Journal of Integer Sequences, Vol. 10 (2007),
Article 07.1.6

Partial Sums of Powers of Prime Factors

Jean-Marie De Koninck
Département de Mathématiques et de Statistique
Université Laval
Québec G1K 7P4
Canada
jmdk@mat.ulaval.ca

Florian Luca
Mathematical Institute, UNAM
Ap. Postal 61-3 (Xangari)
CP 58 089
Morelia, Michoacán
Mexico
fluca@matmor.unam.mx

Abstract

Given integers $k \geq 2$ and $\ell \geq 3$, let $S^*_{k,\ell}$ stand for the set of those positive integers n which can be written as $n = p_1^k + p_2^k + \cdots + p_\ell^k$, where $p_1, p_2, \ldots, p_\ell$ are distinct prime factors of n. We study the properties of the sets $S^*_{k,\ell}$ and we show in particular that, given any odd $\ell \geq 3$, $\# \bigcup_{k=2}^{\infty} S^*_{k,\ell} = +\infty$.

1 Introduction

In [1], we studied those numbers with at least two distinct prime factors which can be expressed as the sum of a fixed power $k \geq 2$ of their prime factors. For instance, given an integer $k \geq 2$, and letting

$$S_k := \{n : \omega(n) \geq 2 \text{ and } n = \sum_{p|n} p^k\},$$

where $\omega(n)$ stands for the number of distinct prime factors of n, one can check that the following 8 numbers belong to S_3:

$$\begin{aligned}
378 &= 2 \cdot 3^3 \cdot 7 = 2^3 + 3^3 + 7^3, \\
2548 &= 2^2 \cdot 7^2 \cdot 13 = 2^3 + 7^3 + 13^3, \\
2\,836\,295 &= 5 \cdot 7 \cdot 11 \cdot 53 \cdot 139 = 5^3 + 7^3 + 11^3 + 53^3 + 139^3, \\
4\,473\,671\,462 &= 2 \cdot 13 \cdot 179 \cdot 593 \cdot 1621 = 2^3 + 13^3 + 179^3 + 593^3 + 1621^3, \\
23\,040\,925\,705 &= 5 \cdot 7 \cdot 167 \cdot 1453 \cdot 2713 = 5^3 + 7^3 + 167^3 + 1453^3 + 2713^3, \\
13\,579\,716\,377\,989 &= 19 \cdot 157 \cdot 173 \cdot 1103 \cdot 23857 = 19^3 + 157^3 + 173^3 + 1103^3 + 23857^3, \\
21\,467\,102\,506\,955 &= 5 \cdot 7^3 \cdot 313 \cdot 1439 \cdot 27791 = 5^3 + 7^3 + 313^3 + 1439^3 + 27791^3 \\
119\,429\,556\,097\,859 &= 7 \cdot 53 \cdot 937 \cdot 6983 \cdot 49199 = 7^3 + 53^3 + 937^3 + 6983^3 + 49199^3.
\end{aligned}$$

In particular, we showed that 378 and 2548 are the only numbers in S_3 with exactly three distinct prime factors.

We did not find any number belonging to S_k for $k = 2$ or $k \geq 4$, although each of these sets may very well be infinite.

In this paper, we examine the sets

$$S_k^* := \{n : \omega(n) \geq 2 \text{ and } n = \sum_{p|n}{}^{*} p^k\} \qquad (k = 2, 3, \ldots),$$

where the star next to the sum indicates that it runs over some subset of primes dividing n. For instance, $870 \in S_2^*$, because

$$870 = 2 \cdot 3 \cdot 5 \cdot 29 = 2^2 + 5^2 + 29^2.$$

Clearly, for each $k \geq 2$, we have $S_k^* \supseteq S_k$. Moreover, given integers $k \geq 2$ and $\ell \geq 3$, let $S_{k,\ell}^*$ stand for the set of those positive integers n which can be written as $n = p_1^k + p_2^k + \cdots + p_\ell^k$, where $p_1, p_2, \ldots, p_\ell$ are distinct prime factors of n, so that for each integer $k \geq 2$,

$$S_k^* = \bigcup_{\ell=3}^{\infty} S_{k,\ell}^*.$$

We study the properties of the sets $S_{k,\ell}^*$ and we show in particular that, given any odd $\ell \geq 3$, the set $\bigcup_{k=2}^{\infty} S_{k,\ell}^*$ is infinite. We treat separately the cases $\ell = 3$ and $\ell \geq 5$, the latter case being our main result.

In what follows, the letter p, with or without subscripts, always denotes a prime number.

2 Preliminary results

We shall first consider the set S_2^*. Note that if $n \in S_2^*$, then $P(n)$, the largest prime divisor of n, must be part of the partial sum of primes which allows n to belong to S_2^*. Indeed, assume the contrary, namely that, for some primes $p_1 < p_2 < \cdots < p_r$,

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} = p_{i_1}^2 + \cdots + p_{i_\ell}^2 \quad \in S_2^*,$$

where $i_1 < i_2 < \cdots < i_\ell \leq r-1$, with $r \geq 3$. Then

$$p_1 \cdots p_{r-2} p_{r-1} p_r \leq n < \ell p_{i_\ell}^2 \leq r p_{r-1}^2,$$

so that $p_1 \cdots p_{r-2} p_r < r p_{r-1} < r p_r$, which implies that $p_1 \cdots p_{r-2} < r$, which is impossible for $r \geq 3$.

While by a parity argument one can easily see that each element of S_k (for any $k \geq 2$) must have an odd number of prime factors, one can observe that elements of S_k^* can on the contrary be written as a sum of an even number of prime powers, as can be seen with $298995972 \in S_2^*$ (see below).

We can show that if Schinzel's Hypothesis is true (see Schinzel [2]), then the set S_3^* is infinite. We shall even prove more.

Theorem 1. *If Schinzel's Hypothesis is true, then* $\#S_{3,3}^* = +\infty$.

Proof. Assume that k is an even integer such that $r = k^2 - 9k + 21$ and $p = k^2 - 7k + 13$ are both primes, then $n = 2rp(r+k) \in S_{3,3}^*$. Indeed, in this case, one can see that

$$n = 2rp(r+k) = 2^3 + r^3 + p^3, \tag{1}$$

since both sides of (1) are equal to $2k^6 - 48k^5 + 492k^4 - 2752k^3 + 8844k^2 - 15456k + 11466$. Now Schinzel's Hypothesis guarantees that there exist infinitely many even k's such that the corresponding numbers r and p are both primes. □

Note that the first such values of k are $k = 2$, 6, 10, 82 and 94. These yield the following four elements of $S_{3,3}^*$ (observing that $k = 2$ and $k = 6$ provide the same number, namely $n = 378$):

$$\begin{aligned}
378 &= 2 \cdot 3^3 \cdot 7 = 2^3 + 3^3 + 7^3, \\
109306 &= 2 \cdot 31 \cdot 41 \cdot 43 = 2^3 + 31^3 + 43^3, \\
450843455098 &= 2 \cdot 6007 \cdot 6089 \cdot 6163 = 2^3 + 6007^3 + 6163^3, \\
1063669417210 &= 2 \cdot 8011 \cdot 8105 \cdot 8191 = 2^3 + 8011^3 + 8191^3.
\end{aligned}$$

Not all elements of S_3^* are generated in this way. For instance, the following numbers also belong to S_3^*:

$$\begin{aligned}
23391460 &= 2^2 \cdot 5 \cdot 23 \cdot 211 \cdot 241 = 2^3 + 211^3 + 241^3, \\
173871316 &= 2^2 \cdot 223 \cdot 421 \cdot 463 = 2^3 + 421^3 + 463^3, \\
126548475909264420 &= 2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 83 \cdot 101 \cdot 45569 \cdot 501931 \\
&= 2^3 + 5^3 + 83^3 + 45569^3 + 501931^3,
\end{aligned}$$

as well as all those elements of S_3 mentioned in Section 1.

Theorem 2. $\# \bigcup_{k=2}^{\infty} S_{k,3}^* = +\infty$.

Proof. This follows immediately from the fact that for each element $n \in S^*_{k,3}$, one can find a corresponding element $n' \in S^*_{k(2r+1),3}$ for $r = 1, 2, \ldots$. Indeed, if $n \in S^*_{k,3}$, then it means that

$$n = p_1^k + p_2^k + p_3^k$$

for some distinct primes divisors p_1, p_2, p_3 of n. In particular, it means that $p_a | (p_b^k + p_c^k)$ for each permutation (a, b, c) of the integers 1, 2 and 3. We claim that, given any positive integer r, the number

$$n' := p_1^{k(2r+1)} + p_2^{k(2r+1)} + p_3^{k(2r+1)}$$

belongs to $S^*_{k(2r+1),3}$. Indeed, we only need to show that $p_a | (p_b^{k(2r+1)} + p_c^{k(2r+1)})$ for each permutation (a, b, c) of the integers 1, 2 and 3. But this follows from the fact that $(p_b^k + p_c^k)$ divides $(p_b^{k(2r+1)} + p_c^{k(2r+1)})$; but since p_a divides $(p_b^k + p_c^k)$, we have that p_a divides $(p_b^{k(2r+1)} + p_c^{k(2r+1)})$ and therefore that $n' \in S^*_{k(2r+1),3}$. Since $378 \in S^*_{3,3}$, the proof is complete. □

Remark. It clearly follows from Theorems 1 and 2 that $\# S^*_{3(2r+1),3} = +\infty$ for any $r \geq 1$.

3 Proof of the main result

Theorem 3. *Given any odd integer $\ell \geq 5$,*

$$\# \bigcup_{k=2}^{\infty} S^*_{k,\ell} = +\infty.$$

This is an immediate consequence of the following two lemmas.

Lemma 3.1. *Let $t = 2s \geq 2$ be an even integer and $p_1, \ldots, p_t$ be primes such that*

(i) $p_i \equiv 3 \pmod 4$ *for all* $i = 1, \ldots, t$.

(ii) $\gcd(p_i, p_j - 1) = 1$ *for all* i, j *in* $\{1, \ldots, t\}$.

(iii) $\gcd(p_i - 1, p_j - 1) = 2$ *for all* $i \neq j$ *in* $\{1, \ldots, t\}$.

Assume furthermore that $a_1, \ldots, a_t$ are integers and $n_1, \ldots, n_t$ are odd positive integers such that

(iv) $\gcd(2n_i + 1, p_i - 1) = 1$ *for all* $i = 1, \ldots, t$.

(v) $p_i \mid \sum_{j=1}^t p_j^{n_i} + a_i^{n_i}$ *for all* $i = 1, \ldots, t$.

(vi) $s = t/2$ *of the* t *numbers* $\left(\dfrac{a_i}{p_i}\right)$ *for* $i = 1, \ldots, t$ *are equal to* 1 *and the other* s *are equal to* -1.

Then there exist infinitely many primes p such that $S^*_{\frac{p-1}{2},t+1}$ contains at least one element.

Proof. Let a be such that

$$a \equiv 2n_i + 1 \pmod{(p_i - 1)/2}, \qquad a \equiv 3 \pmod 4, \qquad a \equiv a_i \pmod{p_i} \tag{2}$$

for all $i = 1, \ldots, t$. The fact that the above integer a exists is a consequence of the Chinese Remainder Theorem and conditions (i)-(iii) above. Since n_i is odd, $(p_i - 1)/2$ is also odd and $a \equiv 3 \pmod 4$, we conclude that the congruence $a \equiv 2n_i + 1 \pmod{(p_i - 1)/2}$ implies $a \equiv 2n_i + 1 \pmod{2(p_i - 1)}$.

Now let $M = 4\prod_{i=1}^{t} \frac{p_i(p_i - 1)}{2}$. Note that the number a is coprime to M by conditions (i)-(iv). Thus, by Dirichlet's Theorem on primes in arithmetic progressions, it follows that there exist infinitely many prime numbers p such that $p \equiv a \pmod M$. It is clear that these primes satisfy the same congruences (2) as a does. Let p be such a prime and set

$$n = \sum_{i=1}^{t} p_i^{(p-1)/2} + p^{(p-1)/2}.$$

Note that since $p \equiv 2n_i + 1 \pmod{2(p_i - 1)}$, we get that $(p-1)/2 \equiv n_i \pmod{p_i - 1}$. Therefore by Fermat's Little Theorem and condition (v) we get

$$n \equiv \sum_{j=1}^{t} p_j^{n_i} + p^{n_i} \equiv \sum_{j=1}^{t} p_j^{n_i} + a_i^{n_i} \equiv 0 \pmod{p_i}$$

for all $i = 1, \ldots, t$. Finally, conditions (i), (v) and the Quadratic Reciprocity Law show that from the $t = 2s$ numbers

$$\left(\frac{p_i}{p}\right) = -\left(\frac{p}{p_i}\right) = -\left(\frac{a_i}{p_i}\right),$$

exactly half of them are 1 and the other half are -1. Thus, half of the numbers $p_i^{(p-1)/2}$ are congruent to 1 modulo p and the other half are congruent to -1 modulo p which shows that n is a multiple of p. Hence, n is a multiple of p_i for $i = 1, \ldots, t$ and of p as well, which implies that $n \in S^*_{\frac{p-1}{2}, t+1}$. □

Lemma 3.2. *If $s \geq 2$ then there exist primes p_i and integers a_i, n_i for $i = 1, \ldots, t$ satisfying the conditions of Lemma 3.1.*

Proof. Observe that $t - 1 \geq 3$. Choose primes $p_1, \ldots, p_{t-1}$ such that $p_i \equiv 11 \pmod{12}$ for all $i = 1, \ldots, t-1$, $\gcd(p_i, p_j - 1) = 1$ for all i, j in $\{1, \ldots, t-1\}$, $\gcd(p_i - 1, p_j - 1) = 2$ for all $i \neq j$ in $\{1, \ldots, t-1\}$ and finally $p_1 + \cdots + p_{t-1}$ is coprime to $p_1 \cdots p_{t-1}$. Note that $N = p_1 + \cdots + p_{t-1}$ is an odd number. Such primes can be easily constructed starting with say $p_1 = 11$ and recursively defining $p_2, \ldots, p_{t-1}$ as primes in suitable arithmetic progressions. Take $n_i = 1$ for $i = 1, \ldots, t$. Let $q_1, \ldots, q_\ell$ be all the primes dividing N. Pick some integers $a_1, \ldots, a_{t-1}$ such that s of the numbers $\left(\frac{-a_i}{p_i}\right)$ are -1 and the other $s - 1$ are 1. Now choose a prime p_t such that $p_t \equiv 11 \pmod{12}$, p_t is coprime to $p_i - 1$ for $i = 1, \ldots, t-1$,

$p_t \equiv -a_i - N \pmod{p_i}$ for $i = 1, \ldots, t-1$, and $\left(\dfrac{q_i}{p_t}\right) = 1$ for all $i = 1, \ldots, \ell$. For these last congruences to be fulfilled, we note that it is enough to choose $p_t \equiv 1 \pmod{q_u}$ if $q_u \equiv 1 \pmod 4$ and $p_t \equiv -1 \pmod{q_u}$ if $q_u \equiv 3 \pmod 4$, where $u = 1, \ldots, \ell$. Notice that this choice is consistent with the fact that $p_t \equiv 11 \pmod{12}$ if it happens that one of the q_u is 3. So far, the primes $p_1, \ldots, p_t$ satisfy conditions (i)–(iii) of Lemma 3.1. Finally, put $a_t = -N$. Note that $\left(\dfrac{-a_t}{p_t}\right) = \prod_{u=1}^{\ell} \left(\dfrac{q_u}{p_t}\right)^{\alpha_u} = 1$. Here, α_u is the exact power of q_u in $-a_t$. Hence, exactly s of the numbers $\left(\dfrac{-a_i}{p_i}\right)$ are 1 and the others are -1 and since all primes p_i are congruent to 3 modulo 4 the same remains true if one replaces $-a_i$ by a_i. Thus, condition (vi) of Lemma 3.1 holds. Now one checks immediately that (v) holds with $n_i = 1$ for all $i = 1, \ldots, t$, because for all $i = 1, \ldots, t-1$ we have

$$\sum_{j=1}^{t} p_j^{n_j} + a_i^{n_i} \equiv N + p_t + a_i \pmod{p_i} \equiv 0 \pmod{p_i},$$

while

$$\sum_{j=1}^{t} p_j^{n_j} + a_t = N + p_t - N \equiv 0 \pmod{p_t},$$

and (iv) is obvious because $2n_i + 1 = 3$ and $p_i - 1 \equiv 10 \pmod{12}$ is not a multiple of 3 for $i = 1, \ldots, t$. □

Remark. The above argument does not work for $s = 1$. Indeed, in this case $t - 1 = 1$, therefore $p_1 + \cdots + p_{t-1} = p_1$ and this is **not** coprime to p_1.

4 Computational results and further remarks

To conduct a search for elements of S_k^*, one can proceed as follows. If $n \in S_{k,\ell}^*$, then there exists a positive number Q and primes $p_1, p_2, \ldots, p_\ell$ such that

$$n = Qp_1 \cdots p_{\ell-1}p_\ell = p_1^k + \cdots + p_{\ell-1}^k + p_\ell^k,$$

so that a necessary condition for n to be in $S_{k,\ell}^*$ is that $p_\ell | (p_1^k + \cdots + p_{\ell-1}^k)$. (Note that some of the p_i's may also divide Q.)

For instance, in order to find $n \in S_{k,3}^*$, we examine the prime factors p of $r^k + q^k$ as $2 \le r < q$ run through the primes up to a given x, and we then check if $Q := \dfrac{r^k + q^k + p^k}{rqp}$ is an integer. If this is so, then the integer $n = Qrqp$ belongs to $S_{k,3}^*$.

Proceeding in this manner (with $\ell = 3, 4$), we found the following elements of S_2^*:

$$\begin{aligned}
870 &= 2 \cdot 3 \cdot 5 \cdot 29 = 2^2 + 5^2 + 29^2, \\
188355 &= 3 \cdot 5 \cdot 29 \cdot 433 = 5^2 + 29^2 + 433^2, \\
298995972 &= 2^2 \cdot 3 \cdot 11 \cdot 131 \cdot 17291 = 3^2 + 11^2 + 131^2 + 17291^2, \\
1152597606 &= 2 \cdot 3 \cdot 5741 \cdot 33461 = 2^2 + 5741^2 + 33461^2, \\
1879906755 &= 3 \cdot 5 \cdot 2897 \cdot 43261 = 5^2 + 2897^2 + 43261^2, \\
5209105541772 &= 2^2 \cdot 3 \cdot 11 \cdot 17291 \cdot 2282281 = 3^2 + 11^2 + 17291^2 + 2282281^2.
\end{aligned}$$

Although we could not find any elements of S_4, we did find some elements of S_4^*, but they are quite large. Here are six of them:

$$\begin{aligned}
107827277891825604 &= 2^2 \cdot 3 \cdot 7 \cdot 31 \cdot 67 \cdot 18121 \cdot 34105993 = 3^4 + 31^4 + 67^4 + 18121^4, \\
486984904149815559698 &= 2 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17 \cdot 157 \cdot 83537 \cdot 14816023 = 2^4 + 17^4 + 83537^4,
\end{aligned}$$

$$\begin{aligned}
&313716322726301830198116071053308704 4 \\
&\quad = 2^2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 191 \cdot 283 \cdot 7541 \cdot 1330865843 \cdot 2086223663996743 \\
&\quad = 3^4 + 7^4 + 191^4 + 1330865843^4, \\
&129500871006614668230506335477000185618 \\
&\quad = 2 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 31 \cdot 241 \cdot 15331 \cdot 21613 \cdot 524149 \cdot 1389403 \cdot 3373402577 \\
&\quad = 2^4 + 241^4 + 3373402577^4, \\
&225611412654969160905328479254197935523312771590 \\
&\quad = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13^2 \cdot 37 \cdot 41 \cdot 109 \cdot 113 \cdot 127 \cdot 151 \cdot 541 \cdot 911 \cdot 5443 \\
&\qquad \cdot 3198662197 \cdot 689192061269 \\
&\quad = 5^4 + 7^4 + 113^4 + 127^4 + 911^4 + 689192061269^4, \\
&17492998726637106830622386354099071096746866616980 \\
&\quad = 2^2 \cdot 5 \cdot 7 \cdot 23 \cdot 31 \cdot 97 \cdot 103 \cdot 373 \cdot 1193 \cdot 8689 \cdot 2045107145539 \cdot 2218209705651794191 \\
&\quad = 2^4 + 103^4 + 373^4 + 1193^4 + 2045107145539^4.
\end{aligned}$$

Note that these numbers provide elements of $S_{4,3}^*$, $S_{4,4}^*$, $S_{4,5}^*$ and $S_{4,6}^*$.

The smallest elements of S_2^*, S_3^*, …, S_{10}^* are the following:

$$\begin{aligned}
870 &= 2 \cdot 3 \cdot 5 \cdot 29 = 2^2 + 5^2 + 29^2 \\
378 &= 2 \cdot 3^3 \cdot 7 = 2^3 + 3^3 + 7^3 \\
107\,827\,277\,891\,825\,604 &= 2^2 \cdot 3 \cdot 7 \cdot 31 \cdot 67 \cdot 18121 \cdot 34105993 = 3^4 + 31^4 + 67^4 + 18121^4 \\
178\,101 &= 3^2 \cdot 7 \cdot 11 \cdot 257 = 3^5 + 7^5 + 11^5 \\
594\,839\,010 &= 2 \cdot 3 \cdot 5 \cdot 17 \cdot 29 \cdot 37 \cdot 1087 = 2^6 + 5^6 + 29^6 \\
275\,223\,438\,741 &= 3 \cdot 23 \cdot 43 \cdot 92761523 = 3^7 + 23^7 + 43^7 \\
26\,584\,448\,904\,822\,018 &= 2 \cdot 3 \cdot 7 \cdot 17 \cdot 19 \cdot 113 \cdot 912733109 = 2^8 + 17^8 + 113^8 \\
40\,373\,802 &= 2 \cdot 3^4 \cdot 7 \cdot 35603 = 2^9 + 3^9 + 7^9 \\
420\,707\,243\,066\,850 &= 2 \cdot 3^2 \cdot 5^2 \cdot 29 \cdot 32238102917 = 2^{10} + 5^{10} + 29^{10}.
\end{aligned}$$

Below is a table of the smallest element $n \in S^*_{k,\ell}$ for $\ell = 3, 4, 5, 6, 7$ (with a convenient k):

ℓ	n
3	$378 = 2 \cdot 3^3 \cdot 7 = 2^3 + 3^3 + 7^3$
4	$298995972 = 2^2 \cdot 3 \cdot 11 \cdot 131 \cdot 17291 = 3^2 + 11^2 + 131^2 + 17291^2$
5	$2\,836\,295 = 5 \cdot 7 \cdot 11 \cdot 53 \cdot 139 = 5^3 + 7^3 + 11^3 + 53^3 + 139^3$
6	(a 48 digit number) $= 5^4 + 7^4 + 113^4 + 127^4 + 911^4 + 689192061269^4$
7	(a 145 digit number) $= 2^{14} + 3^{14} + 5^{14} + 11^{14} + 29^{14} + 149^{14} + 19809551197^{14}$

Theorem 3 provides a way to construct infinitely many elements of $S^*_{k,\ell}$ given any fixed positive odd integer ℓ. However, in practice, the elements obtained are very large. Indeed, take the case $k = 5$. With the notation of Lemma 1, we have $t = 4$; one can then choose $\{p_1, p_2, p_3, p_4\} = \{11, 47, 59, 227\}$. As suggested in Lemma 2, let $n_i = 1$ for $i = 1, 2, 3, 4$. An appropriate set of integers a_i's is given by $\{a_1, a_2, a_3, a_4\} = \{8, 32, 10, 110\}$, which gives $\left\{\left(\frac{a_i}{p_i}\right) : i = 1, 2, 3, 4\right\} = \{-1, 1, -1, 1\}$. Looking for a solution a of the set of congruences

$$\begin{cases} a \equiv 3 \pmod{\frac{p_i - 1}{2}} & (i = 1, 2, 3, 4) \\ a \equiv 3 \pmod{4} & \\ a \equiv a_i \pmod{p_i} & (i = 1, 2, 3, 4) \end{cases},$$

we obtain $a = 4\,619\,585\,064\,883$. With $M = 4 \prod_{i=1}^{4} \frac{p_i(p_i - 1)}{2} = 10\,437\,648\,923\,020$, we notice that indeed $\gcd(a, M) = 1$. As the smallest prime number $p \equiv a \pmod{M}$, we find $p = 10M + a = 108\,996\,074\,295\,083$. This means that the smallest integer $n \in S^*_{k,5}$ constructed with our algorithm is given by

$$n = \sum_{i=1}^{4} p_i^{\frac{p-1}{2}} + p^{\frac{p-1}{2}},$$

which is quite a large integer since

$$n \approx p^{\frac{p-1}{2}} \approx (10^{14})^{\frac{1}{2} \cdot 10^{14}} \approx 10^{7 \cdot 10^{14}}.$$

References

[1] J. M. De Koninck and F. Luca, Integers representable as the sum of powers of their prime factors, *Functiones et Approximatio* **33** (2005), 57–72.

[2] A. Schinzel and W. Sierpiński, Sur certaines hypothèses concernant les nombres premiers, *Acta Arith.* **4** (1958), 185–208. Corrigendum: **5** (1959), 259.

2000 *Mathematics Subject Classification*: Primary 11A41; Secondary 11A25.
Keywords: prime factorization.

Received January 5 2006; revised version received December 30 2006. Published in *Journal of Integer Sequences*, December 30 2006.

Return to Journal of Integer Sequences home page.