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The Compositions of Differential Operations and the Gateaux Directional Derivative

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Abstract

This paper deals with the number of meaningful compositions of higher order of differential operations and the Gateaux directional derivative.

1 The compositions of differential operations of the space $\mathbb{R}^3$

In the real three-dimensional space $\mathbb{R}^3$ we consider the following sets:

$$\mathrm{A}_0 = \{f : \mathbb{R}^3 \longrightarrow \mathbb{R} \mid f \in C^\infty(\mathbb{R}^3)\} \quad \text{and} \quad \mathrm{A}_1 = \{\vec{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \mid \vec{f} \in \vec{C}^\infty(\mathbb{R}^3)\}. \tag{1}$$

It is customary in vector analysis to consider $m = 3$ basic differential operations on A_0 and A_1 [1], namely:

$$\begin{aligned}
\operatorname{grad} f &= \nabla_1 f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) : \mathrm{A}_0 \longrightarrow \mathrm{A}_1,\\
\operatorname{curl} \vec{f} &= \nabla_2 \vec{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) : \mathrm{A}_1 \longrightarrow \mathrm{A}_1,\\
\operatorname{div} \vec{f} &= \nabla_3 \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} : \mathrm{A}_1 \longrightarrow \mathrm{A}_0 .
\end{aligned} \tag{2}$$

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Let us present the number of meaningful compositions of higher order over the set $\mathcal{A}_3 = \{\nabla_1, \nabla_2, \nabla_3\}$. It is familiar fact that there are $m = 5$ compositions of the second order [2, p. 161]:

$$\begin{aligned}
&\Delta f = \operatorname{div}\operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl}\operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{grad}\operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl}\operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div}\operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0.
\end{aligned} \tag{3}$$

Malešević [3] proved that there are $m = 8$ compositions of the third order:

$$\begin{aligned}
&\operatorname{grad}\operatorname{div}\operatorname{grad} f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl}\operatorname{curl}\operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{div}\operatorname{grad}\operatorname{div} \vec{f} = \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl}\operatorname{curl}\operatorname{grad} f = \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div}\operatorname{curl}\operatorname{grad} f = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,\\
&\operatorname{div}\operatorname{curl}\operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{grad}\operatorname{div}\operatorname{curl} \vec{f} = \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0},\\
&\operatorname{curl}\operatorname{grad}\operatorname{div} \vec{f} = \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}.
\end{aligned} \tag{4}$$

If $\mathtt{f}(k)$ is the number of compositions of the k^{th} order, then Malešević [4] proved

$$\mathtt{f}(k) = F_{k+3}, \tag{5}$$

where F_k is k^{th} Fibonacci number.

2 The compositions of the differential operations and Gateaux directional derivative of the space $\mathbb{R}^3$

Let $f \in \mathrm{A}_0$ be a scalar function and $\vec{e} = (e_1, e_2, e_3) \in \mathbb{R}^3$ be a unit vector. The *Gateaux directional derivative* in direction $\vec{e}$ is defined by [5, p. 71]:

$$\operatorname{dir}_{\vec{e}} f = \nabla_0 f = \nabla_1 f \cdot \vec{e} = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 : \mathrm{A}_0 \longrightarrow \mathrm{A}_0. \tag{6}$$

Let us determine the number of meaningful compositions of higher order over the set $\mathcal{B}_3 = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$. There exist $m = 8$ compositions of the second order:

$$\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big),\\
&\Delta f = \operatorname{div} \operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_3 \vec{f} = \big(\nabla_1 \circ \nabla_3 \vec{f}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0;
\end{aligned} \tag{7}$$

and there exist $m = 16$ compositions of the third order:

$$\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 \circ \nabla_0 f,\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 \circ \nabla_0 f,\\
&\operatorname{div} \operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_3 \circ \nabla_1 \circ \nabla_0 f,\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \operatorname{grad} f = \nabla_0 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{grad} \operatorname{div} \operatorname{grad} f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_0 \circ \nabla_3 \vec{f},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_0 \circ \nabla_3 \vec{f},\\
&\operatorname{div} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_2 \circ \nabla_1 \circ \nabla_0 \vec{f} = \vec{0},\\
&\operatorname{curl} \operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \operatorname{grad} f = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,\\
&\operatorname{div} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \operatorname{curl} \vec{f} = \nabla_0 \circ \nabla_3 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{grad} \operatorname{div} \operatorname{curl} \vec{f} = \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0},\\
&\operatorname{curl} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}.
\end{aligned} \tag{8}$$

Further on we shall use the method from the paper [4]. Let us define a binary relation σ "*to be in composition*": $\nabla_i \,\sigma\, \nabla_j$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful. Then Cayley table of the relation σ is determined by

σ	∇_0	∇_1	∇_2	∇_3
∇_0	$\top$	$\top$	$\bot$	$\bot$
∇_1	$\bot$	$\bot$	$\top$	$\top$
∇_2	$\bot$	$\bot$	$\top$	$\top$
∇_3	$\top$	$\top$	$\bot$	$\bot$

(9)

Let us denote by ∇_{-1} nowhere-defined function, where domain and range are empty sets [3] and let $\nabla_{-1} \,\sigma\, \nabla_i$ hold for $i = 0, 1, 2, 3$. If G is graph which is determined by the relation σ, then graph of paths of G is the tree with the root ∇_{-1} (Fig. 1).

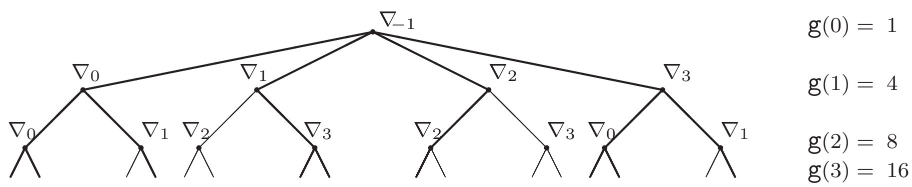

Fig. 1

Let $\mathsf{g}(k)$ be the number of meaningful compositions of the k^{th} order of the functions from $\mathcal{B}_3$ and let $\mathsf{g}_i(k)$ be the number of meaningful compositions of the k^{th} order beginning from the left by ∇_i. Then $\mathsf{g}(k) = \mathsf{g}_0(k) + \mathsf{g}_1(k) + \mathsf{g}_2(k) + \mathsf{g}_3(k)$. Based on the partial self similarity of the tree (Fig. 1) we obtain equalities

$$\begin{aligned}
\mathsf{g}_0(k) &= \mathsf{g}_0(k-1) + \mathsf{g}_1(k-1),\\
\mathsf{g}_1(k) &= \mathsf{g}_2(k-1) + \mathsf{g}_3(k-1),\\
\mathsf{g}_2(k) &= \mathsf{g}_2(k-1) + \mathsf{g}_3(k-1),\\
\mathsf{g}_3(k) &= \mathsf{g}_0(k-1) + \mathsf{g}_1(k-1).
\end{aligned} \tag{10}$$

Hence, the recurrence for $\mathsf{g}(k)$ is

$$\mathsf{g}(k) = 2\,\mathsf{g}(k-1) \tag{11}$$

and because $\mathsf{g}(1) = 4$ we have

$$\mathsf{g}(k) = 2^{k+1}. \tag{12}$$

3 The compositions of differential operations of the space $\mathbb{R}^{\mathbf{n}}$

Let us present the number of meaningful compositions of differential operations in the vector analysis of the space $\mathbb{R}^n$, where differential operations ∇_r $(r = 1, \ldots, n)$ are defined on corresponding non-empty sets A_s ($s = 1, \ldots, m$ and $m = \lfloor n/2 \rfloor$, $n \geq 3$) according to the papers [4], [6]:

$$\begin{array}{ll} \mathcal{A}_n \ (n{=}2m): & \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\ & \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\ & \vdots \\ & \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\ & \vdots \\ & \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\ & \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\ & \vdots \\ & \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\ & \vdots \\ & \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\ & \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0, \end{array} \qquad \begin{array}{ll} \mathcal{A}_n \ (n{=}2m{+}1): & \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\ & \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\ & \vdots \\ & \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\ & \vdots \\ & \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\ & \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_m \\ & \nabla_{m+2} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\ & \vdots \\ & \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\ & \vdots \\ & \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\ & \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0. \end{array} \tag{13}$$

Let us define *higher order differential operations* as meaningful compositions of higher order of differential operations from the set $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$. The number of higher order differential operations is given according to the paper [4]. Furthermore, let us define a binary relation ρ "*to be in composition*": $\nabla_i \, \rho \, \nabla_j$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful. Then Cayley table of the relation ρ is determined by

$$\nabla_i \, \rho \, \nabla_j = \begin{cases} \top & , \quad (j = i+1) \vee (i+j = n+1); \\ \bot & , \quad \text{otherwise.} \end{cases} \tag{14}$$

Let $\mathtt{A} = [a_{ij}] \in \{0,1\}^{n \times n}$ be the adjacency matrix associated with the graph which is determined by the relation ρ. Malešević [6] proved the following statements.

Theorem 3.1. *Let $P_n(\lambda) = |\mathtt{A} - \lambda \mathtt{I}| = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n$ be the characteristic polynomial of the matrix $\mathtt{A}$ and $v_n = [\,1 \ldots 1\,]_{1 \times n}$. If $\mathtt{f}(k)$ is the number of the k^{th} order differential operations, then the following formulas hold:*

$$\mathtt{f}(k) = v_n \cdot \mathtt{A}^{k-1} \cdot v_n^T \tag{15}$$

and

$$\alpha_0 \mathtt{f}(k) + \alpha_1 \mathtt{f}(k-1) + \cdots + \alpha_n \mathtt{f}(k-n) = 0 \quad (k > n). \tag{16}$$

Lemma 3.2. *Let $P_n(\lambda)$ be the characteristic polynomial of the matrix* $\mathtt{A}$. *Then the following recurrence holds:*

$$P_n(\lambda) = \lambda^2\big(P_{n-2}(\lambda) - P_{n-4}(\lambda)\big). \tag{17}$$

Lemma 3.3. *Let $P_n(\lambda)$ be the characteristic polynomial of the matrix* $\mathtt{A}$. *Then it has the following explicit form:*

$$P_n(\lambda) = \begin{cases} \displaystyle\sum_{k=1}^{\lfloor \frac{n+2}{4} \rfloor + 1} (-1)^{k-1} \binom{\frac{n}{2}-k+2}{k-1} \lambda^{n-2k+2} & , \; n=2m; \\ \displaystyle\sum_{k=1}^{\lfloor \frac{n+2}{4} \rfloor + 2} (-1)^{k-1} \left(\binom{\frac{n+3}{2}-k}{k-1} + \binom{\frac{n+3}{2}-k}{k-2} \lambda \right) \lambda^{n-2k+2} & , \; n=2m+1. \end{cases} \tag{18}$$

From previous statements one can obtain the recurrences in the table, [4]:

Dimension	Recurrence for the number of the k^{th} order differential operations
$n = 3$	$\mathtt{f}(k) = \mathtt{f}(k-1) + \mathtt{f}(k-2)$
$n = 4$	$\mathtt{f}(k) = 2\mathtt{f}(k-2)$
$n = 5$	$\mathtt{f}(k) = \mathtt{f}(k-1) + 2\mathtt{f}(k-2) - \mathtt{f}(k-3)$
$n = 6$	$\mathtt{f}(k) = 3\mathtt{f}(k-2) - \mathtt{f}(k-4)$
$n = 7$	$\mathtt{f}(k) = \mathtt{f}(k-1) + 3\mathtt{f}(k-2) - 2\mathtt{f}(k-3) - \mathtt{f}(k-4)$
$n = 8$	$\mathtt{f}(k) = 4\mathtt{f}(k-2) - 3\mathtt{f}(k-4)$
$n = 9$	$\mathtt{f}(k) = \mathtt{f}(k-1) + 4\mathtt{f}(k-2) - 3\mathtt{f}(k-3) - 3\mathtt{f}(k-4) + \mathtt{f}(k-5)$
$n = 10$	$\mathtt{f}(k) = 5\mathtt{f}(k-2) - 6\mathtt{f}(k-4) + \mathtt{f}(k-6)$

The values of the function $\mathtt{f}(k)$, for small values of the argument k, are given in the database of integer sequences [8] as the following sequences A020701 ($n = 3$), A090989 ($n = 4$), A090990 ($n = 5$), A090991 ($n = 6$), A090992 ($n = 7$), A090993 ($n = 8$), A090994 ($n = 9$), A090995 ($n = 10$).

4 The compositions of differential operations and Gateaux directional derivative of the space $\mathbb{R}^n$

Let $f \in \mathrm{A}_0$ be a scalar function and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. The *Gateaux directional derivative* in direction $\vec{e}$ is defined by [5, p. 71]:

$$\mathrm{dir}_{\vec{e}}\, f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\, e_k : \mathrm{A}_0 \longrightarrow \mathrm{A}_0. \tag{19}$$

Let us extend the set of differential operations $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$ with Gateaux directional derivative to the set $\mathcal{B}_n = \mathcal{A}_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$:

$$
\mathcal{B}_n\ (n{=}2m)\colon \begin{array}{l}
\nabla_0 : \mathrm{A}_0 \to \mathrm{A}_0\\
\nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1\\
\nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2\\
\vdots\\
\nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i\\
\vdots\\
\nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m\\
\nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_{m-1}\\
\vdots\\
\nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j\\
\vdots\\
\nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1\\
\nabla_n : \mathrm{A}_1 \to \mathrm{A}_0,
\end{array}
\qquad
\mathcal{B}_n\ (n{=}2m{+}1)\colon \begin{array}{l}
\nabla_0 : \mathrm{A}_0 \to \mathrm{A}_0\\
\nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1\\
\nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2\\
\vdots\\
\nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i\\
\vdots\\
\nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m\\
\nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_m\\
\nabla_{m+2} : \mathrm{A}_m \to \mathrm{A}_{m-1}\\
\vdots\\
\nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j\\
\vdots\\
\nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1\\
\nabla_n : \mathrm{A}_1 \to \mathrm{A}_0.
\end{array}
\tag{20}
$$

Let us define *higher order differential operations with Gateaux derivative* as the meaningful compositions of higher order of the functions from the set $\mathcal{B}_n = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. Our aim is to determine the number of higher order differential operations with Gateaux derivative. Let us define a binary relation σ "*to be in composition*":

$$
\nabla_i\,\sigma\,\nabla_j = \left\{ \begin{array}{l} \top\ ,\ (i{=}0 \wedge j{=}0) \vee (i{=}n \wedge j{=}0) \vee (j{=}i{+}1) \vee (i{+}j{=}n{+}1);\\ \bot\ ,\ \text{otherwise.} \end{array} \right. \tag{21}
$$

and let $\mathtt{B} = [b_{ij}] \in \{0,1\}^{(n+1)\times n}$ be the adjacency matrix associated with the graph which is determined by relation σ. So, analogously to the paper [6], the following statements hold.

Theorem 4.1. *Let $Q_n(\lambda) = |\mathtt{B} - \lambda\mathtt{I}| = \beta_0\lambda^{n+1} + \beta_1\lambda^n + \cdots + \beta_{n+1}$ be the characteristic polynomial of the matrix $\mathtt{B}$ and $v_{n+1} = [\,1\ \ldots\ 1\,]_{1\times(n+1)}$. If $\mathsf{g}(k)$ is the number of the k^{th} order differential operations with Gateaux derivative, then the following formulas hold:*

$$
\mathsf{g}(k) = v_{n+1} \cdot \mathtt{B}^{k-1} \cdot v_{n+1}^T \tag{22}
$$

and

$$
\beta_0\mathsf{g}(k) + \beta_1\mathsf{g}(k-1) + \cdots + \beta_{n+1}\mathsf{g}(k-(n+1)) = 0 \quad (k > n{+}1). \tag{23}
$$

Lemma 4.2. *Let $Q_n(\lambda)$ and $P_n(\lambda)$ be the characteristic polynomials of the matrices $\mathtt{B}$ and $\mathtt{A}$ respectively. Then the following equality holds:*

$$
Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda). \tag{24}
$$

Proof. Let us calculate the characteristic polynomial

$$Q_n(\lambda) = |\mathtt{B} - \lambda\mathtt{I}| = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{25}$$

Expanding the determinant $Q_n(\lambda)$ by the first column we have

$$Q_n(\lambda) = (1-\lambda)P_n(\lambda) + (-1)^{n+2}D_n(\lambda), \tag{26}$$

where

$$D_n(\lambda) = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \dots & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \end{vmatrix}. \tag{27}$$

Let us expand the determinant $D_n(\lambda)$ by the first row and then in the next step, multiply the first row by -1 and add it to the last row. We obtain the determinant of order $n-1$:

$$D_n(\lambda) = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & -\lambda & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -\lambda & 0 \end{vmatrix}. \tag{28}$$

Expanding the previous determinant by the last column we have

$$D_n(\lambda) = (-1)^n \begin{vmatrix} -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{29}$$

If we expand the previous determinant by the last row and if we expand the obtained determinant by the first column, we have the determinant of order $n-4$:

$$D_n(\lambda) = (-1)^n \lambda^2 \begin{vmatrix} -\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & \ldots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{30}$$

In other words

$$D_n(\lambda) = (-1)^n \lambda^2 P_{n-4}(\lambda). \tag{31}$$

From equalities (31) and (26) there follows:

$$Q_n(\lambda) = (1-\lambda) P_n(\lambda) + \lambda^2 P_{n-4}(\lambda). \tag{32}$$

On the basis of Lemma 3.2. the following equality holds:

$$Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda). \tag{33}$$

□

Lemma 4.3. *Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix* B*. Then the following recurrence holds:*

$$Q_n(\lambda) = \lambda^2 \big(Q_{n-2}(\lambda) - Q_{n-4}(\lambda) \big). \tag{34}$$

Proof. On the basis of Lemma 3.2. and Lemma 4.2. the Lemma follows. □

Lemma 4.4. *Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix* B*. Then it has the following explicit form*:

$$Q_n(\lambda) = \begin{cases} (\lambda-2) \displaystyle\sum_{k=1}^{\lfloor \frac{n}{4} \rfloor + 1} (-1)^{k-1} \binom{\frac{n+1}{2} - k}{k-1} \lambda^{n-2k+2} & , \ n=2m+1; \\ \displaystyle\sum_{k=1}^{\lfloor \frac{n+3}{4} \rfloor + 2} (-1)^{k-1} \left(\binom{\frac{n}{2}-k+2}{k-1} + \binom{\frac{n}{2}-k+2}{k-2} \lambda \right) \lambda^{n-2k+3} & , \ n=2m. \end{cases} \tag{35}$$

Proof. On the basis of Lemma 3.3 and Lemma 4.2. the Lemma follows. □

The recurrences for dimensions $n = 3, 4, \ldots, 10$ are obtained by means of Malešević-Jovović [7] and they are given in the table below.

Dimension	Recurrence for the num. of the k^{th} order diff. operations with Gateaux derivative
$n = 3$	$\mathsf{g}(k) = 2\mathsf{g}(k-1)$
$n = 4$	$\mathsf{g}(k) = \mathsf{g}(k-1) + 2\mathsf{g}(k-2) - \mathsf{g}(k-3)$
$n = 5$	$\mathsf{g}(k) = 2\mathsf{g}(k-1) + \mathsf{g}(k-2) - 2\mathsf{g}(k-3)$
$n = 6$	$\mathsf{g}(k) = \mathsf{g}(k-1) + 3\mathsf{g}(k-2) - 2\mathsf{g}(k-3) - \mathsf{g}(k-4)$
$n = 7$	$\mathsf{g}(k) = 2\mathsf{g}(k-1) + 2\mathsf{g}(k-2) - 4\mathsf{g}(k-3)$
$n = 8$	$\mathsf{g}(k) = \mathsf{g}(k-1) + 4\mathsf{g}(k-2) - 3\mathsf{g}(k-3) - 3\mathsf{g}(k-4) + \mathsf{g}(k-5)$
$n = 9$	$\mathsf{g}(k) = 2\mathsf{g}(k-1) + 3\mathsf{g}(k-2) - 6\mathsf{g}(k-3) - \mathsf{g}(k-4) + 2\mathsf{g}(k-5)$
$n = 10$	$\mathsf{g}(k) = \mathsf{g}(k-1) + 5\mathsf{g}(k-2) - 4\mathsf{g}(k-3) - 6\mathsf{g}(k-4) + 3\mathsf{g}(k-5) + \mathsf{g}(k-6)$

The values of the function $\mathsf{g}(k)$, for small values of the argument k, are given in the database of integer sequences [8] as the following sequences A000079 ($n = 3$), A090990 ($n = 4$), A007283 ($n = 5$), A090992 ($n = 6$), A000079 ($n = 7$), A090994 ($n = 8$), A020714 ($n = 9$), A129638 ($n = 10$).

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(Concerned with sequences A000079, A007283, A020701, A020714, A090989, A090990, A090991, A090992, A090993, A090994, A090995, and A129638.)

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