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Automatic Enumeration of Regular Objects

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Abstract

We describe a framework for systematic enumeration of families combinatorial structures that possess a certain regularity. More precisely, we describe how to obtain the differential equations satisfied by their generating series. These differential equations are then used to determine the initial terms in the counting sequence and for asymptotic analysis. The key tool is the scalar product for symmetric functions.

1 Introduction

Some classes of combinatorial objects naturally possess a substantial amount of symmetry and when formal sums of monomials encoding some parameter of interest are taken over the entire class, symmetric functions, or symmetric series appear. There has been some recent activity to determine how to extract enumerative series of sparse sub-families of these classes directly from the symmetric functions. The principle can be illustrated with one well studied example, the subset of labelled graphs in which the degree of each vertex is a fixed value, say k, known as the *k-regular graphs*. Here, we encode a graph by its degree sequence. When we consider the sum of this encoding over all graphs, they are encoded by the infinite product

$$G(x_1, x_2, \ldots) = \prod_{i<j}(1 + x_i x_j). \tag{1}$$

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This is a well known symmetric series. Further, we remark that the coefficient of $x_1^k x_2^k \cdots x_n^k$ in the formal power series development gives the number of labelled k-regular graphs on n vertices.

Such a coefficient extraction can be set up as a multidimensional Cauchy integral, as described by McKay for regular tournaments and Eulerian digraphs [20], and by McKay and Wormald for graphs with a fixed degree sequence [22]. However, in general this may not be a useful or practical formula.

Indeed these techniques are well developed, and provide general formulae that we cannot currently obtain with the methods here, but they are difficult to make systematic, as they contain a saddle point analysis to make the asymptotic estimate that may be quite fine and specific to the problem.

The primary goal here is to lucidly illustrate how techniques for computing the scalar product of symmetric functions in [7] can be a part of an essentially algorithmic process for asymptotic analysis. At the heart of the method is the fact that the scalar product of symmetric functions preserves a notion of D-finiteness [14], and, thanks to the algorithms in [7], this result is effective.

We begin with a short recollection of symmetric series and D-finiteness, and a brief discussion on some places that D-finite symmetric series appear in combinatorics. We analyse graphs with fixed finite degree sets, and hypergraphs. Finally, in Section 5, we have the results of our semi-automated asymptotic analysis of these classes.

2 Symmetric series and D-finiteness

We provide a basic summary of symmetric functions in order to establish notation. The reader is directed to MacDonald's book [19] for full details.

Denote by $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of the integer n. This means that $n = \lambda_1 + \cdots + \lambda_k$ and $\lambda_1 \geq \cdots \geq \lambda_k > 0$, which we also denote $\lambda \vdash n$. Partitions serve as indices for the five principal symmetric function families that we use: homogeneous (h_λ), power (p_λ), monomial (m_λ), elementary (e_λ), and Schur (s_λ). These are series in the infinite set of variables, $x_1, x_2, \ldots$ over a field K of characteristic 0. When the indices are restricted to all partitions of the same positive integer n, any of the five families forms a basis for the vector space of symmetric polynomials of degree n in $x_1, x_2, \ldots$ On the other hand, the family of p_i's indexed by the integers $i \in \mathbb{N}$ generates the algebra Λ of symmetric functions over K: $\Lambda = K[p_1, p_2, \ldots]$. Furthermore, the p_i's are algebraically independent over $\mathbb{Z}$.

Generating series of symmetric functions live in the larger ring of symmetric series, $K[t][[p_1, p_2, \ldots]]$. There, we have the generating series of homogeneous and elementary functions:

$$H(t) = \sum_n h_n t^n = \exp\left(\sum_i p_i \frac{t^i}{i}\right), \qquad E(t) = \sum_n e_n t^n = \exp\left(\sum_i (-1)^i p_i \frac{t^i}{i}\right).$$

We often refer to $H = H(1)$ and $E = E(1)$.

Alternatively, the power notation $\lambda = 1^{n_1} \cdots k^{n_k}$ for partitions indicates that i occurs

n_i times in λ, for $i = 1, 2, \ldots, k$. The normalization constant

$$z_\lambda := 1^{n_1} n_1! \cdots k^{n_k} n_k!$$

plays the role of the square of a norm of p_λ in the following important formula:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda, \tag{2}$$

where $\delta_{\lambda,\mu}$ is 1 if $\lambda = \mu$ and 0 otherwise.

The scalar product is a basic tool for coefficient extraction. Indeed, if we write $F(x_1, x_2, \ldots)$ in the form $\sum_\lambda f_\lambda m_\lambda$, then the coefficient of $x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ in F is $f_\lambda = \langle F, h_\lambda \rangle$. Moreover, when $\lambda = 1^n$, the identity $h_{1^n} = p_{1^n}$ yields a simple way to compute this coefficient when F is written in the basis of the p's. When viewed at the level of generating series, this fact gives the following theorem:

Theorem 2.1 (Gessel[14]; Goulden & Jackson[15]). *Let θ be the K-algebra homomorphism from the algebra of symmetric functions over K to the algebra $K[[t]]$ of formal power series in t defined by $\theta(p_1) = t$, $\theta(p_n) = 0$ for $n > 1$. Then if F is a symmetric function,*

$$\theta(F) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!},$$

where a_n is the coefficient of $x_1 \cdots x_n$ in F.

To end our brief recollections of symmetric functions recall that plethysm is a way to compose symmetric functions. An inner law of Λ, denoted $u[v]$ for u, v in Λ, it satisfies the following rules [29], with $u, v, w \in \Lambda$ and α, β in K

$$(\alpha u + \beta v)[w] = \alpha u[w] + \beta v[w], \quad (uv)[w] = u[w]v[w],$$

and if $w = \sum_\lambda c_\lambda p_\lambda$ then $p_n[w] = \sum_\lambda c_\lambda p_{(n\lambda_1)} p_{(n\lambda_2)} \ldots$. For example, consider that $w[p_n] = p_n[w]$, and in particular that $p_n[p_m] = p_{nm}$. In a mnemonic way:

$$w[p_n] = w(p_{1n}, p_{2n}, \ldots, p_{kn}, \ldots) \qquad \text{whenever} \qquad w = w(p_1, p_2, \ldots, p_k, \ldots).$$

2.1 D-finite multivariate series

Recall that a series $F \in K[[x_1, \ldots, x_n]]$ is *D-finite* in $x_1, \ldots, x_n$ when the set of all partial derivatives and their iterates, $\partial^{i_1 + \cdots + i_n} F / \partial x_1^{i_1} \cdots \partial x_n^{i_n}$, spans a finite-dimensional vector space over the field $K(x_1, \ldots, x_n)$. A *D-finite description* of a series F is a set of differential equations that establishes this property. A typical example of such a set is a system of n differential equations of the form

$$q_1(x) f(x) + q_2(x) \frac{\partial f}{\partial x_i}(x) + \cdots + q_k(x) \frac{\partial^k f}{\partial x_i^k}(x) = 0,$$

where i ranges over $1, \ldots, n$, each q_j is in $K(x_1, \ldots, x_n)$ for $1 \leq j \leq k$, and k and q_j depend on i.

Such a system is a typical example of a D-finite description of a functions, and often this will be the preferred form for manipulating f. In truth we can accept any basis that generates the vector space of partial derivatives, but in the applications below, this form is particularly easy to obtain.

2.2 D-finite symmetric series

The following definition of D-finiteness of series in an infinite number of variables is given by Gessel [14], who had symmetric functions in mind. A series $F \in K[[x_1, x_2, \ldots]]$ is *D-finite* in the x_i if the specialization to 0 of all but a finite (arbitrary) choice, S, of the variable set results in a D-finite function (in the finite sense). In this case, many of the properties of the finite multivariate case hold true. One exception is closure under algebraic substitution, which requires additional hypotheses.

The definition is then tailored to symmetric series by considering the algebra of symmetric series as generated over K by the set $\{p_1, p_2, \ldots\}$: a symmetric series is called *D-finite* when it is D-finite in the p_i's[2].

> **Example.** Both $H(t)$ and $E(t)$ are D-finite symmetric functions, as for any specialization of all but a finite number of the p_i's to 0 results in an exponential of a polynomial. Similarly, $\exp(h_k t)$ is D-finite because $h_k = \sum_{\lambda \vdash k} p_\lambda$ is a polynomial in the p_is.

The closure under Hadamard product of D-finite series [18] yields the consequence:

Theorem 2.2 (Gessel)**.** *Let f and g be elements of $K[t_1, \ldots, t_k][[p_1, p_2, \ldots]]$, D-finite in the p_i's and t_j's, and suppose that g involves only finitely many of the p_i's. Then $\langle f, g\rangle$ is D-finite in the t_j's provided it is well-defined as a power series.*

2.3 Effective calculation and algorithms

In our initial study [7] we gave an algorithm which, given a D-finite descriptions of two functions satisfying the hypothesis of Theorem 2.2, determines a D-finite description of the series of the scalar product. Henceforth, we shall refer to this algorithm as scalar_de. As we noted in [7], a second algorithm, hammond, based on the work of Goulden, Jackson and Reilly [15] applies in the case when $g = \exp(h_n t)$, which we shall see is precisely how one can extract the exponential generating series of sub-classes with "regularity". They are implemented in Maple, are are available for public distribution at the website `http://www.math.sfu.ca/~mmishna`. Maple worksheets illustrating the calculations presented are also available at that same site.

3 D-finite symmetric series appear naturally in combinatorics

Species theory (in the sense of [3, 17]) is a formalism for defining and manipulating combinatorial structures that relates classes to encoding series. An important connection to our work here is that the series for structures we consider are D-finite symmetric series, and many of the natural combinatorial actions preserve D-finiteness on the level of these series.

The reader unfamiliar with species is heartily encouraged to consult [3]. A *species* associates to every set a family of structures in a way such that two sets of the same cardinality

[2] This is interestingly enough *not* equivalent to D-finiteness with respect to either the h or e basis.

Object	Series	Symmetric function	Object	Series	Symmetric function
2-sets	Γ_{E_2}	$e_2 = \frac{p_1^2}{2} - \frac{p_2}{2}$	2-multisets	Z_{E_2}	$h_2 = \frac{p_1^2}{2} + \frac{p_2}{2}$
3-sets	Γ_{E_3}	e_3	3-multisets	Z_{E_3}	h_3
4-sets	Γ_{E_4}	e_4	4-multisets	Z_{E_4}	h_4
k-sets	Γ_{E_k}	e_k	k-multisets	Z_{E_k}	h_k
3-cycles	Z_{C_3}	$\frac{p_1^3}{3} + \frac{p_3}{3}$	triples	Z_{X^3}	p_1^3
4-cycles	Z_{C_4}	$\frac{p_1^4}{4} + \frac{p_2^2}{12} + \frac{p_4}{12}$	4-arrays	Z_{X^4}	p_1^4
5-cycles	Z_{C_5}	$\frac{p_1^5}{5} + \frac{p_5}{30}$	5-arrays	Z_{X^5}	p_1^5
k-cycles	Z_{C_k}	$\sum_{cd=k} \phi(d) \frac{p_d^c}{k!}$	k-arrays	Z_{X^k}	p_i^k

Table 1: Index series of small species and their corresponding symmetric functions

yield the same family, upto isomorphism. For example, the species of sets E on the underlying set U is simply $\mathsf{E}[U] = U$. The species of lists $\mathsf{L}[U] = \{(x_1, x_2, \ldots, x_n) : x_i \in U, n = 0, 1, 2 \ldots\}$, is the set of finite ordered collections of elements. The atomic species, $X[U]$ is U if U contains a single element, and is empty otherwise.

The theory of species develops a rigorous formalism that allows a sort of calculus of combinatorial families. For example we construct lists of length 4 from our atomic species via multiplication: $\mathsf{L}_4[U] = X^4[U]$.

The key feature that we use are that for every combinatorial family (species) F that one can define, there is an associated cycle index series Z_{F} and an asymmetric cycle index series Γ_{F}, both of which are symmetric series. Recall for any species F its cycle index series Z_{F} is the series in $\mathbb{C}[\![p_1, p_2, \ldots]\!]$ given by

$$Z_{\mathsf{F}}(p_1, p_2, \ldots) := \sum_n \sum_{\lambda \vdash n} \operatorname{Fix} \mathsf{F}[\lambda] \frac{p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}}{z_\lambda}, \tag{3}$$

where the value of $\operatorname{Fix} \mathsf{F}[\lambda]$ is the number of structures of F that remain fixed under some labelling permutation of type[3] λ, and m_k gives the number of parts of λ equal to k.

The definition of the *asymmetry index series* of a species F, denoted Γ_{F}, as introduced by Labelle [3] is related, but more subtle. The series Γ behaves analytically in much the same way as the cycle index series, notably, substitution (in almost all cases) is reflected by plethysm, etc. Essentially, this series counts the objects with no internal symmetry. Table 1 contains some small examples of both series.

In a fashion similar to the cycle index series, Γ_{F} arises through the enumeration of colorings of asymmetric F-structures.

A notable example is the species of sets, E. Recall for any finite set U we have that $\mathsf{E}[U] = U$. The two series above turn out to be $Z_{\mathsf{E}} = \exp(\sum_n p_n/n) = \sum_n h_n$ and $\Gamma_{\mathsf{E}} = \exp(\sum_n (-1)^n p_n/n) = \sum_n e_n$.

The primary advantage of this approach, as is true with any generating series approach, is that natural combinatorial operations (set, cartesian product, substitution) coincide with

[3] A permutation of type $(1^{m_1}, 2^{m_2}, \ldots)$ has m_1 fixed points, m_2 cycles of length 2, etc.

straighforward analytic operations (sum, product, plethystic substitution)[4].

The *exponential generating series* of a species F is the sum $\mathsf{F}(t) = \sum_n |\mathsf{F}[n]| \frac{t^n}{n!}$, where $|\mathsf{F}[n]|$ is the number of structures of type F on a set of size n. The *ordinary generating function*, $\widetilde{\mathsf{F}}(t)$, is the sum $\mathsf{F}(t) = \sum_n \mathrm{Orb}(\mathsf{F}[n])t^n$, where $\mathrm{Orb}(\mathsf{F}[n])$ is the number structures of F on a set of size n distinct up to relabelling. Also recall the notation $[x^n]f(x)$ refers to the coefficient of $[x^n]$ in the expansion of $f(x)$. This definition extends likewise to monomials.

The next result is essentially a collection of known results and basic facts of D-finite series.

Theorem 3.1. *Suppose F is a species such that Z_{F} is a D-finite symmetric series and write $p_n = x_1^n + x_2^n + \ldots$. Then all of the following series are D-finite with respect to t:*

1. *The exponential generating function $\mathsf{F}(t)$;*
2. *The ordinary generating function $\widetilde{\mathsf{F}}(t)$, if the additional condition that $Z_{\mathsf{F}}(p_1, p_2, \ldots)$ is D-finite with respect to the x_i variables is also true;*
3. *The series $\sum_n \left([x_1^k \cdots x_n^k] Z_{\mathsf{F}}\right) \frac{t^n}{n!}$, for fixed k;*
4. *The series $\sum_n \sum_{\bar{k} \in S^n} \left(\left[x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}\right] Z_{\mathsf{F}}\right) t^n/n!$, for any finite set $S \subset \mathbb{N}$.*

Proof. The first two parts are proved using two basic results about cycle index series:

$$\mathsf{F}(t) = Z_{\mathsf{F}}(t, 0, 0, \ldots) \quad \text{and} \quad \widetilde{\mathsf{F}}(t) = Z_{\mathsf{F}}(t, t^2, t^3, \ldots)$$

The first specialization is well-known to preserve D-finiteness [28] for any n. The additional condition on the second item is sufficient to prove the D-finiteness since the stated substitution is the same as $x_1 \mapsto t$, and $x_i \mapsto 0$ otherwise.

The third item of the proposition is proved by the expression

$$\sum_n \left([x_1^k \cdots x_n^k] Z_{\mathsf{F}}\right) \frac{t^n}{n!} = \langle Z_{\mathsf{F}}, \exp(th_k) \rangle \,,$$

which is D-finite by Theorem 2.2.

The final item of the proposition is true because the series is equal to

$$\left\langle Z_{\mathsf{F}}, \exp(t \sum_{i \in S} h_i) \right\rangle,$$

which is also D-finite by Theorem 2.2. $\square$

We have one large class of species for which the cycle index series is D-finite. All of our examples come from this class.

Theorem 3.2. *Let E be the species of sets and let P be a polynomial species with finite support*[5]*. Then $\mathsf{F} = \mathsf{E} \circ \mathsf{P}$ describes a species for which Z_{F} is a D-finite symmetric series and provided that $\mathsf{P}(0) = 0$, Γ_{F} is also a D-finite symmetric series.*

[4] This is slightly less true with the asymmetry index series, but true enough for our purposes.

[5] Species that can be written as polynomials of *molecular species*. For example, every species in Table 1 is polynomial.

Proof. If P is a polynomial species, then its cycle index series is a polynomial in the p_i's, say $P(p_1, \ldots, p_n)$. Composition of species is reflected in the cycle index series by plethysm, thus

$$Z_{\mathsf{F}} = \exp(\sum_k p_k/k)[P(p_1, \ldots, p_n)] = \exp\left(\sum P(p_k, p_{2k}, \ldots, p_{nk})/k\right).$$

For any specialization of all but a finite number of p_i to zero, this gives an exponential of a polynomial, which is clearly D-finite. Thus, Z_{F} is D-finite. We can similarly show that Γ_{F} is also D-finite under the stated conditions, since the composition also results in a plethystic composition. □

Our concluding remarks in Section 6.2 address the more general question of combinatorial criteria on a species F that ensure that Z_{F} or Γ_{F} are D-finite.

4 Using species to describe regular graph-like structures

Ultimately our goal is to generalize the well-studied case of k-regular graphs to other structures whose cycle index series are D-finite. To do so, we express the graph encoding by degree sequence as symmetric series, and describe how to find such a representation in general using species theory.

In Eq. (1) we define $G(x_1, x_2, \ldots)$ as the encoding over all graphs of their degree sequence and we express this as an infinite product. It turns out that this series is equivalent to $E[e_2]$, which is equivalent to $\Gamma_{\mathsf{E}\circ\mathsf{E}_2}$. The equivalent series for multigraphs (with loops) is equal to $H[h_2] = Z_{\mathsf{E}\circ\mathsf{E}_2}$, and thus suspecting an explanation via species, we investigate this connection. Specifically, how do we construct symmetric function equations to describe the generating functions of different families of objects, such as hypergraphs.

We begin with the remark that $\mathsf{F} = \mathsf{E} \circ \mathsf{E}_2$ is *not* the species of graphs. It is the species of partitions into 2-sets. For example, $\{\{1,4\},\{2,6\},\{3,7\},\{5,8\}\}$ is an element of F, and we should not think that this is the graph on 8 vertices, with four edges, rather it just gives the basic structure, i.e. four edges.

We express the Polya cycle index in the power series symmetric function. As a series in the symmetric x_i indeterminates, it is an inventory of distinct (non-isomorphic) colorings of the elements of the species. For example, the non-isomorphic colorings (by positive integers, say) of the set $\{a, b\}$ is the set of maps $\{(a,b) \mapsto (i,j) \in \mathbb{N}^2 : i \leq j\}$, and the inventory of all such colorings is $\sum_{i\leq j} x_i x_j = h_2$.

A coloring of an element in $\mathsf{E} \circ \mathsf{E}_2$ gives rise to a graph (See Figure 1) and two colorings are isomorphic if one is a graph relabelling of the other. The monomial encoding a coloring indicates how many time each color was used, that is, in how many edges the color appears, that is, the degree of the vertex represented by the color.

We restate this correspondence. The species $\mathsf{E} \circ \mathsf{E}_2$ indicates the structure– sets of pairs. The cycle index series $Z_{\mathsf{E}\circ\mathsf{E}_2}$ encodes non-isomorphic colorings of elements, which are in turn equivalent to labelled multi-graphs. The sets of pairs indicate edges, and the colors indicate vertices.

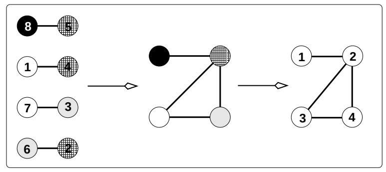

Figure 1: The graph associated with the colored set partition $\{\{1_3, 4_2\}, \{2_2, 6_4\}, \{3_4, 7_3\}, \{5_2, 8_1\}\}$

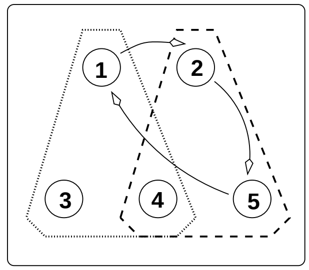

Figure 2: A structure composed of a set of smaller structures (cycles and 3-sets).

For many applications, like regular graphs, we would like to count colorings without repetition. In this case, we do not allow repetition of a color in a given object, hence to encode a k-set, each color appears exactly once, and this is precisely the notion of asymmetry in the asymmetry cycle index, and thus we use Γ instead of Z. Remark, and thus

$$\Gamma_{\mathsf{E}_k} = e_k \quad \text{and thus,} \quad \Gamma_{\mathsf{E}} = E.$$

Taking the same species $\mathsf{E} \circ \mathsf{E}_2$ as above, and using the asymmetry index series with a similar argument, we get that $\Gamma_{\mathsf{E}\circ\mathsf{E}_2} = E[e_2]$ encodes simple graphs without loops on the set of colors precisely as is determined by Eq. (1): $E[e_2] = \prod_{i<j}(1 + x_i x_j)$. This gives us a way to have *direct access to monomial encodings of combinatorial objects*, as symmetric functions expressed in common bases, like the power sum basis. These two series are compatible and, one can show that graphs with loops are encoded by $E[h_2]$, and graphs with multiple edges, but no loops are given by $H[e_2]$.

More generally, we can consider any structure that is built as a set of objects from a finite set of classes. Figure 2 shows a more general object built as a set of cycles and sets. In this framework it is encoded by the monomial $x_1^2 x_2^2 x_3 x_4^2 x_5^2$, and thus we see that regularity in this situation refers to the number of times each label appears in one of the smaller substructures.

Using this framework we can examine other species of structures built up from smaller objects. These species are such that both Z_{F} and Γ_{F} give rise to interesting combinatorial objects.

We can produce enumerative results for objects all of the same flavour: labelled sets of objects in which there is a certain regularity. We begin with a natural generalization of k-regular graphs, and then we consider other types of objects such as hyper-graphs, and directed graphs.

i,j	Initial terms in the counting sequence
1,2	1,0,1,4,18,112,820,6912,66178,708256,8372754,108306280,1521077404
1,3	1, 0, 1, 0, 8, 0, 730, 0, 188790, 0, 102737670, 0, 102172297920,0
1,4	1,0,1,0,3,6,30,1011,38920,1920348,116400186,8580463110,757574641296
2,3	1,0,0,1,10,112,1760,35150,848932,24243520,805036704,30649435140
2,4	1,0,0,1,3,38,730,20670,781578,37885204,2289786624,168879532980
3,4	1,0,0,0,1,26,820,35150,1944530,133948836,11234051976,1127512146540

Table 2: Counting sequences for $\{i, j\}$-regular graphs for small values of i and j.

4.1 S-regular graphs

A graph is *S-regular* if the set of vertex degrees in the graph is a subset of S. For example, a graph is $\{i, j\}$-regular if every vertex is of degree i or j.

It does not seem that the asymptotic enumeration of these objects has been directly considered before. It is, in some sense, a variation of the asymptotic number of labelled graphs with a given degree sequence, which has been considered by Bender and Canfield [2] and McKay and Wormald [22], and may very well be computable from this.

Thus, the scalar product that represents the generating series for the number of $\{i, j\}$-regular graphs is given by incorporating this factor, which ultimately greatly simplifies the calculation. The exponential generating series for the number of $\{i, j\}$-regular graphs is given by

$$G_{i,j}(t) = \big\langle E[e_2], \exp\left(t(h_i + h_j)\right) \big\rangle.$$

This is clearly D-finite, and computable using scalar_de (although not hammond). Furthermore, by a similar computation, we have the following result.

Theorem 4.1. *The number of S-regular graphs is D-finite for any finite $S \subset \mathbb{N}$, and its exponential generating series is given by the scalar product of symmetric functions,*

$$G_S(t) = \left\langle E[e_2], \exp\left(\sum_{i\in S} h_i t\right)\right\rangle.$$

Table 2 offers the initial counting sequence for some small values of k and j. Each one corresponds to a known differential equation satisfied by it generating function. In Table 3 we compute the asymptotic number of some of these graphs.

We make one simple observation. The $\{k, k+2\}$-regular graphs are isomorphic to the $k+2$-regular graphs with loops, by simply adding loops to the vertices of degree k. This gives a family of identities

$$\langle E[e_2], \exp\left(t(h_i + h_{i+2})\right)\rangle = \langle E[h_2], \exp(t h_{i+2})\rangle\,.$$

4.2 Set covers and uniform hypergraphs

We now illustrate the method on another family of objects, which results in set covers and uniform hypergraphs. An n-set is a set of cardinality n.

Definition 1 (k-cover of a set)**.** *A collection of r-sets $\mathcal{B} = \{B_1, \ldots, B_r\}$ is an r-*cover *of S if $\bigcup_{i=1}^{r} B_i = S$. If $S = [n] = \{1, 2, \ldots, n\}$, then it is an r-cover of n. A cover is* restrictive *if all of the B_i are distinct. An r-cover is k-*regular *if any given element occurs in exactly k-subsets.*

A combinatorial argument shows that the number of distinct covers for a set of n elements is

$$\frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{2^n-k},$$

which is clearly not P-recursive (equivalently, its generating series is not D-finite.)

Devitt and Jackson [10] give a generating function for the number of k-regular r-covers of $[n]$, a notion introduced by Comtet [8]. Further, they prove that the number of arithmetic operations required to actually calculate the number of k-covers of an n set by their method is bounded by $cn^k \log n$. Results for fixed k, specifically $k = 2, 3$ were treated by Comtet [8] and Bender [1] respectively.

We can derive enumeration formulas. For example, a k-regular graph on n vertices is a restrictive k-cover of $[n]$ into 2-sets. In general, calculating the generating function for restrictive k-covers of $[n]$ into j-sets can be expressed as

$$\left\langle \Gamma_{\mathsf{E}\circ\mathsf{E}_j}(p_1, p_2, \ldots), \sum_n h_k^n t^n \right\rangle = \left\langle E[e_j], \sum_n h_k^n t^n \right\rangle.$$

To determine k-covers with mixed-cardinality sets, say both i and j, we calculate

$$\left\langle \Gamma_{\mathsf{E}\circ(\mathsf{E}_i+\mathsf{E}_j)}(p_1, p_2, \ldots), \sum_n h_k^n t^n \right\rangle = \left\langle E[e_i + e_j], \sum_n h_k^n t^n \right\rangle.$$

This yields the following simple consequence of Theorem 3.1.

Corollary 4.2. *Let S be a finite set of integers. For fixed n, and fixed k, the exponential generating function for k-regular S-covers of sets is D-finite, and is given by the scalar product*

$$\left\langle E[\sum_{s\in S} e_s], \exp(h_k t) \right\rangle.$$

Example. We can express the problem of counting distinct restrictive 2-covers of a set of cardinality n by sets of cardinality less than 5 as a scalar product. Denote the exponential generating function of such set covers, by $S(t)$. We have,

$$S(t) = \langle E[e_1 + e_2 + e_3 + e_4], \exp(th_2)\rangle.$$

This problem is perfectly suited to either of our algorithms. We can determine this differential equation, and the initial terms of the counting sequence:

$$1, 0, 1, 8, 80, 1037, 17200, 350682, 8544641, 243758420, 8010360039.$$

It is worthwhile to remark that for a fixed j, the set coverings by j-sets are equivalent to loopless j-uniform hypergraphs without multiplicities. These are encoded by $E[e_j]$. If we wish to encode hypergraphs with loops, we replace e_j by h_j, and if we wish to encode hypergraphs with multiplicities we replace E by H.

5 Asymptotic analysis

Now that we have established how to determine the differential equations satisfied by regular families of combinatorial objects, we process these differential equations to obtain asymptotic enumeration results.

Asymptotic enumeration of regular graphs is a topic that has received a great deal of attention. Indeed, as Gropp [16] points out, the basic problem of regular graph enumeration was considered before graphs were even "invented", over 120 years ago. We first see some explicit results for graphs with fixed degree sequences in the work of Read [25, 26], however, these are rumoured to be "difficult to penetrate". Nonetheless, one can determine an asymptotic expression for the number of 3-regular graphs. Bender and Caufield [2] produce the first general asymptotic formula for the number of k-regular graphs on n vertices, and Bollobás produces a similar result by a more probabilistic approach that generalizes with ease to treat hypergraphs. Next, work by McKay [20] and McKay and Wormald [23] consider the problem of k which is not fixed, but rather a function of n, and they achieve a formula that they believe to be true in general, valid as $n \to \infty$ uniformly for $1 \leq k = o(n^{1/2})$

$$g_k(n) \sim \frac{(nk)!}{(nk/2)! 2^{nk/2} (k!)^n} \exp\left(- \frac{k^2 - 1}{4} - \frac{k^3}{12n} + O(k^2/n) \right). \tag{4}$$

This resembles Bollobás' asymptotic formula [4] for labelled k-regular r-uniform hypergraphs, on n vertices when $\frac{nk}{r}$ is an integer

$$g_k^{(r)}(n) \sim \frac{(nk)!}{(nk/r)!(r!)^{(nk/r)}(k!)^n} \exp\left(-(r-1)(k-1)/2\right).$$

He also gives a formula for hypergraphs in which hyperedges only have single vertex intersections, which gives the constant in the McKay and Wormald formula for $r = 2$.

The asymptotic enumeration problem of regular graphs has been treated with a variety of methods, such as the multidimensional Cauchy integral technique mentioned earlier [20], a "switching" technique based on inclusion exclusion [21, 23], and some direct combinatorial arguments on the equivalent problem of symmetric (0,1) matrices with fixed row sum [2].

Bollobás [5] remarks that the number of k-regular unlabelled graphs grows asymptotically like $l_n/n!$ as n tends to infinity and where l_n is the number of labelled regular graphs. Intuitively, this is due to the fact that, for most large graphs with no isolated vertices, and at most one vertex of maximal degree, the automorphism group consists of only the identity automorphism.

The enumeration of other configurations is relatively untreated. Gessel remarked [14] that the exponential generating functions of k-regular r-uniform hypergraphs (with and without loops, with and without multiplicities) are D-finite and the differential equations they satisfy are obtainable via the scalar product. Domocoş [11] determines a scalar product form for the generating series of minimal coverings that are multipartite hypergraphs.

Here we continue to treat a variety of configurations. The results are tabulated in Table 3 and Table 4 and allow for a comparison across objects rather than regularity parameter. All of the results were automatically generated.

5.1 Technique

Our method is a classical singularity analysis of formal solutions of the linear differential equations. It is precisely the same method we used in our analysis of k-uniform Young tableaux [7], and thus we do not repeat the details here. Instead, after a short description of the major steps, we present the fruits of our analyses. A Maple worksheet of the computations is available at
`http://www.math.sfu.ca/~mmishna`.

In the simplest cases, essentially the cases we could analyze directly with combinatorial arguments, we can solve the differential equation and do an asymptotic analysis on the solution. In the more complex cases, we first convert our differential equation to the recurrence satisfied by the coefficients. Our series are D-finite, and thus such a sequence is bounded by a rational power of $n!$, and thus, we scale our sequence until it is convergent, and this allows us a more precise analysis. We convert this recurrence back to a differential equation, and determine the roots of the polynomial that is the coefficient of the leading term. From this we can calculate the dominant singularity, and determine a power series solution to the differential equation around this point. From this, we analyze the solution to determine an asymptotic expression for the coefficients. This can be done automatically using tools from Maple, specifically DEtools and gfun. Finally, by generating sufficiently many terms in the sequence, we compare with the formula to determine a value for the constant.

In the tables that follow, the Sloane sequence number refer to the counting sequence as indexed in the Sloane On-line Encyclopedia of Integer Sequences [27].

Furthermore, we mention only the dominant term of the asymptotic expression, but we could get the subsequent terms, save for the appropriate constants. This is a consequence of the principal weakness of our method– we cannot generate exact expressions for the constants.

We remark that, as presented in Table 3, graphs of different types have the same asymptotic development, but differ only in the constants. There are a few observations we can make based on this data. We see that allowing repetitions influences only the constant of the asymptotic expansion, and often only slightly. We see this again in the cycle covers in Table 4. The formulas look different than the graph formulas we presented earlier, however if we expand the factorials with Stirling's formula for $n!$, we quickly see that they are the same.

k		1	2	3
Formula		$\left(\frac{1}{2}\right)^{\frac{n}{2}} \frac{n!}{(n/2)!}$	$\frac{n!}{\sqrt{n}}$	$\left(\frac{3}{2}\right)^{\frac{n}{2}} \frac{n!(n/2)!}{n}$
Simple graphs	Sloane #	A001147	A001205	A002829
$E[e_2]$	Constant	1	$e^{-\frac{3}{4}}/\sqrt{\pi}$	.043
Graphs with loops	Sloane #	A001147	A108246	A110039
$E[h_2]$	Constant	1	$e^{-\frac{1}{4}}/\sqrt{\pi}$	.318
Multigraphs	Sloane #	A001147	A002137	A108243
$H[e_2]$	Constant	1	$e^{\frac{1}{4}}/\sqrt{\pi}$	.318
Multigraphs with loops	Sloane #	A001147	A002135	A005814
$H[h_2]$	Constant	1	$e^{\frac{3}{4}}/\sqrt{\pi}$	2.35

Table 3: Asymptotic enumeration formulas for different classes of k-regular graphs. Formulas for 1- and 3- regular are valid only for even n.

k or S	Restrictions	Sloane #	Formula	Constant
S-regular graphs				
$\{1,2\}$		A00986	$n^{-\frac{1}{2}} e^{\sqrt{2n}} n!$	$e^{\frac{-3}{2}}/2\sqrt{\pi}$
$\{2,3\}$		A110040	$n^{-\frac{3}{4}} \left(\frac{\sqrt{3}}{2}\right)^n e^{\sqrt{3n}} n!^{3/2}$	0.007
$\{1,3\}$	$n = 0 \mod 2$	A110039	$n^{-1} \left(\frac{3}{2}\right)^{n/2} n!(n/2)!$	0.43
$\{1,2,3\}$		A110041	$n^{-\frac{3}{4}} \left(\frac{\sqrt{3}}{2}\right)^n e^{\sqrt{3n}} n!^{3/2}$	0.05
k-regular 3-uniform hyper-graphs: $E[e_3]$				
1	$n = 0 \mod 3$	A025035	$\left(\frac{1}{3!}\right)^{n/3} \frac{n!}{(n/3)!}$	1
2	$n = 0 \mod 3$	A110100	$n^{-1} \left(\frac{3}{2}\right)^{n/3} n!(n/3)!$	0.175
3		A110101	$n^{-1} \left(\frac{3}{4}\right)^{n} n!^2$	0.037
k-regular 4-uniform hyper-graphs: $E[e_4]$				
1	$n = 0 \mod 4$	A110102	$\left(\frac{1}{4!}\right)^{n/4} \frac{n!}{(n/4)!}$	1
2	$n = 0 \mod 2$	A110103	$n^{-1} \left(\frac{2}{3}\right)^{n/2} n!(n/2)!$	0.100

Table 4: Asymptotic enumeration of different classes of regular objects

Although we are able to compute the differential equations for the generating functions of the classes of 4-regular (multi-) graphs (with loops), their asymptotic analysis is more complicated to do in an automated fashion, because a saddle point analysis arises. This is an obvious starting point for future work. Most of the tools are already implemented, it is mostly a question of understanding them, and determining how to best automate them.

We could make further observations by considering directed versions of any of these structures. For directed graphs, we need only consider $\mathsf{E} \circ \mathsf{L}_2$, where L_k is the species of lists of length k, and we could generalize hypergraphs in a number of ways; by putting an order, or even an orientation on each "edge" using the species of cycles or lists, as in the directed graph case.

6 Comments, conclusions and perspectives

6.1 Asymptotic expansions of different families of functions

Coefficients of taylor expansions of algebraic functions have a known kind of expansion, that can in fact we used to establish the transcendence of some series [12]. We can also describe asymptotic discrepancy criteria for coefficients of D-finite functions.

As we remarked earlier, coefficients of D-finite series are also restricted in their asymptotic growth. A more complete version of this criteria is following theorem presented in Wimp and Zeilberger [30].

Theorem 6.1 (Wimp and Zeilberger)**.** *Suppose that $f(t) = \sum_{n\geq 0} f_n t^n$ is a D-finite series in in $\mathbb{C}[[t]]$. Then, for sufficiently large n, the coefficients f_n have an asymptotic expansion that is a sum of terms of the form*

$$\lambda (n!)^{r/s} \exp(Q(n^{1/m}))\omega^n n^\alpha (\log n)^k,$$

where $r, s, m, k \in \mathbb{N}$, Q is a polynomial and λ, ω, α, are complex numbers.

We may ask ourselves, have our examples encompassed the full asymptotic potential of D-finite sequences? We are very curious about the combinatorial structure of families that do have such expansions. Are D-finite species sufficient to consider?

For example, in our earlier study of k-uniform Young Tableaux [7], which are enumerated by the scalar product $\langle H[e_1 + e_2], \exp(h_k t)\rangle$ have a conjectured form (verified for $k = 1..4$) of

$$y_n^{[k]} \sim \frac{1}{\sqrt{2}} \left(\frac{e^{k-2}}{2\pi}\right)^{k/4} n!^{k/2-1} \left(\frac{k^{k/2}}{k!}\right)^n \frac{\exp(\sqrt{kn})}{n^{k/4}}, \qquad n \to \infty.$$

This is an exact conjecture, more complete than the examples we have presented here, although it results from the same kind of calculation, and presumably if we completed the complex saddle point analyses required for the $k = 4$ cases, we might be able to guess such a form for our examples.

It is also of interest to note that while in some cases operations of summation and integration preserve D-finiteness, G_r, the class of *all* regular graphs is not D-finite. The same is true of all the classes we have presented here: Although for any k, the subclass of

k-regular objects is D-finite, the larger subclass of regular objects is not. This is interesting and can help us refine our notion of D-finiteness.

6.2 D-finite species?

We have thus far restrained ourselves from defining a notion of D-finite species. Ideally, such a theory would contain two main components: A “D-finite species” should satisfy some sort of system combinatorial differential equations with polynomial coefficients; and symmetric series of such species should be D-finite. We would then expect to be able to have theorems of the form:

1. If F and G are D-finite species, then so are $\mathsf{F}+\mathsf{G}$, and $\mathsf{F}\cdot\mathsf{G}$, F';

2. If G is a polynomial species, (in particular, if its cycle index series is a polynomial), then $\mathsf{F}\circ\mathsf{G}$ is a D-finite species;

3. If F satisfies an “algebraic equation” of species, including for instance, equations of the form $\mathsf{F}=\mathsf{X}\mathsf{P}(\mathsf{X},\mathsf{F})$ for polynomial species P, then F is a D-finite species.

A candidate definition is given in [24], however more work remains to be done.

Finally, much work has been done to characterize combinatorial classes of objects with rational and algebraic generating series (see [6] for a recent summary), and hopefully this work is a step towards such a characterization for D-finite generating functions. We are encouraged by a recent thought of Flajolet, Gerhold and Salvy [13],

> *Almost every thing is non-holonomic unless it is holonomic by design.*

(Series are holonomic if and only if they are D-finite.) They follow this with the remark that there are several surprising exceptions to this rule, notably k-regular graphs. Hopefully we have demonstrated that this isn’t so surprising; That in fact, there are deep reasons underlying the D-finiteness of objects with this sort of regularity, and that furthermore, this D-finiteness can be exploited in an automatic way.

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