The Compositions of Differential Operations and the Gateaux Directional Derivative

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Abstract
This paper deals with the number of meaningful compositions of higher order of differential operations and the Gateaux directional derivative.

1 The compositions of differential operations of the space $\mathbb{R}^3$

In the real three-dimensional space $\mathbb{R}^3$ we consider the following sets:

$$A_0 = \{ f: \mathbb{R}^3 \rightarrow \mathbb{R} | f \in C^\infty(\mathbb{R}^3) \} \quad \text{and} \quad A_1 = \{ \vec{f}: \mathbb{R}^3 \rightarrow \mathbb{R}^3 | \vec{f} \in \vec{C}^\infty(\mathbb{R}^3) \}. \quad (1)$$

It is customary in vector analysis to consider $m = 3$ basic differential operations on $A_0$ and $A_1$ [1], namely:

$$\nabla_1 f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right): A_0 \rightarrow A_1,$$

$$\nabla_2 \vec{f} = \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right): A_1 \rightarrow A_1, \quad (2)$$

$$\nabla_3 \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}: A_1 \rightarrow A_0.$$
Let us present the number of meaningful compositions of higher order over the set \( \mathcal{A}_3 = \{\nabla_1, \nabla_2, \nabla_3\} \). It is familiar fact that there are \( m = 5 \) compositions of the second order \( \cite{2}, p. 161 \):

\[
\Delta f = \text{div} \text{grad} f = \nabla_3 \circ \nabla_1 f,
\]

\[
\text{curl} \text{curl} f = \nabla_2 \circ \nabla_2 f,
\]

\[
\text{grad} \text{div} f = \nabla_1 \circ \nabla_3 f,
\]

\[
\text{curl} \text{grad} f = \nabla_2 \circ \nabla_1 f = 0,
\]

\[
\text{div} \text{curl} f = \nabla_3 \circ \nabla_2 f = 0.
\]

Malešević \( \cite{3} \) proved that there are \( m = 8 \) compositions of the third order:

\[
\text{grad} \text{div} \text{grad} f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,
\]

\[
\text{curl} \text{curl} \text{curl} f = \nabla_2 \circ \nabla_2 \circ \nabla_2 f,
\]

\[
\text{div} \text{grad} \text{div} f = \nabla_3 \circ \nabla_1 \circ \nabla_3 f,
\]

\[
\text{curl} \text{grad} \text{curl} f = \nabla_2 \circ \nabla_1 \circ \nabla_3 f = 0,
\]

\[
\text{div} \text{curl} \text{grad} f = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,
\]

\[
\text{div} \text{curl} \text{curl} f = \nabla_3 \circ \nabla_2 \circ \nabla_2 f = 0,
\]

\[
\text{grad} \text{div} \text{curl} f = \nabla_1 \circ \nabla_3 \circ \nabla_2 f = 0,
\]

\[
\text{curl} \text{grad} \text{div} f = \nabla_2 \circ \nabla_1 \circ \nabla_3 f = 0.
\]

If \( f(k) \) is the number of compositions of the \( k \)th order, then Malešević \( \cite{4} \) proved

\[
f(k) = F_{k+3},
\]

where \( F_k \) is \( k \)th Fibonacci number.

### 2 The compositions of the differential operations and Gateaux directional derivative of the space \( \mathbb{R}^3 \)

Let \( f \in A_0 \) be a scalar function and \( \vec{e} = (e_1, e_2, e_3) \in \mathbb{R}^3 \) be a unit vector. The \textit{Gateaux directional derivative} in direction \( \vec{e} \) is defined by \( \cite{5}, p. 71 \):

\[
dir_{\vec{e}} f = \nabla_0 f = \nabla_1 f \cdot \vec{e} = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 : A_0 \to A_0.
\]
Let us determine the number of meaningful compositions of higher order over the set $B_3 = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$. There exist $m = 8$ compositions of the second order:

\[
\begin{align*}
\text{dir}_e \text{dir}_e f &= \nabla_0 \circ \nabla_0 f = \nabla_1 (\nabla_1 f \cdot \vec{e}) \cdot \vec{e}, \\
\text{grad dir}_e f &= \nabla_1 \circ \nabla_0 f = \nabla_1 (\nabla_1 f \cdot \vec{e}), \\
\Delta f &= \text{div grad} f = \nabla_3 \circ \nabla_1 f, \\
\text{curl curl} \vec{f} &= \nabla_2 \circ \nabla_2 \vec{f}, \\
\text{dir}_e \text{div} \vec{f} &= \nabla_0 \circ \nabla_3 \vec{f} = (\nabla_1 \circ \nabla_3 \vec{f}) \cdot \vec{e}, \\
\text{grad div} \vec{f} &= \nabla_1 \circ \nabla_3 \vec{f}, \\
\text{curl grad} \vec{f} &= \nabla_2 \circ \nabla_1 f = \vec{0}, \\
\text{div curl} \vec{f} &= \nabla_3 \circ \nabla_2 \vec{f} = 0; \\
\end{align*}
\]

and there exist $m = 16$ compositions of the third order:

\[
\begin{align*}
\text{dir}_e \text{dir}_e \text{dir}_e f &= \nabla_0 \circ \nabla_0 \circ \nabla_0 f, \\
\text{grad dir}_e \text{dir}_e f &= \nabla_1 \circ \nabla_0 \circ \nabla_0 f, \\
\text{div grad dir}_e f &= \nabla_3 \circ \nabla_1 \circ \nabla_0 f, \\
\text{dir}_e \text{div grad} f &= \nabla_0 \circ \nabla_3 \circ \nabla_1 f, \\
\text{grad div grad} f &= \nabla_1 \circ \nabla_3 \circ \nabla_1 f, \\
\text{curl curl curl} \vec{f} &= \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f}, \\
\text{dir}_e \text{dir}_e \text{div} \vec{f} &= \nabla_0 \circ \nabla_0 \circ \nabla_3 \vec{f}, \\
\text{grad dir}_e \text{div} \vec{f} &= \nabla_1 \circ \nabla_0 \circ \nabla_3 \vec{f}, \\
\text{div grad div} \vec{f} &= \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f}, \\
\text{curl grad dir} \vec{f} &= \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}, \\
\text{curl curl grad} \vec{f} &= \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0}, \\
\text{div curl grad} \vec{f} &= \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0, \\
\text{div curl curl} \vec{f} &= \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0, \\
\text{dir}_e \text{div curl} \vec{f} &= \nabla_0 \circ \nabla_3 \circ \nabla_2 \vec{f} = 0, \\
\text{grad div curl} \vec{f} &= \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0}, \\
\text{curl grad div} \vec{f} &= \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}. \\
\end{align*}
\]
Further on we shall use the method from the paper [4]. Let us define a binary relation \( \sigma \) “to be in composition”: \( \nabla_i \sigma \nabla_j \) iff the composition \( \nabla_j \circ \nabla_i \) is meaningful. Then Cayley table of the relation \( \sigma \) is determined by

\[
\begin{array}{c|cccc}
\sigma & \nabla_0 & \nabla_1 & \nabla_2 & \nabla_3 \\
\hline
\nabla_0 & T & T & \bot & \bot \\
\nabla_1 & \bot & \bot & T & T \\
\nabla_2 & \bot & \bot & T & T \\
\nabla_3 & T & T & \bot & \bot \\
\end{array}
\] (9)

Let us denote by \( \nabla_{-1} \) nowhere-defined function, where domain and range are empty sets [3] and let \( \nabla_{-1} \sigma \nabla_i \) hold for \( i = 0, 1, 2, 3 \). If \( G \) is graph which is determined by the relation \( \sigma \), then graph of paths of \( G \) is the tree with the root \( \nabla_{-1} \) (Fig. 1).

![Fig. 1](image)

Let \( g(k) \) be the number of meaningful compositions of the \( k \)th order of the functions from \( \mathcal{B}_3 \) and let \( g_i(k) \) be the number of meaningful compositions of the \( k \)th order beginning from the left by \( \nabla_i \). Then \( g(k) = g_0(k) + g_1(k) + g_2(k) + g_3(k) \). Based on the partial self similarity of the tree (Fig. 1) we obtain equalities

\[
\begin{align*}
g_0(k) &= g_0(k-1) + g_1(k-1), \\
g_1(k) &= g_2(k-1) + g_3(k-1), \\
g_2(k) &= g_2(k-1) + g_3(k-1), \\
g_3(k) &= g_0(k-1) + g_1(k-1).
\end{align*}
\] (10)

Hence, the recurrence for \( g(k) \) is

\[
g(k) = 2g(k-1)
\] (11)

and because \( g(1) = 4 \) we have

\[
g(k) = 2^{k+1}.
\] (12)

3 The compositions of differential operations of the space \( \mathbb{R}^n \)

Let us present the number of meaningful compositions of differential operations in the vector analysis of the space \( \mathbb{R}^n \), where differential operations \( \nabla_r \ (r = 1, \ldots, n) \) are defined on
Let us define higher order differential operations as meaningful compositions of higher order
differential operations from the set $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$. The number of higher order
differential operations is given according to the paper [4]. Furthermore, let us define a
binary relation $\rho$ “to be in composition”: $\nabla_i \rho \nabla_j$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful.
Then Cayley table of the relation $\rho$ is determined by

$$
\nabla_i \rho \nabla_j = \begin{cases} 
\top & , (j = i + 1) \lor (i + j = n + 1); \\
\bot & , \text{otherwise.}
\end{cases}
$$

Let $A = [a_{ij}] \in \{0, 1\}^{n \times n}$ be the adjacency matrix associated with the graph which is
determined by the relation $\rho$. Malešević [6] proved the following statements.

**Theorem 3.1.** Let $P_n(\lambda) = [A - \lambda I] = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n$ be the characteristic polynomial
of the matrix $A$ and $v_n = [1 \ldots 1]_{1 \times n}$. If $f(k)$ is the number of the $k^{th}$ order differential
operations, then the following formulas hold:

$$
f(k) = v_n \cdot A^{k-1} \cdot v_n^T \quad (15)
$$

and

$$
\alpha_0 f(k) + \alpha_1 f(k-1) + \cdots + \alpha_n f(k-n) = 0 \quad (k > n). \quad (16)
$$
Lemma 3.2. Let $P_n(\lambda)$ be the characteristic polynomial of the matrix $A$. Then the following recurrence holds:

$$P_n(\lambda) = \lambda^2(P_{n-2}(\lambda) - P_{n-4}(\lambda)).$$  \hfill (17)

Lemma 3.3. Let $P_n(\lambda)$ be the characteristic polynomial of the matrix $A$. Then it has the following explicit form:

$$P_n(\lambda) = \begin{cases} \sum_{k=1}^{\lfloor \frac{n+2}{4} \rfloor + 1} (-1)^{k-1} \left( \begin{array}{c} \frac{n-k+2}{2} \\ k-1 \end{array} \right) \lambda^{n-2k+2}, & n = 2m; \\ \sum_{k=1}^{\lfloor \frac{n+2}{4} \rfloor + 2} (-1)^{k-1} \left( \begin{array}{c} \frac{n+3-k}{2} \\ k-1 \end{array} \right) + \left( \begin{array}{c} \frac{n+3-k}{2} \\ k-2 \end{array} \right) \lambda^{n-2k+2}, & n = 2m+1. \end{cases}$$  \hfill (18)

From previous statements one can obtain the recurrences in the table, [4]:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Recurrence for the number of the $k^{\text{th}}$ order differential operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 3$</td>
<td>$f(k) = f(k-1) + f(k-2)$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$f(k) = 2f(k-2)$</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>$f(k) = f(k-1) + 2f(k-2) - f(k-3)$</td>
</tr>
<tr>
<td>$n = 6$</td>
<td>$f(k) = 3f(k-2) - f(k-4)$</td>
</tr>
<tr>
<td>$n = 7$</td>
<td>$f(k) = f(k-1) + 3f(k-2) - 2f(k-3) - f(k-4)$</td>
</tr>
<tr>
<td>$n = 8$</td>
<td>$f(k) = 4f(k-2) - 3f(k-4)$</td>
</tr>
<tr>
<td>$n = 9$</td>
<td>$f(k) = f(k-1) + 4f(k-2) - 3f(k-3) - 3f(k-4) + f(k-5)$</td>
</tr>
<tr>
<td>$n = 10$</td>
<td>$f(k) = 5f(k-2) - 6f(k-4) + f(k-6)$</td>
</tr>
</tbody>
</table>

The values of the function $f(k)$, for small values of the argument $k$, are given in the database of integer sequences [8] as the following sequences: A020701 ($n = 3$), A090989 ($n = 4$), A090990 ($n = 5$), A090991 ($n = 6$), A090992 ($n = 7$), A090993 ($n = 8$), A090994 ($n = 9$), A090995 ($n = 10$).

4 The compositions of differential operations and Gateaux directional derivative of the space $\mathbb{R}^n$

Let $f \in A_0$ be a scalar function and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. The Gateaux directional derivative in direction $\vec{e}$ is defined by [5, p. 71]:

$$\text{dir}_{\vec{e}} f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} e_k : A_0 \rightarrow A_0.$$  \hfill (19)
Let us extend the set of differential operations $A_n = \{\nabla_1, \ldots, \nabla_n\}$ with Gateaux directional derivative to the set $B_n = A_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$:

$$
\begin{align*}
B_n (n=2m): & \quad \nabla_0 : A_0 \rightarrow A_0 \\
& \nabla_1 : A_0 \rightarrow A_1 \\
& \nabla_2 : A_1 \rightarrow A_2 \\
& \vdots \\
& \nabla_i : A_{i-1} \rightarrow A_i \\
& \vdots \\
& \nabla_m : A_{m-1} \rightarrow A_m \\
& \nabla_{m+1} : A_m \rightarrow A_{m-1} \\
& \vdots \\
& \nabla_{n-j} : A_{j+1} \rightarrow A_j \\
& \vdots \\
& \nabla_{n-1} : A_2 \rightarrow A_1 \\
& \nabla_n : A_1 \rightarrow A_0, \\
\end{align*}
$$

$$
\begin{align*}
B_n (n=2m+1): & \quad \nabla_0 : A_0 \rightarrow A_0 \\
& \nabla_1 : A_0 \rightarrow A_1 \\
& \nabla_2 : A_1 \rightarrow A_2 \\
& \vdots \\
& \nabla_i : A_{i-1} \rightarrow A_i \\
& \vdots \\
& \nabla_m : A_{m-1} \rightarrow A_m \\
& \nabla_{m+1} : A_m \rightarrow A_{m-1} \\
& \vdots \\
& \nabla_{n-j} : A_{j+1} \rightarrow A_j \\
& \vdots \\
& \nabla_{n-1} : A_2 \rightarrow A_1 \\
& \nabla_n : A_1 \rightarrow A_0. \\
\end{align*}
$$

(20)

Let us define higher order differential operations with Gateaux derivative as the meaningful compositions of higher order of the functions from the set $B_n = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. Our aim is to determine the number of higher order differential operations with Gateaux derivative. Let us define a binary relation $\sigma$ “to be in composition”:

$$
\nabla_i \sigma \nabla_j = \begin{cases} 
T, & (i = 0 \land j = 0) \lor (i = n \land j = 0) \lor (j = i + 1) \lor (i + j = n + 1); \\
\bot, & \text{otherwise.}
\end{cases}
$$

(21)

and let $B = [b_{ij}] \in \{0, 1\}^{(n+1) \times n}$ be the adjacency matrix associated with the graph which is determined by relation $\sigma$. So, analogously to the paper [6], the following statements hold.

**Theorem 4.1.** Let $Q_n(\lambda) = |B - \lambda I| = \beta_0 \lambda^{n+1} + \beta_1 \lambda^n + \cdots + \beta_{n+1}$ be the characteristic polynomial of the matrix $B$ and $v_{n+1} = [1 \ldots 1]_{1 \times (n+1)}$. If $g(k)$ is the number of the $k^{th}$ order differential operations with Gateaux derivative, then the following formulas hold:

$$
g(k) = v_{n+1} \cdot B^{k-1} \cdot v_{n+1}^T
$$

(22)

and

$$
\beta_0 g(k) + \beta_1 g(k - 1) + \cdots + \beta_{n+1} g(k - (n + 1)) = 0 \quad (k > n + 1).
$$

(23)

**Lemma 4.2.** Let $Q_n(\lambda)$ and $P_n(\lambda)$ be the characteristic polynomials of the matrices $B$ and $A$ respectively. Then the following equality holds:

$$
Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda).
$$

(24)
Proof. Let us calculate the characteristic polynomial

\[ Q_n(\lambda) = |B - \lambda I| = \begin{vmatrix}
1 - \lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & -\lambda & 1 & \ldots & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & \ldots & -\lambda & 1 & 0 \\
1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda \\
\end{vmatrix}. \tag{25}\]

Expanding the determinant \( Q_n(\lambda) \) by the first column we have

\[ Q_n(\lambda) = (1 - \lambda)P_n(\lambda) + (-1)^{n+2}D_n(\lambda), \tag{26}\]

where

\[ D_n(\lambda) = \begin{vmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & \ldots & -\lambda & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & \ldots & 0 & -\lambda & 1 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\
\end{vmatrix}. \tag{27}\]

Let us expand the determinant \( D_n(\lambda) \) by the first row and then in the next step, multiply the first row by \(-1\) and add it to the last row. We obtain the determinant of order \( n - 1 \):

\[ D_n(\lambda) = \begin{vmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
-\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & \ldots & -\lambda & 1 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 0 \\
\end{vmatrix}. \tag{28}\]

Expanding the previous determinant by the last column we have

\[ D_n(\lambda) = (-1)^n \begin{vmatrix}
-\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\
0 & 0 & -\lambda & 1 & \ldots & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda \\
\end{vmatrix}. \tag{29}\]
If we expand the previous determinant by the last row and if we expand the obtained determinant by the first column, we have the determinant of order $n - 4$:

$$D_n(\lambda) = (-1)^n \lambda^2$$

\[
\begin{vmatrix}
-\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\
0 & 0 & -\lambda & 1 & \ldots & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda \\
\end{vmatrix}
\]  

(30)

In other words

$$D_n(\lambda) = (-1)^n \lambda^2 P_{n-4}(\lambda).$$

(31)

From equalities (31) and (26) there follows:

$$Q_n(\lambda) = (1 - \lambda) P_n(\lambda) + \lambda^2 P_{n-4}(\lambda).$$

(32)

On the basis of Lemma 3.2, the following equality holds:

$$Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda).$$

(33)

**Lemma 4.3.** Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix $B$. Then the following recurrence holds:

$$Q_n(\lambda) = \lambda^2 (Q_{n-2}(\lambda) - Q_{n-4}(\lambda)).$$

(34)

**Proof.** On the basis of Lemma 3.2 and Lemma 4.2, the Lemma follows.

**Lemma 4.4.** Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix $B$. Then it has the following explicit form:

$$Q_n(\lambda) = \begin{cases} 
(\lambda - 2) \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \binom{n+1}{2k+1} \lambda^{n-2k+1}, & n = 2m+1; \\
\sum_{k=1}^{\lfloor (n+3)/4 \rfloor} (-1)^{k-1} \left( \binom{n-k+2}{k-1} + \binom{n-k+2}{k-2} \right) \lambda^{n-2k+3}, & n = 2m.
\end{cases}$$

(35)

**Proof.** On the basis of Lemma 3.3 and Lemma 4.2, the Lemma follows.

The recurrences for dimensions $n = 3, 4, \ldots, 10$ are obtained by means of Malešević-Jovović [7] and they are given in the table below.
<table>
<thead>
<tr>
<th>Dimension</th>
<th>Recurrence for the num. of the $k^{th}$ order diff. operations with Gateaux derivative</th>
</tr>
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<tr>
<td>$n = 3$</td>
<td>$g(k) = 2g(k-1)$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$g(k) = g(k-1) + 2g(k-2) - g(k-3)$</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>$g(k) = 2g(k-1) + g(k-2) - 2g(k-3)$</td>
</tr>
<tr>
<td>$n = 6$</td>
<td>$g(k) = g(k-1) + 3g(k-2) - 2g(k-3) - g(k-4)$</td>
</tr>
<tr>
<td>$n = 7$</td>
<td>$g(k) = 2g(k-1) + 2g(k-2) - 2g(k-3)$</td>
</tr>
<tr>
<td>$n = 8$</td>
<td>$g(k) = g(k-1) + 4g(k-2) - 3g(k-3) - 3g(k-4) + g(k-5)$</td>
</tr>
<tr>
<td>$n = 9$</td>
<td>$g(k) = 2g(k-1) + 3g(k-2) - 6g(k-3) - g(k-4) + 2g(k-5)$</td>
</tr>
<tr>
<td>$n = 10$</td>
<td>$g(k) = g(k-1) + 5g(k-2) - 4g(k-3) - 6g(k-4) + 3g(k-5) + g(k-6)$</td>
</tr>
</tbody>
</table>

The values of the function $g(k)$, for small values of the argument $k$, are given in the database of integer sequences [8] as the following sequences $A000079$ ($n = 3$), $A090990$ ($n = 4$), $A007283$ ($n = 5$), $A090992$ ($n = 6$), $A000079$ ($n = 7$), $A090994$ ($n = 8$), $A020714$ ($n = 9$), $A129638$ ($n = 10$).

References


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