successive tests, each of which is failed with probability $1/q$. If $s < wE$, then it moreover has to pass the next test. Accordingly,

$$\mathrm{volume}(\mathrm{Eis}(\mathcal{O}, e, s)) = q^{-e-s+d(p,e_0,e,s)}(1 - q^{-1})(1 - \delta_s^{wE} q^{-1}), \tag{53}$$

with δ_s^{wE} either 1 or 0 according to whether $s < wE$ or $s = wE$, as in Section 3.

The Krasner mass formula and proof of (52). For $K \in \mathrm{Fields}_F(e, 1, s)$, let U_K be its set of uniformizers and let $\mathrm{Eis}(\mathcal{O}, e, s)_K$ be the set of its defining Eisenstein polynomials. Consider again the degree $|\mathrm{Aut}(K/F)|$ cover $U_K \to \mathrm{Eis}(\mathcal{O}, e, s)_K$ of (46). By a Jacobian computation [23], one has

$$\mathrm{mass}(K) = q^{s+e} \frac{\mathrm{volume}(\mathrm{Eis}(\mathcal{O}, e, s)_K)}{1 - q^{-1}}. \tag{54}$$

Summing (54) over $K \in \mathrm{Fields}_F(e, 1, s)$ and eliminating volumes via (53) gives

$$\phi_F(e, 1, s) = q^{d(p,e_0,e,s)}(1 - \delta_s^{wE} q^{-1}). \tag{55}$$

which is the Krasner mass formula [15].

The case of general f reduces to the totally ramified case $f = 1$ via

$$\mathrm{Fields}_F(e, f, s) = \mathrm{Fields}_{F_f}(e, 1, s), \tag{56}$$

where F_f is the unique up to isomorphism degree f extension of F. One has

$$\phi_F(e, f, s) = \frac{1}{f} \phi_{F_f}(e, 1, s) = \frac{1}{f} q^{f\, d(p,e_0,e,s)}(1 - \delta_s^{wE} q^{-f}). \tag{57}$$

Here the first equality of (57) holds because of (56) and the fact that $|\mathrm{Aut}(F_f/F)| = f$. The second equality of (57) holds because of (55), with F replaced by F_f on the left and hence q replaced by q^f on the right.

The quantities $\phi_{\mathcal{F}}(e, f, s)$ and $\phi_F(e, f, s)$ agree because their explicit formulas in (21) and (57) agree. Replacing $\mathcal{F}$ by F in the evaluation (22)-(24) of $\Phi_{\mathcal{F}}(x, y, z)$ then shows that $\Phi_F(x, y, z)$ evaluates to the same explicit formula.

10 p-adic algebras: geometric packets

One obvious difference between the standard $\Lambda_{\mathbb{R}}(x)$, $\Lambda_{\mathbb{C}}(x)$ and our more complicated $\Lambda_Q(x)$ is that the coefficients of the former decay while the coefficients of the latter grow. Another important difference is that the former have non-integral coefficients while the latter, and even the underlying $\Lambda_{p,e_0,f_0}(x, y, z)$, have integer coefficients. In this section, we take a new perspective which explains this integrality conceptually. While the previous section justifies our definitions in Section 2 and 3 numerically, this section goes farther and justifies our combinatorial definitions set-theoretically, explaining the natural objects to which wild partitions correspond.

Algebras over F^{un}. For the new perspective, we fix a maximal unramified extension F^{un}/F, with ring of integers $\mathcal{O}^{\mathrm{un}}$ and associated residual extension $\overline{\kappa}/\kappa$. We let $\sigma \in \mathrm{Gal}(\overline{\kappa}/\kappa) = \mathrm{Gal}(F^{\mathrm{un}}/F)$ be the Frobenius element.

The theory of Eisenstein series over $\mathcal{O}$ goes through without change over $\mathcal{O}^{\mathrm{un}}$. Accordingly, the important set $\mathrm{Fields}_{F^{\mathrm{un}},e}$ of degree e field extensions of F^{un} is identified with the quotient of $\mathrm{Eis}(\mathcal{O}^{\mathrm{un}}, e)$ modulo an equivalence relation $\sim$, where $g_1(x) \sim g_2(x)$ if and only if $F^{\mathrm{un}}[x]/g_1(x)$ and $F^{\mathrm{un}}[x]/g_2(x)$ are isomorphic. One likewise has $\mathrm{Fields}_{F^{\mathrm{un}}}(e, s) = \mathrm{Eis}(\mathcal{O}^{\mathrm{un}}, e, s)/\sim$, these sets being non-empty exactly for $s \in \mathrm{Ore}(p, e_0, e)$. Thus

$$\mathrm{Fields}_{F^{\mathrm{un}}} = \coprod_{e=1}^{\infty} \coprod_{s \in \mathrm{Ore}(p,e_0,e)} \mathrm{Fields}_{F^{\mathrm{un}}}(e, s). \tag{58}$$

The Frobenius element σ acts compatibly on both sides of (58) by taking $K = F^{\mathrm{un}}[x]/\sum a_i x^i$ to $K^\sigma = F^{\mathrm{un}}[x]/\sum a_i^\sigma x^i$. In turn, as for any ground field, $\mathrm{Algebras}_{F^{\mathrm{un}}}$ is the free abelian monoid generated by $\mathrm{Fields}_{F^{\mathrm{un}}}$. We denote by $\mathrm{Algebras}_{F^{\mathrm{un}}}^\sigma$ the set of σ-fixed points, as usual.

Geometric packets. The mass formulas of the previous section transfer to cardinality formulas in our new context as follows. If $K \in \mathrm{Algebras}_F$, then one has its corresponding base-changed algebra $K^{\mathrm{un}} \in \mathrm{Algebras}_{F^{\mathrm{un}}}^\sigma$. Explicitly, if $K = F[x]/g(x)$ then $K^{\mathrm{un}} = F^{\mathrm{un}}[x]/g(x)$. The fiber of the map

$$\mathrm{Algebras}_F \to \mathrm{Algebras}_{F^{\mathrm{un}}}^\sigma \tag{59}$$

above a point $L \in \mathrm{Algebras}_{F^{\mathrm{un}}}^\sigma$ is the set of all $K \in \mathrm{Algebras}_F$ with $K^{\mathrm{un}} \cong L$, i.e. the set of all models of L. These fibers are the geometric packets of the section title. If K_1 and K_2 are in the same fiber, one says they are geometrically equivalent.

The main point letting one convert mass formulas to cardinality formulas is that every geometric packet has total mass one. This is a standard fact from descent theory, but we review the proof here because of the critical role it plays for us. Fix $L \in \mathrm{Algebras}_{F^{\mathrm{un}}}$ and let $A = \mathrm{Aut}(L/F^{\mathrm{un}})$. Let $\mathcal{A} = \mathrm{Aut}(L/F)$. Then one has a short-exact sequence

$$A \hookrightarrow \mathcal{A} \twoheadrightarrow \mathrm{Gal}(\overline{\kappa}/\kappa). \tag{60}$$

Let A^1 be the set of preimages of σ in $\mathcal{A}$. So A acts by conjugation on A^1. Then each $\rho \in A^1$ determines a model L^ρ/F of L/F^{un}. In fact, L^ρ is just the fixed algebra of the subgroup of $\mathcal{A}$ generated by ρ. Also elements of A^1 determine isomorphic models if and only if they differ by conjugation by an element of A; thus the isomorphism class of L^ρ is determined by the conjugacy class $[\rho]$ of ρ. Also, the automorphism group of L^ρ/F is the subgroup of A which fixes ρ. Finally, one has always

$$\sum_\rho \frac{1}{|\mathrm{Aut}(L^\rho/F)|} = \sum_\rho \frac{|[\rho]|}{|A|} = 1, \tag{61}$$

each sum being over representatives of the conjugacy classes in A^1. The last equality in (61) follows because A^1 is partitioned into the classes $[\rho]$ and $|A| = |A^1|$.

Wild partitions and F^{un}-algebras. We have established

$$\left|W(p, e_0, e, s)^{\sigma^m}\right| = \left|\mathrm{Fields}_{F^{\mathrm{un}}}(e, s)^{\sigma^m}\right| \tag{62}$$

as both sides are $q^{f\, d(p,e_0,e,s)}(1 - \delta_s^{wE} q^{-f})$, the left side via (21) and the right side via (57) and the mass one principle. As all orbits on both sides are finite, this implies

$$W(p, e_0, e, s) \cong \mathrm{Fields}_{F^{\mathrm{un}}}(e, s) \tag{63}$$

as σ-sets. A choice of bijections (63) induces a bijection from (p, e_0, f_0)-wild partitions to $\mathrm{Algebras}^\sigma_{F^{\mathrm{un}}}$. In particular, λ_{F,n,c_t,c_w} is the number of F^{un}-algebras of degree n, tame conductor c_t, and wild conductor c_w; this is the promised conceptual explanation of the integrality of λ_{F,n,c_t,c_w}.

Explicit bijections For wild partitions to truly index geometric algebras, one would need to choose explicit σ-invariant bijections from $W(p, e_0, e, s)$ to $\mathrm{Fields}_{F^{\mathrm{un}}}(e, s)$ for each (e, s). We do not need explicit bijections for our purposes, but we describe the simple case $F = \mathbb{Q}_p$ and $e = p$ to give a first indication of how the general case would look. Our description is taken from [2], which also describes the case $e = p$ for general F. In our setting, the possible Swan conductors are $\mathrm{Ore}(p, 1, p) = \{1, \ldots, p\}$. Always the associated dimension is $d(p, 1, p, s) = 1$. If $s < p$ then an explicit σ-equivariant bijection is

$$\begin{aligned} W(p,1,p,s) = \overline{\mathbb{F}}_p^\times &\to \mathrm{Fields}_{F^{\mathrm{un}}}(p, s), \\ a &\to \mathbb{Q}_p^{\mathrm{un}}[x]/(x^p + p\tilde{a}x^s + p). \end{aligned}$$

For $s = p$, an explicit σ-equivariant bijection is

$$\begin{aligned} W(p,1,p,p) = \overline{\mathbb{F}}_p &\to \mathrm{Fields}_{F^{\mathrm{un}}}(p, p), \\ a &\to \mathbb{Q}_p^{\mathrm{un}}[x]/(x^p + p + p^2\tilde{a}). \end{aligned}$$

In each case, $\tilde{a} \in \mathbb{Z}_p^{\mathrm{un}}$ is any lift of a.

Internal structure of geometric packets. Let $L \in \text{Algebras}^{\sigma}_{F^{\text{un}}}$. In many cases, the corresponding geometric packet $\{K_1, \ldots, K_g\} \subset \text{Algebras}_F$ consists of a single algebra of mass one. For example, suppose L has degree p and Swan conductor not divisible by $p-1$. Then automatically K_1/F is not a Galois extension [2] and this forces $|\text{Aut}(K_1/F)| = 1$.

On the other hand, in many other cases $g > 1$. For example, let L be a product of m factors of F^{un}. Then the geometric packet of models for L consists of algebras F_μ, where μ is a partition of m. Here if $\mu = \mu_1 + \cdots + \mu_h$ then $F_\mu = F_{\mu_1} \times \cdots \times F_{\mu_h}$, where, as before, F_f denotes the degree f unramified extension of F. Then $|\text{Aut}(F_\mu/F)| = \prod_k k^{m_k} m_k!$, where m_k is the number of times k appears in μ and Equation (61) becomes the class equation for the symmetric group S_m.

When a packet $\{K_1, \ldots, K_g\}$ contains a totally ramified field then all its elements are totally ramified fields. The database of local fields [12] contains many instances. For example, suppose K_1 is a sextic field with automorphism group S_3 and hence mass $1/6$. Then its packet is $\{K_1, K_2, K_3\}$ where K_2 has mass $1/2$ and K_3 has mass $1/3$. For $i = 1, 2, 3$, the corresponding Galois closures K_i^g have Galois group $\text{Gal}(K_i^g/F)$ with S_3 as inertia subgroup and the cyclic group C_i as corresponding quotient. For $F = \mathbb{Q}_3$, the database presents five such packets.

The packets just discussed correspond to the irreducible partitions of Section 2. More generally suppose a packet $\{K_1, \ldots, K_g\}$ contains a field with residual degree f. Then the K_i which are fields all have residual degree f and their total mass is $1/f$. These packets correspond to the isotypical partitions of Section 2.

11 Number fields

The sets $\text{Fields}_{F,n,S}$**.** Let $F[x] = \mathbb{Q}[x]/g_0(x)$ be a number field. Let $\mathcal{S}(F)$ be its set of places, indexing the set of completions of F. Thus $\mathcal{S}(\mathbb{Q}) = \{\infty, 2, 3, 5, \ldots\}$ and $\mathcal{S}(F)$ maps surjectively to $\mathcal{S}(\mathbb{Q})$.

For $S \subseteq \mathcal{S}(F)$, let $\text{Fields}_{F,n,S}$ be the set of isomorphism classes of degree n field extensions K/F ramified entirely within S. The ramification condition is then that for all $v \in \mathcal{S}(F) - S$, and all $w \in \mathcal{S}(K)$ over S, the local extension K_w/F_v is unramified. In this context, we view $\mathbb{C}/\mathbb{R}$ as ramified.

An extension K/F has a Galois closure K^g and hence a Galois group $\text{Gal}(K^g/F)$; if $K = F[x]/g(x)$, then K^g is by definition a splitting field of $g(x)$. The largest that $\text{Gal}(K^g/F)$ can be is the full symmetric group S_n, with $n = [K : F]$. For $n \geq 3$, the second largest that $\text{Gal}(K^g/F)$ can be is the alternating group A_n. We have a decomposition

$$\text{Fields}_{F,n,S} = \text{Fields}^{\text{sym}}_{F,n,S} \coprod \text{Fields}^{\text{alt}}_{F,n,S} \coprod \text{Fields}^{\text{small}}_{F,n,S} \tag{64}$$

and also write $\text{Fields}^{\text{big}}_{F,n,S}$ to indicate the union of the first two parts. We write also $\text{Fields}^{s}_{F,S} = \coprod_n \text{Fields}^{s}_{F,n,S}$ for any superscript s. In practice, it is easy to decide whether a given field $F[x]/g(x)$ has big or small Galois group. One quick way is to factor $g(x)$ in sufficiently many completions F_v and use information from the degrees of the factor fields K_w; for most v, this reduces to a calculation in the residue field of F_v. For $n \geq 8$, a group-theoretical result of Jordan [9] suffices: the Galois group is big if and only if

$$\text{the degree of } K_w/F_v \text{ is a prime in } (n/2, n-2) \tag{65}$$

for some K_w/F_v. Many other criteria can be brought to bear as well. The computations are guided by the principle that the factor partitions for v not ramified in K are equidistributed in the set of partitions of n according to the measure induced from the Haar measure on $\mathrm{Gal}(K^g/F)$.

We are interested in the case of S finite. Then a classical fact is that the sets $\mathrm{Fields}_{F,n,S}$ are all finite. Analogously to the local situation, it is natural to define the mass of a field K to be $1/|\mathrm{Aut}(K/F)|$. From (64) we have $\phi_{F,n,S} = \phi^{\mathrm{big}}_{F,n,S} + \phi^{\mathrm{small}}_{F,n,S}$. Our main concern is $\phi^{\mathrm{big}}_{F,n,S}$, which is just the cardinality of $\mathrm{Fields}^{\mathrm{big}}_{F,n,S}$ when $n \geq 4$.

For S all of $\mathcal{S}(F)$, a principle in number theory is that the group S_n is very common, in many rigorous senses. One might at first expect that the sets $\mathrm{Fields}_{F,n,S}$ would behave like smaller versions of the set $\mathrm{Fields}_{F,n,\mathcal{S}(F)}$, so that most fields in $\mathrm{Fields}_{F,n,S}$ would be in $\mathrm{Fields}^{\mathrm{sym}}_{F,n,S}$. This section argues that the evidence points in the opposite direction, at least when one fixes S and considers all n simultaneously.

Ease of constructing fields in $\mathrm{Fields}^{\mathrm{small}}_{F,S}$**.** One has $\mathrm{Fields}_{\mathbb{Q},S} = \{\mathbb{Q}\}$ for $S = \{\}$ or $S = \{\infty\}$. Otherwise, $\mathrm{Fields}_{\mathbb{Q},S}$ is infinite, as it at least contains the real cyclotomic fields $\mathbb{Q}[x]/\Phi^+_{p^k}(x)$ for all p in S and all positive k. Since the Galois group of $\mathbb{Q}[x]/\Phi^+_{p^k}(x)$ is abelian, these fields are in $\mathrm{Fields}^{\mathrm{small}}_{\mathbb{Q},S}$ whenever their degree is ≥ 4.

There is an elaborate theory for describing the part of $\mathrm{Fields}^{\mathrm{small}}_{F,S}$ consisting of fields K such that $\mathrm{Gal}(K^g/\mathbb{Q})$ is solvable [14, 18]. This theory says that as soon as S is large enough, $\mathrm{Fields}^{\mathrm{small}}_{F,S}$ is very large indeed. For example, let L be a maximal pro-2-extension of $\mathbb{Q}$ ramified only within $S = \{\infty, 2\}$. Then Markshaitis' theorem [14, Example 11.18] says $\mathrm{Gal}(L/\mathbb{Q})$ is the free pro-2 product of $\mathbb{Z}/2$ and $\mathbb{Z}_2$. Accordingly $\phi^{\mathrm{small}}_{\mathbb{Q},2^k,\{\infty,2\}}$ grows exponentially with $n = 2^k$.

There are also general techniques for constructing non-solvable fields in $\mathrm{Fields}^{\mathrm{small}}_{F,S}$. For example using modular forms gives fields with Galois groups with $PSL_2(\mathbb{F}_{\ell^f})$ as a simple subquotient. Already this technique shows that the Galois group corresponding to any $\mathrm{Fields}_{\mathbb{Q},\{\infty,p,\ell\}}$ has infinitely many simple subquotients different from A_n. The ABC-construction of [20] shows that one can likewise expect infinitely many simple subquotients of the form $PSp_{2k}(\mathbb{F}_\ell)$ involved in $\mathrm{Fields}_{\mathbb{Q},S}$ for S large enough, e.g. $S = \{\infty, 2, 3\}$.

Difficulty of constructing fields in $\mathrm{Fields}^{\mathrm{big}}_{F,S}$**.** All known constructional techniques for fields with Galois group all of A_n or S_n have only modest control over ramifying primes. The most well-known technique, and one of the best, is uses trinomials. For example, take $F = \mathbb{Q}$ and for $t \in \mathbb{Q} - \{0, 1\}$ consider the polynomial

$$g_{n,t}(x) = x^n - ntx + (n-1)t. \tag{66}$$

Its discriminant is

$$D_{n,t} = (-1)^{(n-1)(n-2)/2} n^n (n-1)^{n-1} t^{n-1}(t-1). \tag{67}$$

Its Galois group is generically S_n or A_n according to whether or not $D_{n,t}$ is a square. If one chooses t such that the denominator of t and the numerator of t and $t-1$ are only divisible by primes dividing $n(n-1)$, then only these primes can ramify in $K_{n,t} = \mathbb{Q}[x]/g_{n,t}(x)$. The problem here is that only finitely many n keep ramification within any given S. Even in the more general context of arbitrary trinomials described in [20, Section 10], with two relatively prime parameters $n > m$, there are only finitely many (n, m) such that all primes dividing $nm(n-m)$ are within S. The recent technique of Chebyshev covers [21] gives larger degree fields, but still to get fields ramified within a given S, one needs an appropriate "numerical accident" such as $2^3 + 1 = 3^2$ for $S = \{\infty, 2, 3\}$.

The finiteness conjecture. Based on the considerations just presented, we make the following conjecture.

Conjecture 11.1. *Let F be a number field and let S be a finite set of places of F. Then the set* $\mathrm{Fields}^{\mathrm{big}}_{F,S}$ *is finite.*

In other words, while $\phi^{\mathrm{small}}_{F,S}$ is usually infinite, we expect $\phi^{\mathrm{big}}_{F,S}$ to always be finite.

Heuristic support for the finiteness conjecture. Bhargava [4] has a heuristic formula for the "expected number" of A_n and S_n fields in a given degree n with a given discriminant d. The asymptotic behavior of Bhargava's heuristic as $|d| \to \infty$ agrees with the previously known Davenport-Heilbronn theorem in $n = 3$. In fundamental work, Bhargava has proven the analogous theorem for $n = 4$ [3] and has announced it for $n = 5$, giving one moderate confidence in the heuristic formula for general n.

Applying Bhargava's heuristic to our situation gives the following "expected number"

$$\phi^{\mathrm{big}}_{F,n,S} \approx \frac{1}{2} \prod_{v \in S} \lambda_{F_v,n}, \tag{68}$$

where here we require that all Archimedean places are in S. Both $\lambda_{\mathbb{R},n}$ and $\lambda_{\mathbb{C},n}$ decrease superexponentially with n, according to (42). For each ultrametric place v of F, the sequence $\lambda_{F_v,n}$ increases only exponentially, with growth factor $Q_v^{1/(p_v-1)}$. All together, $\prod_{v \in S} \lambda_{F_v,n}$ decreases superexponentially with n. This is much more than the mere convergence of $\sum_n \prod_{v \in S} \lambda_{F_v,n}$ which would be enough to heuristically support Conjecture 11.1. Note that one has a heuristic product formula (68) only when one appropriately separates by Galois groups. For example, Bhargava's results show that S_4 and D_4 need to be treated separately.

We understand the factor $1/2$ in (68) in two different ways, depending on whether $n = 2$ or $n \geq 3$. The case $n = 2$ is best first explained in the simplified setting $F = \mathbb{Q}$ and $\{\infty, 2\} \subseteq S$. Then $\lambda_{\mathbb{Q}_v,2}$ is 1, 4, or 2 according to whether v is ∞, 2, or otherwise. The set $\mathrm{Fields}_{\mathbb{Q},2,S}$ has exactly $2^{|S|} - 1$ elements, each with mass $1/2$. So its total mass is $2^{|S|-1} - 2^{-1}$ while (68) is $2^{|S|-1}$. The agreement would be perfect if we worked instead with $\mathrm{Algebras}_{\mathbb{Q},2,S}$ to account for the trivial alternating group A_2. For general F, the $1/2$ in the case $n = 2$ likewise comes from the fact that fields in $\mathrm{Fields}_{F,2,S}$ have mass $1/2$ rather than the usual 1. For the cases $n \geq 3$, one uses the local signs $(2, d_v)HW(K_v)$ associated to $K/F \in \mathrm{Algebras}_{F,n,S}$ and a place $v \in S$. While these signs are all 1 in the case $n = 2$, in general they can be 1 or -1.

For $n \geq 3$, the 1/2 in the mass formula corresponds to the fact that the product of the local signs is 1 for any K in $\mathrm{Fields}_{F,n,S}$. For more explicit information on these signs in the case $F = \mathbb{Q}$, see [12, Section 3.3].

Comparison with computational results over $\mathbb{Q}$. Figure 6 summarizes known facts about the case $(F, S) = (\mathbb{Q}, \{\infty, 2, 3\})$. For n = 1, 2, 3, 4, 5, 6, and 7, Jones and Roberts [10], [11] evaluated $\phi^{\mathrm{big}}_{\mathbb{Q},\{2,3,\infty\}}$ to 1, 3.5, 8.$\overline{3}$, 22, 5, 54, and 10. Roberts [20] found more fields in degrees $n = 8, 9$ and also the S_{32} field $\mathbb{Q}[x]/(x^{32} + 2^{16}3^5x^5 + 2^{13}3^9)$ with discriminant $2^{191}3^{112}$. Jones is finding more fields in degrees 8 and 9 by an ongoing computer search. Malle and Roberts [16] found 300 more fields in degree $9 \leq n \leq 33$ and discussed the issue of finiteness of $\phi^{\mathrm{big}}_{\mathbb{Q},S}$ noncommittally as an open question. Roberts [21] found 43 more fields in degrees $12 \leq n \leq 64$.

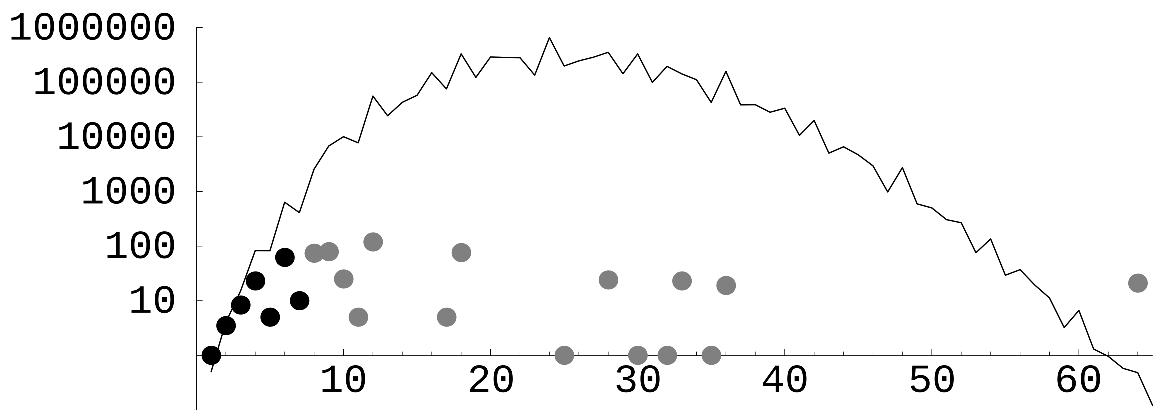


Figure 6: Evaluations (black) and lower bounds (gray) for $\phi^{\mathrm{big}}_{\mathbb{Q},n,\{\infty,2,3\}}$, compared with $\frac{1}{2}\lambda_{\mathbb{R},n}\lambda_{2,n}\lambda_{3,n}$, with logarithmic vertical scale.

Low discriminant phenomena and exceptional fields. Figure 6 also compares the above computational results with the more theoretical quantity $\frac{1}{2}\lambda_{\mathbb{R},n}\lambda_{2,n}\lambda_{3,n}$. Although we are confident in Conjecture 11.1 on a qualitative level, the situation remains enigmatic on a quantitative level.

We interpret the poor agreement in degrees ≤ 7 as reflecting the fact that Bhargava's heuristic does not take into account low discriminant phenomena. Experimentally, these low discriminant phenomena always seem to give fewer fields, with the case of cubics quantitatively explained by Roberts [19] using a negative secondary term. Our guess is that the very poor agreement in medium degrees is due to two factors, the same low discriminant phenomena and the incompleteness of the current list of fields.

On the other hand, the poor agreement in degree 64 is in the other direction. We interpret this disagreement as an indication that the constructional method of [21] is very special in nature. Define a field in $\mathrm{Fields}^{\mathrm{big}}_{F,n,S}$ to be *exceptional* if $\phi^{\mathrm{big}}_{F,n',S} < 1$ for all $n' \geq n$. The starting point $N(F,S)$ of the exceptional range is not as artificial as may first seem, because the decay of $\phi^{\text{big}}_{F,n,S}$ is rapid once it begins.

For $F = \mathbb{Q}$ and $S = \{\infty, 2, 3\}$, $\{\infty, 3, 5\}$, and $\{\infty, 2, 5\}$ the exceptional range starts at $N(\mathbb{Q}, S) = 62$, 38, and 49 respectively. The field constructed in [21] of degree 100, Galois group A_{100}, and discriminant of the form 3^a5^b is well into the exceptional range. Similarly, the five fields constructed there of degrees 2666 through 15875 and discriminant of the form $\pm 2^a5^b$ are exceptional if, as strongly expected, their Galois groups are the full symmetric group on the degree.

Comparison with computational results over quadratic fields. In general, let S be a set of rational places containing ∞. Let F be any degree n_0 number field. For v a place of $\mathbb{Q}$, let $\lambda_{F_v,n} = \prod \lambda_{F_w,n}$, the product being over places w mapping to v. Then $\lambda^{1/n}_{F_\infty,n} \sim (e/n)^{n_0}$, independently of the splitting behavior of ∞ in F. Similarly, $\lambda^{1/n}_{F_p,n} \sim p^{n_0/(p-1)}$ independently of the splitting behavior of p. However, looking at the subexponential factors suggests that $\lambda_{F_v,n}$ is at its lowest if F_v is a field and increases substantially as F_v tends towards the split algebra $\mathbb{Q}_v^{n_0}$.

To illustrate this in practice, let F be the field $F_d = \mathbb{Q}(\sqrt{d})$, where d varies over the set $\{-6, -3, -2, -1, 2, 3, 6\}$. Let $S_d \subset \mathcal{S}(F_d)$ be the set of places mapping to ∞, 2, or 3 in $\mathcal{S}(\mathbb{Q})$. So $|S_d| = 3$ for $d \in \{-6, -3, -1\}$ as none of ∞, 2, or 3 is split in F_d. In contrast, $|S_d| = 4$ if $d \in \{-2, 2, 3, 6\}$ as 3 is split in F_{-2} and ∞ is split in the remaining cases.

Table 3: Evaluations and lower bounds for $\phi^{\text{big}}_{\mathbb{Q}(\sqrt{d}),n,\{\infty,2,3\}_d}$ in italics under d compared with $\frac{1}{2}\lambda_{\mathbb{Q}(\sqrt{d})_\infty,n}\lambda_{\mathbb{Q}(\sqrt{d})_2,n}\lambda_{\mathbb{Q}(\sqrt{d})_3,n}$ in plain type under $\prod$.

							No v split				3 split		∞ split			
n	$\mathbb{C}$	$\mathbb{R}\cdot\mathbb{R}$	4	$2\cdot 2$	9	$3\cdot 3$	-6	-3	-1	$\prod$	-2	$\prod$	2	3	6	$\prod$
2	0.500	1.000	8	16	2	4	*3.5*	*3.5*	*3.5*	4	*7.5*	8	*7.5*	*7.5*	*7.5*	8
3	0.167	0.444	9	25	27	81	$\mathit{14.\overline{3}}$	$\mathit{14.\overline{3}}$	$\mathit{14.\overline{3}}$	20	$\mathit{46.\overline{3}}$	61	$\mathit{40.\overline{3}}$	$\mathit{40.\overline{3}}$	$\mathit{40.\overline{3}}$	54
4	0.042	0.174	272	1296	29	121	*87*	*87*	*87*	164	*343*	686	*385*	*385*	*385*	685
5	0.008	0.047	280	1600	55	361		*17*	*21*	64		421	$\geq$ *87*			361

Table 3 compares the two sides of (68) in degrees $2 \leq n \leq 5$. The first pair of columns gives $\lambda_{\mathbb{C},n} < \lambda^2_{\mathbb{R},n}$ to three decimal places. The next two pairs of columns likewise give $\lambda_{4,n} < \lambda^2_{2,n}$ and $\lambda_{9,n} < \lambda^2_{3,n}$. The column $\lambda^2_{2,n}$ is given for the sake of uniformity, but is not needed as 2 does not split in any of the F_d. Next follow masses $\phi^{\text{big}}_{\mathbb{Q}(\sqrt{d}),n,\{\infty,2,3\}_d}$ in italics, with the corresponding product $\frac{1}{2}\lambda_{\mathbb{Q}(\sqrt{d})_\infty,n}\lambda_{\mathbb{Q}(\sqrt{d})_2,n}\lambda_{\mathbb{Q}(\sqrt{d})_3,n}$ rounded to the nearest integer in regular type to its right. The italicized masses are computed from [10] for $n = 2, 3$ fields and from [5] and associated ongoing searches for $n = 4, 5$. For each n, the references list degree $2n$ extensions of $\mathbb{Q}$ with a corresponding degree $2n$ permutation group G. We extracted those with subfields isomorphic to F_d. Each such field K gives either a single extension of F_d or two conjugate extensions, according to whether the Galois closure K^g of K over F_d is

Galois or not over $\mathbb{Q}$. The Galois groups G giving one and two fields are as follows:

$2n$	One field	Two conjugate fields
4	C_4, V	D_4
6	$A_3C_2^*,\ S_3^tC_2^*,\ S_3C_2$	$T5^*,\ T9,\ T10,\ T13$
8	$A_4C_2^*,\ S_4^tC_2^*,\ S_4C_2$	$T33^*,\ T34,\ T41,\ T42^*,\ T45,\ T46,\ T47$
10	$A_5C_2^*,\ S_5^tC_2^*,\ S_5C_2$	$T40^*,\ T41,\ T42,\ T43$

Here the last group on each list is always $S_n^2.C_2$. The starred groups are the ones giving rise to $\mathrm{Gal}(K^g/\mathbb{Q}(\sqrt{d})) \cong A_n$. This distinction plays a role in the construction of Table 3 only for $n = 3$, as in this case A_3 fields are counted with mass $1/3$; if we were counting the total number of fields then the $n = 3$ entries left to right would be *17*, *23*, *17*, *49*, *41*, *41*, and *43*.

Table 3 reflects again how (68) does not take into account low discriminant phenomena and consequently $\frac{1}{2}\prod_{v\in S}\lambda_{\mathbb{Q}(\sqrt{d})_v,n}$ is an overestimate for the total mass $\phi^{\mathrm{big}}_{F_d,n,S_d}$. However, Table 3 also illustrates a uniformity in the overestimation, as in degrees 2, 3, 4, and 5 the factor is approximately 1, 1.3, 2, and 3.5, independent of d. Thus the principle that splitting primes in the base field increase the number of fields is clearly visible at the level of actual fields.

12 Contrast with positive characteristic

It is standard in number theory to talk about global fields, meaning either number fields as in Section 11 or function fields in one variable over a finite field. By the latter, one means fields F which can be presented as finite extensions of some rational function field $\mathbb{F}_q(t)$. One can talk uniformly about completions of these global fields, getting local fields. In characteristic zero, the local fields are exactly the ones considered in Sections 7-10. In positive characteristic p, all local fields are isomorphic to the Laurent series field $\mathbb{F}_q((u))$, for q some power of p. Often in number theory, the characteristic zero and the positive characteristic situations are quite similar; this principle is well illustrated by our comments on the Serre mass formula below. However with regards to both our specialization $(y, z) = (1, 1)$ and our finiteness conjecture, the case of positive characteristic presents a sharp contrast.

Allowing $e_0 = \infty$. To accommodate positive characteristic, Sections 2 and 3 can be extended by allowing $e_0 = \infty$ also. Ore numbers in this situation are simpler. Namely, as before, $\mathrm{Ore}(p, \infty, e)$ is $\{0\}$ in the tame case when e is not a multiple of p. However in the complementary wild case, the Ore table consists of a single block with infinitely many rows. Thus $\mathrm{Ore}(p, \infty, e)$ is always the complete set of positive integers which are not multiples of p; this is a radical difference because now the number of Ore numbers is infinite. The dimension associated to an Ore number s is simply $d(p, \infty, e, s) = \lceil s/p\rceil$. Since all Ore numbers s are non-maximal, $W(p, \infty, e, s)$ always consists of the vectors in $\overline{\mathbb{F}}_p^{d(p,\infty,e,s)}$ with first component nonzero. The function $\Lambda_{p,\infty,f_0}(x, y, z)$ is defined as before. Writing $s = s_0p + s_1$, with $1 \leq s_1 \leq p-1$, it takes the explicit form

$$\Lambda_{p,\infty,f_0}(x,y,z) = \prod_{s_0=0}^{\infty} \prod_{s_1=1}^{p-1} \frac{1 - q^{s_0} x^e y^{e-1} z^{ps_0+s_1}}{1 - q^{s_0+1} x^e y^{e-1} z^{ps_0+s_1}}, \tag{69}$$

with $q = p^{f_0}$, as always.

A contrast between the two specializations. The two situations in Section 4 now present a sharp contrast. The Serre mass formula continues to hold: $\Lambda_{p,\infty,f_0}(x,1,1/q) = \Lambda(x)$, with essentially the same proof. Our formula $\Lambda_{p,\infty,f_0}(x,1,1) = \Lambda_Q(x)$ becomes completely degenerate as $Q = p^{e_0 f_0}$ needs to be regarded as ∞. The coefficient of x^n on both sides is λ_n if $n < p$ and ∞ otherwise. There is no reasonable analog of Section 5 or 6.

Similar local behavior. Section 7-10 go through in the positive characteristic setting so that in particular the field-theoretic quantity $\Lambda_F(x,y,z)$ agrees with the directly defined $\Lambda_{p,\infty,f_0}(x,y,z)$ of (69). Ramification can be understood from a different perspective in the characteristic p setting. Namely if $K/F = \mathbb{F}_q((v))/\mathbb{F}_q((u))$, then one can write $u = \sum_{i=1}^{\infty} b_i v^i$. The smallest i such that $b_i \neq 0$ is just the degree $n = e$. The smallest i not divisible by p such that $b_i \neq 0$ is $1 + c$, where $c = c_t + c_w = (n-1) + c_w$; this i is just the first exponent such that the corresponding derivative $i b_i v^{i-1}$ is non-zero.

Ease of constructing of fields in $\mathrm{Fields}^{\mathrm{big}}_{F,n,S}$**.** The statement of Conjecture 11.1 makes sense with "number field" replaced by "function field" but it is false in general. As a general source of field extensions of $F = \mathbb{F}_p(t)$, take

$$g(x) = a(x)^p + x + b(t) \tag{70}$$

with $a(x)$ and $b(t)$ running over polynomials with coefficients in $\mathbb{F}_p$. One has always $g'(x) = 1$ and so the polynomial discriminant (43) is $(-1)^{n(n-1)/2}$ with n the degree of $g(x)$. Thus $F[x]/g(x)$ defines a separable algebra which can only be ramified at the the "infinite" place $F_\infty = \mathbb{F}_p((1/t))$ of $\mathbb{F}_p(t)$. If the algebra $F_\infty[x]/g(x)$ has conductor c then $F[x]/g(x)$ is the function field of a curve of genus $1 - n + c/2$ by the Riemann-Hurwitz formula.

As a simple counterexample to the analog of Conjecture 11.1, consider the very special case $g_k(x) = x^2 + x + t^{2k+1}$ for k a non-negative integer. Changing variables via $x = 1/(ys^k)$ and $t = 1/s$ the equation becomes $y^2 + s^{k+1}y + s$. This is an Eisenstein polynomial over the ring $\mathbb{F}_2[[s]]$ with conductor $c = c_t + c_w = (1) + (2k+1) = 2k+2$. So $F[x]/(x^2+x+t^{2k+1})$ is the function field of a genus k curve. Thus the $F[x]/g_k(x)$ form an infinite collection of fields in $\mathrm{Fields}^{\mathrm{big}}_{F,2,\{\infty\}}$. For general p and $a(x)$ of degree $k = 1$, the polynomial $g(x) = a(x)^p + x + b(t)$ likewise defines an Artin-Schreier extension of $\mathbb{F}_p(t)$. For $k > 1$ we expect that "almost all" specializations of the family (70) to extensions of $\mathbb{F}_p(t)$ have Galois group all of A_n or S_n.

Another simple specialization of (70) is $g_{p,k}(x) = x^{kp} - x + t$ for p a prime and k a positive integer. Changing variables via $x = 1/y$ and $t = 1/s$, the equation becomes $y^{kp} - sy^{kp-1} + s$. This is an Eisenstein polynomial with conductor $c = c_t + c_w = (kp-1) + (kp-1) = 2kp - 2$. Thus $\mathbb{F}_p(t)[x]/g_{p,k}(x)$ is the function field of a genus zero curve, a fact which is also clear by the single global observation that $F[x]/g_{p,k}(x) = \mathbb{F}_p(x)$, as $t = -x^{kp} - x$. If $k = p^j$ then one has $g_{p,p^j}(x) = h_j(x) + t$ with $h_j(x)$ the Artin-Schreier polynomial $h_1(x) = x^p - x$ composed with itself j times. Thus the Galois group of $g_{p,p^j}(x)$ is solvable. In the complementary case, computations with Frobenius elements using Jordan's criterion (65) suggest that the Galois group of $g_{p,k}(x)$ is all of A_{pk} or S_{pk} except in the cases $g_{3,4}(x) = x^{12} - x + t$ and $g_{2,12}(x) = x^{24} - x + t$. Here the Galois groups are known to be the Mathieu group M_{11} in its degree 12 representation and the Mathieu group M_{24}, respectively [1, Theorems 6.6 and 6.3]. So the evidence is strong that for each p, the field $\mathbb{F}_p(t)[x]/g_{p,k}(x)$ is in $\text{Fields}^{\text{big}}_{\mathbb{F}_p(t),pk,\{\infty\}}$ for infinitely many k. Abhyankar has studied many similar genus zero families; typically the focus is extracting rare examples with small Galois groups from families with generic Galois group A_n or S_n.

The same heuristic which supports Conjecture 11.1 in characteristic zero gives two reasons why the corresponding statement fails in positive characteristic. Again one can interpret $\frac{1}{2}\prod_{v\in S}\lambda_{F_v,n}$ as the expected total mass of fields in $\text{Fields}^{\text{big}}_{F,n,S}$. But in the function field case if $n \geq p$ then each of the factors $\lambda_{F_v,n}$ is itself infinite; this is the phenomenon behind the existence of the family $g_k(x)$ above. Even if one bounds ramification somehow so that each $\lambda_{F_v,n}$ is replaced by a finite number $\lambda^*_{F_v,n}$, the numbers $\frac{1}{2}\prod_{v\in S}\lambda^*_{F_v,n}$ can still increase due to the lack of Archimedean places. This is the phenomenon behind the existence of the family of polynomials $g_{p,k}(x)$.

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