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Efficient Lower Bounds on the Number of Repetition-free Words

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Abstract

We propose a new effective method for obtaining lower bounds on the number of repetition-free words over a finite alphabet.

1 Introduction

Repetition-free words over finite alphabets are a traditional object of research in combinatorics of words. A word w over an alphabet Σ is a finite sequence $a_1 \cdots a_n$ of symbols from Σ. The number n is called the *length* of w and is denoted by $|w|$. The symbol a_i of w is denoted by $w[i]$. A word $a_i \cdots a_j$, where $1 \le i \le j \le n$, is called a *factor* of w and is denoted by $w[i : j]$. For any $i = 1, \ldots, n$ the factor $w[1 : i]$ ($w[i : n]$) is called a *prefix* (a *suffix*) of w. A positive integer p is called a *period* of w if $a_i = a_{i+p}$ for each $i = 1, \ldots, n - p$. If p is the minimal period of w, the ratio n/p is called the *exponent* of w. Two words w', w'' over Σ are called *isomorphic* if $|w'| = |w''|$ and there exists a bijection $\sigma : \Sigma \longrightarrow \Sigma$ such that $w''[i] = \sigma(w'[i])$, $i = 1, \ldots, |w'|$. The set of all words over Σ is denoted by Σ^*.

For any finite set A we denote by $|A|$ the number of elements of A. Let W be an arbitrary set of words. This set is called *factorial* if for any word w from W all factors of w are also contained in W. We denote by $W(n)$ the subset of W consisting of all words of length n. If W is factorial then it is not difficult to show (see, e.g., [4, 1]) that there exists the limit

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$\lim_{n\to\infty} \sqrt[n]{|W(n)|}$ which is called the *growth rate* of words from W. For any word v and any $n \geq |v|$ we denote by $W^{(v)}(n)$ the set of all words from $W(n)$ which contain v as a suffix.

By a *repetition* we mean any word of exponent greater than 1. The best known example of a repetition is a *square*; that is, a word of the form uu, where u is an arbitrary nonempty word. Avoiding ambiguity[2], by the *period* of the square uu we mean the length of u. In an analogous way, a *cube* is a word of the form uuu for a nonempty word u, a the *period* of this cube is also the length of u. A word is called *square-free* (*cube-free*) if it contains no squares (cubes) as factors. It is easy to see that there are no binary square-free words of length more than 3. On the other hand, by the classical results of Thue [19, 20], there exist ternary square-free words of arbitrary length and binary cube-free words of arbitrary length. If we denote by $S^{\langle \mathrm{sf} \rangle}(n)$ the number of ternary square-free words of length n and by $S^{\langle \mathrm{cf} \rangle}(n)$ the number of binary cube-free words of length n, we then have that $S^{\langle \mathrm{sf} \rangle}(n) > 0$ and $S^{\langle \mathrm{cf} \rangle}(n) > 0$ for any n. For ternary square-free words this result was strengthened by Dejean in [7]. She found ternary words of arbitrary length which have no factors with exponents greater than 7/4. On the other hand, she showed that any long enough ternary word contains a factor with an exponent greater than or equal to 7/4. Thus, the number 7/4 is the minimal limit for exponents of prohibited factors in arbitrarily long ternary words. For this reason we call ternary words having no factors with exponents greater than 7/4 *minimally repetitive* ternary words. Dejean conjectured also that the minimal limit for exponents of prohibited factors in arbitrarily long words over a k-letter alphabet is equal to 7/5 for $k = 4$ and $k/k-1$ for $k \geq 5$. This conjecture was proved for $k = 4$ by Pansiot [17], for $5 \leq k \leq 11$ by Moulin Ollagnier [14], for $12 \leq k \leq 14$ by Mohammad-Noori and Currie [13], and for $k \geq 38$ by Carpi [5]. Denote by $S^{\langle \mathrm{lf} \rangle}(n)$ the number of minimally repetitive ternary words of length n. It follows from the result of Dejean that $S^{\langle \mathrm{lf} \rangle}(n) > 0$ for any n.

Note that the set of all ternary square-free words, the set of all binary cube-free words, and the set of all minimally repetitive ternary words are factorial. So there exist the growth rates $\gamma^{\langle \mathrm{sf} \rangle} = \lim_{n\to\infty} \sqrt[n]{S^{\langle \mathrm{sf} \rangle}(n)}$, $\gamma^{\langle \mathrm{cf} \rangle} = \lim_{n\to\infty} \sqrt[n]{S^{\langle \mathrm{cf} \rangle}(n)}$, $\gamma^{\langle \mathrm{lf} \rangle} = \lim_{n\to\infty} \sqrt[n]{S^{\langle \mathrm{lf} \rangle}(n)}$ of words from these sets. Brandenburg proved in [3] that the values $S^{\langle \mathrm{sf} \rangle}(n)$ and $S^{\langle \mathrm{cf} \rangle}(n)$ grew exponentially with n, namely, $S^{\langle \mathrm{sf} \rangle}(n) \geq 6 \cdot 1.032^n$ and $S^{\langle \mathrm{cf} \rangle}(n) \geq 2 \cdot 1.080^n$, i. e. $\gamma^{\langle \mathrm{sf} \rangle} \geq 1.032$ and $\gamma^{\langle \mathrm{cf} \rangle} \geq 1.080$. Later the lower bound for $\gamma^{\langle \mathrm{sf} \rangle}$ was improved consecutively[3] by Ekhad, Zeilberger, Grimm, and Sun in [9, 10, 18]. The best upper bounds known at present $\gamma^{\langle \mathrm{sf} \rangle} < 1.30178858$ and $\gamma^{\langle \mathrm{cf} \rangle} < 1.4576$ are obtained by Ochem and Edlin in [16] and [8] respectively. In [15] Ochem established the exponential growth of the number of minimally repetitive words over either a three-letter or a four-letter alphabet. However, this result does not give any significant lower bound for $\gamma^{\langle \mathrm{lf} \rangle}$.

In [12] we proposed a new method for obtaining lower bounds on the number of repetition-free words. This method is essentially based on inductive estimation of the number of words which contain repetitions as factors. Using this method, we obtained the bounds $\gamma^{\langle \mathrm{sf} \rangle} \geq 1.30125$ and $\gamma^{\langle \mathrm{cf} \rangle} \geq 1.456975$. The main drawback of the proposed method was the large size of computer computations required. In particular, for this reason we did

[2] Note that the period of a square is not necessarily the minimal period of this word.

[3] A review of results on the estimations for the number of repetition-free words can be found in [2].

not manage to obtain a lower bound for $\gamma^{\langle \mathrm{lf} \rangle}$ by the proposed method. In this paper we propose a modification of the given method which requires a much less size of computer computations. Using this modification, we obtain the bounds $\gamma^{\langle \mathrm{sf} \rangle} \geq 1.30173$, $\gamma^{\langle \mathrm{lf} \rangle} \geq 1.245$, and $\gamma^{\langle \mathrm{cf} \rangle} \geq 1.457567$. Comparing the obtained lower bounds for $\gamma^{\langle \mathrm{sf} \rangle}$ and $\gamma^{\langle \mathrm{cf} \rangle}$ with the known upper bounds, one can conclude that we have estimated $\gamma^{\langle \mathrm{sf} \rangle}$ and $\gamma^{\langle \mathrm{cf} \rangle}$ within a precision of 0.0001, which demonstrates the high efficiency of the proposed modification.

2 Estimation for the number of ternary square-free words

For obtaining a lower bound on $\gamma^{\langle \mathrm{sf} \rangle}$ we consider the alphabet $\Sigma_3 = \{0, 1, 2\}$. Denote the set of all square-free words from Σ_3^* by $\mathcal{F}$. Let m be a natural number, $m > 2$, and w', w'' be two words from $\mathcal{F}(m)$. We call the word w'' a *descendant* of the word w' if $w'[2:m] = w''[1:m-1]$ and $w'w''[m] = w'[1]w'' \in \mathcal{F}(m+1)$. The word w' is called in this case an *ancestor* of the word w''. Further, we introduce a notion of closed words in the following inductive way. A word w from $\mathcal{F}(m)$ is called *right closed* (*left closed*) if and only if this word satisfies one of the two following conditions:

(a) **Basis of induction.** w has no descendants (ancestors);

(b) **Inductive step.** All descendants (ancestors) of w are right closed (left closed).

A word is *closed* if it is either right closed or left closed. Denote by $\mathcal{L}_m$ the set of all words from Σ_3^* which do not contain closed words from $\mathcal{F}(m)$ as factors. By $\mathcal{F}_m$ we denote the set of all square-free words from $\mathcal{L}_m$. Note that a word w is closed if and only if any word isomorphic to w is also closed. So we have the following obvious fact.

Proposition 2.1. *For any isomorphic words w', w'' and any $n \geq |w'|$ the equality $|\mathcal{F}_m^{(w')}(n)| = |\mathcal{F}_m^{(w'')}(n)|$ holds.*

By $\mathcal{F}'(m)$ we denote the set of all words w from $\mathcal{F}(m)$ such that $w[1] = 0$ and $w[2] = 1$. It is obvious that for any word w from $\mathcal{F}(m)$ there exists a single word from $\mathcal{F}'(m)$ which is isomorphic to w. Let w', w'' be two words from $\mathcal{F}'(m)$. We call the word w'' a *quasi-descendant* of the word w' if w'' is isomorphic to some descendant of w'. The word w' is called in this case a *quasi-ancestor* of the word w''. Let $\mathcal{F}''(m)$ be the set of all words from $\mathcal{F}'(m)$ which are not closed. Since ternary square-free words of arbitrary length exist, $\mathcal{F}''(m)$ is not empty for any m. Denote $s = |\mathcal{F}''(m)|$. We enumerate all words from $\mathcal{F}''(m)$ by numbers $1, 2, \ldots, s$ in lexicographic order and denote i-th word of the set $\mathcal{F}''(m)$ by w_i, $i = 1, \ldots, s$. Then we define a matrix $\Delta_m = (\delta_{ij})$ of size $s \times s$ in the following way: $\delta_{ij} = 1$ if and only if w_i is an quasi-ancestor of w_j; otherwise $\delta_{ij} = 0$. Note that Δ_m is a nonnegative matrix, so, by Perron-Frobenius theorem, for Δ_m there exists some maximal in modulus eigenvalue r which is a nonnegative real number. Moreover, we can find some eigenvector $\tilde{x} = (x_1; \ldots; x_s)$ with nonnegative components which corresponds to r. Assume that $r > 1$ and all components of $\tilde{x}$ are positive. For $n \geq m$ define $S_m^{\langle \mathrm{sf} \rangle}(n) = \sum_{i=1}^{s} x_i \cdot |\mathcal{F}_m^{(w_i)}(n)|$. In an inductive way we estimate $S_m^{\langle \mathrm{sf} \rangle}(n+1)$ by $S_m^{\langle \mathrm{sf} \rangle}(n)$.

First we estimate $|\mathcal{F}_m^{(w_i)}(n+1)|$ for $i = 1, \ldots, s$. It is obvious that

$$|\mathcal{F}_m^{(w_i)}(n+1)| = |\mathcal{G}^{(w_i)}(n+1)| - |\mathcal{H}^{(w_i)}(n+1)|, \tag{1}$$

where $\mathcal{G}^{(w_i)}(n+1)$ is the set of all words w from $\mathcal{L}_m^{(w_i)}(n+1)$ such that the words $w[1:n]$ and $w[n-m+1:n+1]$ are square-free, and $\mathcal{H}^{(w_i)}(n+1)$ is the set of all words from $\mathcal{G}^{(w_i)}(n+1)$ which contain some square as a suffix. Denote by $\pi(i)$ the set of all quasi-ancestors of w_i. Taking into account Proposition 2.1, it is easy to see that

$$|\mathcal{G}^{(w_i)}(n+1)| = \sum_{w\in\pi(i)} |\mathcal{F}_m^{(w)}(n)|. \tag{2}$$

We now estimate $|\mathcal{H}^{(w_i)}(n+1)|$. For any word w from $\mathcal{H}^{(w_i)}(n+1)$ we can find the minimal square which is a suffix of w. Denote the period of this square by $\lambda(w)$. It is obvious that $\lfloor (m+1)/2 \rfloor < \lambda(w) \le \lfloor (n+1)/2 \rfloor$. Denote by $\mathcal{H}_j^{(w_i)}(n+1)$ the set of all words w from $\mathcal{H}^{(w_i)}(n+1)$ such that $\lambda(w) = j$. Then

$$|\mathcal{H}^{(w_i)}(n+1)| = \sum_{\lfloor (m+1)/2 \rfloor < j \le \lfloor (n+1)/2 \rfloor} |\mathcal{H}_j^{(w_i)}(n+1)|. \tag{3}$$

Take some integer $p \ge m$ and assume that $n \ge 2p$. Consider a set $\mathcal{H}_j^{(w_i)}(n+1)$ where $j \le p$. Let w be an arbitrary word from this set. Then the suffix $w[n-2j+2:n+1]$ is a square which contains neither closed words from $\mathcal{F}(m)$ nor other squares as factors and contains the word w_i as a suffix. Let $v_1, \ldots, v_t$ be all possible squares of period j which satisfy the given conditions. Denote by $\mathcal{H}_{j,k}^{(w_i)}(n+1)$ the set of all words from $\mathcal{H}_j^{(w_i)}(n+1)$ which contain the square v_k as a suffix, $k = 1, \ldots, t$. Let $w \in \mathcal{H}_{j,k}^{(w_i)}(n+1)$. Since the prefix $w[1:n]$ is square-free, in this case we have $w[n-2j+1] \ne w[n-2j+2] = v_k[1]$ and $w[n-2j+1] \ne w[n-j+1] = v_k[j]$. Moreover, $v_k[1] = w[n-j+2] \ne w[n-j+1] = v_k[j]$. Thus, the symbol $w[n-2j+1]$ is determined uniquely by v_k as the symbol from Σ_3 which is different from the two distinct symbols $v_k[1]$ and $v_k[j]$. Denoting this symbol by b_k, we conclude that v_k determines uniquely the factor $w[n-2j+1:n]$ as the word $b_k v_k[1:2j-1]$. Therefore, if this word is not square-free or contains a closed word from $\mathcal{F}(m)$ as a factor then $\mathcal{H}_{j,k}^{(w_i)}(n+1) = \emptyset$. Let $b_k v_k[1:2j-1] \in \mathcal{F}_m$. Then define $u_k = b_k v_k[1:m-1] = w[n-2j+1:n-2j+m]$. Since w is determined uniquely by the prefix $w[1:n-2j]$, we have $|\mathcal{H}_{j,k}^{(w_i)}(n+1)| \le |\mathcal{F}_m^{(u_k)}(n-2j+m)|$. Denote by u'_k the word from $\mathcal{F}'(m)$ which is isomorphic to u_k. Then, by Proposition 2.1, we have

$$|\mathcal{H}_{j,k}^{(w_i)}(n+1)| \le |\mathcal{F}_m^{(u'_k)}(n-2j+m)|.$$

Thus, denoting by $U_j(w_i)$ the set of all words[4] u'_k, we obtain that

$$|\mathcal{H}_j^{(w_i)}(n+1)| = \sum_{k=1}^{t} |\mathcal{H}_{j,k}^{(w_i)}(n+1)| \le \sum_{u\in U_j(w_i)} |\mathcal{F}_m^{(u)}(n-2j+m)|. \tag{4}$$

[4]Note that among words u'_k we can have identical words, i.e., the same word can be counted several times in $U_j(w_i)$.

Consider now the set $\mathcal{H}_j^{(w_i)}(n+1)$ for $j > p$. Note that in this case $j > m$. Let w be an arbitrary word from $\mathcal{H}_j^{(w_i)}(n+1)$. Then for w we have $w[n-2j+2 : n-j+1] = w[n-j+2 : n+1]$, i.e., $w[n-j-m+2 : n-j+1] = w_i$ and w is determined uniquely by the prefix $w[1 : n-j-m+1]$. Therefore, in this case the inequality

$$|\mathcal{H}_j^{(w_i)}(n+1)| \le |\mathcal{F}_m^{(w_i)}(n-j+1)|. \tag{5}$$

holds. Using inequalities (4) and (5) in (3), we obtain

$$|\mathcal{H}^{(w_i)}(n+1)| \le A_p^{(w_i)}(n+1) + B_p^{(w_i)}(n+1) \tag{6}$$

where

$$\begin{aligned} A_p^{(w_i)}(n+1) &= \sum_{\lfloor (m+1)/2 \rfloor < j \le p} \left(\sum_{u \in U_j(w_i)} |\mathcal{F}_m^{(u)}(n-2j+m)| \right), \\ B_p^{(w_i)}(n+1) &= \sum_{p < j \le \lfloor (n+1)/2 \rfloor} |\mathcal{F}_m^{(w_i)}(n-j+1)|. \end{aligned}$$

We now estimate $S_m^{\langle \mathrm{sf} \rangle}(n+1)$. Using equality (1), we have

$$S_m^{\langle \mathrm{sf} \rangle}(n+1) = \sum_{i=1}^{s} x_i \cdot |\mathcal{G}^{(w_i)}(n+1)| - \sum_{i=1}^{s} x_i \cdot |\mathcal{H}^{(w_i)}(n+1)|. \tag{7}$$

Recall that $\tilde{x}$ is a eigenvector of Δ_m for the eigenvalue r. So, applying equality (2), we obtain

$$\begin{aligned} &\textstyle\sum_{i=1}^{s} x_i \cdot |\mathcal{G}^{(w_i)}(n+1)| = \sum_{i=1}^{s} \big(x_i \cdot \sum_{w \in \pi(i)} |\mathcal{F}_m^{(w)}(n)| \big) \\ &= (x_1; x_2; \dots; x_s) \begin{pmatrix} \delta_{11} & \delta_{21} & \dots & \delta_{s1} \\ \delta_{12} & \delta_{22} & \dots & \delta_{s2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1s} & \delta_{2s} & \dots & \delta_{ss} \end{pmatrix} \begin{pmatrix} |\mathcal{F}_m^{(w_1)}(n)| \\ |\mathcal{F}_m^{(w_2)}(n)| \\ \vdots \\ |\mathcal{F}_m^{(w_s)}(n)| \end{pmatrix} \\ &= r \cdot (x_1; x_2; \dots; x_s) \begin{pmatrix} |\mathcal{F}_m^{(w_1)}(n)| \\ |\mathcal{F}_m^{(w_2)}(n)| \\ \vdots \\ |\mathcal{F}_m^{(w_s)}(n)| \end{pmatrix} = r \cdot S_m^{\langle \mathrm{sf} \rangle}(n). \end{aligned} \tag{8}$$

For estimating the second sum in the right side of (7) we use equality (6):

$$\sum_{i=1}^{s} x_i \cdot |\mathcal{H}^{(w_i)}(n+1)| \le \sum_{i=1}^{s} x_i \cdot A_p^{(w_i)}(n+1) + \sum_{i=1}^{s} x_i \cdot B_p^{(w_i)}(n+1) \tag{9}$$

where

$$\begin{aligned} \sum_{i=1}^{s} x_i \cdot B_p^{(w_i)}(n+1) &= \sum_{p < j \le \lfloor (n+1)/2 \rfloor} \left(\sum_{i=1}^{s} x_i \cdot |\mathcal{F}_m^{(w_i)}(n-j+1)| \right) \\ &= \sum_{p < j \le \lfloor (n+1)/2 \rfloor} S_m^{\langle \mathrm{sf} \rangle}(n-j+1). \end{aligned} \tag{10}$$

For $k=1,\dots,s$ denote by $\zeta_j^{(k)}(w_i)$ the number of words w_k in the set $U_j(w_i)$, and define $\eta_k(j)$ as $\sum_{i=1}^s x_i\cdot\zeta_j^{(k)}(w_i)$. Note that any word from $U_j(w_i)$ belongs to $\mathcal{F}''(m)$. Hence

$$\begin{aligned}\sum_{i=1}^s x_i\cdot A_p^{(w_i)}(n+1) &= \sum_{\lfloor (m+1)/2\rfloor<j\le p}\ \sum_{i=1}^s x_i\left(\sum_{u\in U_j(w_i)}|\mathcal{F}_m^{(u)}(n-2j+m)|\right)\\ &= \sum_{\lfloor (m+1)/2\rfloor<j\le p}\ \sum_{k=1}^s \eta_k(j)\cdot|\mathcal{F}_m^{(w_k)}(n-2j+m)|. \qquad (11)\end{aligned}$$

Take some integer $q\ge 2p-m$ and assume that $n\ge q+m$. For the sake of convenience we present sum (11) as

$$\sum_{d=d_0}^q\ \sum_{k=1}^s \eta'_k(d)\cdot|\mathcal{F}_m^{(w_k)}(n-d)|$$

where $\eta'_k(d)=\eta_k\big((d+m)/2\big)$ if $d+m$ is even, $\eta'_k(d)=0$ otherwise, and $d_0=2\cdot\lfloor (m+3)/2\rfloor-m$. We majorize this sum by some sum $\sum_{d=d_0}^q\rho_d\cdot S_m^{\langle\mathrm{sf}\rangle}(n-d)$ in the following way. We compute consecutively coefficients ρ_d of this sum for $d=d_0,d_0+1,\dots,q$. For each $d=d_0,d_0+1,\dots,q-1$ together with the number ρ_d we compute also numbers $\eta''_1(d+1),\dots,\eta''_s(d+1)$ such that

$$\sum_{j=d_0}^{d+1}\sum_{k=1}^s\eta'_k(j)\cdot|\mathcal{F}_m^{(w_k)}(n-j)|\le\sum_{k=1}^s\eta''_k(d+1)\cdot|\mathcal{F}_m^{(w_k)}(n-d-1)|+\sum_{j=d_0}^d\rho_j\cdot S_m^{\langle\mathrm{sf}\rangle}(n-j). \quad (12)$$

For $d=d_0$ we take $\rho_{d_0}=\min_{1\le k\le s}(\eta'_k(d_0)/x_k)$. Then

$$\sum_{k=1}^s\eta'_k(d_0)\cdot|\mathcal{F}_m^{(w_k)}(n-d_0)|=\rho_{d_0}\cdot S_m^{\langle\mathrm{sf}\rangle}(n-d_0)+\sum_{k=1}^s\nu_k\cdot|\mathcal{F}_m^{(w_k)}(n-d_0)|$$

where $\nu_k=\eta'_k(d_0)-\rho_{d_0}\cdot x_k$, $k=1,\dots,s$. Denote by $\tilde\nu$ the vector $(\nu_1;\dots;\nu_s)$ and consider the vector $\tilde\nu'=\Delta_m\tilde\nu$. Let $\tilde\nu'=(\nu'_1;\dots;\nu'_s)$. It follows from (1) and (2) that

$$|\mathcal{F}_m^{(w_k)}(n-d_0)|\le|\mathcal{G}^{(w_i)}(n-d_0)|=\sum_{w\in\pi(k)}|\mathcal{F}_m^{(w)}(n-d_0-1)|$$

for any $k=1,\dots,s$. Note also that $\nu_k\ge 0$ for $k=1,\dots,s$. Hence

$$\begin{aligned}\sum_{k=1}^s\nu_k\cdot|\mathcal{F}_m^{(w_k)}(n-d_0)| &\le \sum_{k=1}^s\left(\nu_k\cdot\sum_{w\in\pi(k)}|\mathcal{F}_m^{(w)}(n-d_0-1)|\right)\\ &= \sum_{k=1}^s\nu'_k\cdot|\mathcal{F}_m^{(w_k)}(n-d_0-1)|.\end{aligned}$$

Thus

$$\sum_{j=d_0}^{d_0+1}\sum_{k=1}^s\eta'_k(j)\cdot|\mathcal{F}_m^{(w_k)}(n-j)|\le\rho_{d_0}\cdot S_m^{\langle\mathrm{sf}\rangle}(n-d_0)+\sum_{k=1}^s\eta''_k(d_0+1)\cdot|\mathcal{F}_m^{(w_k)}(n-d_0-1)| \quad (13)$$

where $\eta''_k(d_0+1) = \eta'_k(d_0+1) + \nu'_k$. Assume now that for some d such that $d_0 < d < q$ we already computed the numbers $\rho_{d_0}, \dots, \rho_{d-1}$ and $\eta''_1(d), \dots, \eta''_s(d)$. Then we take $\rho_d = \min_{1\le k\le s}(\eta''_k(d)/x_k)$, $\tilde{\nu} = (\eta''_1(d) - \rho_d \cdot x_1, \dots, \eta''_s(d) - \rho_d \cdot x_s)$, and $\tilde{\nu}' = \Delta_m \tilde{\nu}$. We take also $\eta''_k(d+1) = \eta'_k(d+1) + \nu'_k$ where ν'_k is the k-th component of the vector $\tilde{\nu}'$, $k = 1, \dots, s$. Analogously to inequality (13), in this case we have the inequality

$$\sum_{k=1}^{s} \left(\eta''_k(d) \cdot |\mathcal{F}_m^{(w_k)}(n-d)| + \eta'_k(d+1) \cdot |\mathcal{F}_m^{(w_k)}(n-d-1)|\right)$$
$$\le \rho_d \cdot S_m^{\langle \mathrm{sf} \rangle}(n-d) + \sum_{k=1}^{s} \eta''_k(d+1) \cdot |\mathcal{F}_m^{(w_k)}(n-d-1)|.$$

This inequality implies that inequality (12) holds for every d. For $d = q$ we take $\rho_q = \max_{1\le k\le s}(\eta''_k(q)/x_k)$. Thus,

$$\sum_{i=1}^{s} x_i A_p^{(w_i)}(n+1) = \sum_{d=d_0}^{q} \sum_{k=1}^{s} \eta'_k(d) \cdot |\mathcal{F}_m^{(w_k)}(n-d)| \le \sum_{d=d_0}^{q} \rho_d \cdot S_m^{\langle \mathrm{sf} \rangle}(n-d). \tag{14}$$

For the sake of convenience denote by $\mathcal{P}_m^{(p,q)}(z)$ the polynomial $\sum_{d=d_0}^{q} \rho_d \cdot z^d$ in a variable z.

Let for some $\alpha > 1$ and each $i = m, m+1, \dots, n-1$ the inequality $S_m^{\langle \mathrm{sf} \rangle}(i+1) \ge \alpha S_m^{\langle \mathrm{sf} \rangle}(i)$ be valid. Then for each $i = m, m+1, \dots, n-1$ we have $S_m^{\langle \mathrm{sf} \rangle}(i) \le S_m^{\langle \mathrm{sf} \rangle}(n)/\alpha^{n-i}$. So relation (14) implies that

$$\sum_{i=1}^{s} x_i \cdot A_p^{(w_i)}(n+1) \le \sum_{d=d_0}^{q} \rho_d \cdot (S_m^{\langle \mathrm{sf} \rangle}(n)/\alpha^d) = \mathcal{P}_m^{(p,q)}(1/\alpha) \cdot S_m^{\langle \mathrm{sf} \rangle}(n).$$

In an analogous way relation (10) implies that

$$\begin{aligned} \sum_{i=1}^{s} x_i B_p^{(w_i)}(n+1) &\le \sum_{p<j\le\lfloor (n+1)/2 \rfloor} S_m^{\langle \mathrm{sf} \rangle}(n)/\alpha^{j-1} < S_m^{\langle \mathrm{sf} \rangle}(n) \cdot \sum_{j=p}^{\infty} 1/\alpha^j \\ &= \frac{S_m^{\langle \mathrm{sf} \rangle}(n)}{\alpha^{p-1}(\alpha-1)}. \end{aligned}$$

Thus, from (9) we have

$$\sum_{i=1}^{s} x_i \cdot |\mathcal{H}^{(w_i)}(n+1)| < S_m^{\langle \mathrm{sf} \rangle}(n) \cdot \left(\mathcal{P}_m^{(p,q)}(1/\alpha) + \frac{1}{\alpha^{p-1}(\alpha-1)}\right).$$

Using this inequality together with equality (8) in (7), we obtain

$$S_m^{\langle \mathrm{sf} \rangle}(n+1) > S_m^{\langle \mathrm{sf} \rangle}(n) \cdot \left(r - \mathcal{P}_m^{(p,q)}(1/\alpha) - \frac{1}{\alpha^{p-1}(\alpha-1)}\right).$$

Therefore, if α satisfy the inequality

$$r - \mathcal{P}_m^{(p,q)}(1/\alpha) - \frac{1}{\alpha^{p-1}(\alpha-1)} \ge \alpha,$$

we obtain inductively that the inequality $S_m^{\langle\mathrm{sf}\rangle}(n+1) \geq \alpha S_m^{\langle\mathrm{sf}\rangle}(n)$ holds for any n. Thus, in this case we have $S_m^{\langle\mathrm{sf}\rangle}(n) = \Omega(\alpha^n)$. Since, obviously, the order of growth of $S^{\langle\mathrm{sf}\rangle}(n)$ is not less than $S_m^{\langle\mathrm{sf}\rangle}(n)$, we then conclude that $S^{\langle\mathrm{sf}\rangle}(n) = \Omega(\alpha^n)$. Hence $\gamma^{\langle\mathrm{sf}\rangle} \geq \alpha$.

For obtaining a concrete lower bound on $\gamma^{\langle\mathrm{sf}\rangle}$ we take the parameters $m = 45$, $p = 52$, $q = 60$. Using computer computations, we have obtained[5] that $|\mathcal{F}''(45)| = 277316$, the maximal in modulus eigenvalue r for Δ_{45} is 1.302011, and all components of the eigenvector corresponding to r are positive. Further, we obtained that

$$\mathcal{P}_{45}^{(52,60)}(z) = 3.759479 \cdot z^{44} + 3.176743 \cdot z^{45} + 6.048526 \cdot z^{46} + \\ 7.120005 \cdot z^{48} + 14.679230 \cdot z^{50} + 41.594270 \cdot z^{52} + 37.431675 \cdot z^{55} + \\ 40.471892 \cdot z^{56} + 32.780085 \cdot z^{58} + 5.235193 \cdot z^{59} + 275.705551 \cdot z^{60}.$$

Let $\alpha = 1.30173$. It is immediately checked that

$$r - \mathcal{P}_{45}^{(52,60)}(1/\alpha) - \frac{1}{\alpha^{51}(\alpha-1)} \geq \alpha.$$

Moreover, the inequalities $S_{45}^{\langle\mathrm{sf}\rangle}(n+1) \geq \alpha S_{45}^{\langle\mathrm{sf}\rangle}(n)$ for each $n = 45, 46, \ldots, q+m-1 = 104$ are verified in the same inductive way as described above with evident modifications following from the restriction $n < q + m$. Thus, we obtaine that $\gamma^{\langle\mathrm{sf}\rangle} \geq 1.30173$.

3 Estimation for the number of minimally repetitive ternary words

For obtaining a lower bound on $\gamma^{\langle\mathrm{lf}\rangle}$ we also consider words over Σ_3. Now denote by $\mathcal{F}$ the set of all minimally repetitive words from Σ_3^*. By a prohibited repetition we mean a word with an exponent greater than 7/4. Let m be a natural number, $m > 2$. Analogously to the case of square-free words, we introduce also the notions of descendant, ancestor, and closed word for minimally repetitive words, and denote by $\mathcal{L}_m$ the set of all words from Σ_3^* which do not contain closed words from $\mathcal{F}(m)$ as factors. Denote also by $\mathcal{F}_m$ the set of all minimally repetitive words from $\mathcal{L}_m$, by $\mathcal{F}'(m)$ the set of all words w from $\mathcal{F}(m)$ such that $w[1] = 0$ and $w[2] = 1$, and by $\mathcal{F}''(m)$ the set of all words from $\mathcal{F}'(m)$ which are not closed. As in the case of square-free words, we introduce the notions of quasi-descendant and quasi-ancestor, define for $\mathcal{F}''(m)$ the matrix Δ_m of size $s \times s$ where $s = |\mathcal{F}''(m)|$, and compute the maximal in modulus eigenvalue r of this matrix. If $r > 1$ and all components of the eigenvector $\tilde{x} = (x_1; \ldots; x_s)$ corresponding to r are positive, then we denote by μ the ratio $\max_i x_i / \min_i x_i$, and for $n \geq m$ consider $S_m^{\langle\mathrm{lf}\rangle}(n) = \sum_{i=1}^{s} x_i \cdot |\mathcal{F}_m^{(w_i)}(n)|$ where w_i is i-th word of the set $\mathcal{F}''(m)$, $i = 1, \ldots, s$.

Analogously to equality (1), for $i = 1, \ldots, s$ we have

$$|\mathcal{F}_m^{(w_i)}(n+1)| = |\mathcal{G}^{(w_i)}(n+1)| - |\mathcal{H}^{(w_i)}(n+1)| \tag{15}$$

[5]In this paper the obtained numerical results are given with precision of 6 decimal digits after the point.

where $\mathcal{G}^{(w_i)}(n+1)$ is the set of all words w from $\mathcal{L}_m^{(w_i)}(n+1)$ such that the words $w[1:n]$ and $w[n-m+1:n+1]$ are minimally repetitive, and $\mathcal{H}^{(w_i)}(n+1)$ is the set of all words from $\mathcal{G}^{(w_i)}(n+1)$ which contain some prohibited repetition as a suffix. Analogously to equality (2), we can obtain

$$|\mathcal{G}^{(w_i)}(n+1)| = \sum_{w\in\pi(i)} |\mathcal{F}_m^{(w)}(n)|$$

where $\pi(i)$ is the set of all quasi-ancestors of w_i. For any word w from $\mathcal{H}^{(w_i)}(n+1)$ denote by $\lambda(w)$ the minimal period of the shortest prohibited repetition which is a suffix of w. Then, analogously to equality (3),

$$|\mathcal{H}^{(w_i)}(n+1)| = \sum_{\lfloor (4m+3)/7\rfloor < j \le \lfloor 4n/7\rfloor} |\mathcal{H}_j^{(w_i)}(n+1)|,$$

where $\mathcal{H}_j^{(w_i)}(n+1)$ is the set of all words w from $\mathcal{H}^{(w_i)}(n+1)$ such that $\lambda(w)=j$. Take some integer $p \ge 4m/3-1$ and assume that $n > \lfloor 7p/4\rfloor$. Let w be an arbitrary word from $\mathcal{H}_j^{(w_i)}(n+1)$ where $j \le p$. Then the suffix $w[n-\lfloor 7p/4\rfloor+1:n+1]$ is a prohibited repetition which contains neither closed words from $\mathcal{F}(m)$ nor other prohibited repetitions as factors and contains the word w_i as a suffix. Let $v_1,\dots,v_t$ be all possible prohibited repetitions with minimal period j which satisfy the given conditions. Denote by $\mathcal{H}_{j,k}^{(w_i)}(n+1)$ the set of all words from $\mathcal{H}_j^{(w_i)}(n+1)$ which contain v_k as a suffix, $k=1,\dots,t$. Let $w \in \mathcal{H}_{j,k}^{(w_i)}(n+1)$. Analogously to the case of square-free words, the symbol $w[n-\lfloor 7j/4\rfloor]$ is determined uniquely by v_k as the symbol from Σ_3 which is different from the two distinct symbols $v_k[1]$ and $v_k[j]$. Denoting this symbol by b_k, we conclude that the factor $w[n-\lfloor 7j/4\rfloor : n]$ is determined uniquely as the word $b_k v_k[1:\lfloor 7j/4\rfloor]$. Let this word belong to $\mathcal{F}_m$. Then we denote by u'_k the word from $\mathcal{F}'(m)$ which is isomorphic to the word $b_k v_k[1:m-1]$. Analogously to inequality (4), one can obtain the inequality

$$|\mathcal{H}_j^{(w_i)}(n+1)| \le \sum_{u\in U_j(w_i)} |\mathcal{F}_m^{(u)}(n+m-\lfloor 7j/4\rfloor-1)|$$

where $U_j(w_i)$ is the set of all words[6] u'_k. Thus,

$$|\mathcal{H}^{(w_i)}(n+1)| \le A_p^{(w_i)}(n+1) + \sum_{p<j\le\lfloor 4n/7\rfloor} |\mathcal{H}_j^{(w_i)}(n+1)| \tag{16}$$

where

$$A_p^{(w_i)}(n+1) = \sum_{\lfloor (4m+3)/7\rfloor < j \le p} \left(\sum_{u\in U_j(w_i)} |\mathcal{F}_m^{(u)}(n+m-\lfloor 7j/4\rfloor-1)| \right).$$

From (15) and (16) we obtain that

$$\begin{aligned} S_m^{\langle \mathrm{lf}\rangle}(n+1) \;\ge\; & \sum_{i=1}^{s} x_i\cdot|\mathcal{G}^{(w_i)}(n+1)| - \sum_{i=1}^{s} x_i\cdot|A_p^{(w_i)}(n+1)| - \\ & \sum_{p<j\le\lfloor 4n/7\rfloor} \left(\sum_{i=1}^{s} x_i\cdot|\mathcal{H}_j^{(w_i)}(n+1)| \right). \end{aligned} \tag{17}$$

[6] Note that, as in the case of square-free words, the same word can be counted several times in $U_j(w_i)$.

Analogously to equality (8), the equality

$$\sum_{i=1}^{s} x_i \cdot |\mathcal{G}^{(w_i)}(n+1)| = r \cdot S_m^{\langle \mathrm{lf} \rangle}(n) \tag{18}$$

is valid. Let $j > p$, i.e., $j \geq 4m/3$. Note that the sets $\mathcal{H}_j^{(w_i)}(n+1)$ are non-overlapping. So we have the obvious inequality

$$\sum_{i=1}^{s} x_i \cdot |\mathcal{H}_j^{(w_i)}(n+1)| \leq |\mathcal{M}_j| \cdot \max_{i=1,\dots,s} x_i \tag{19}$$

where $\mathcal{M}_j = \bigcup_{i=1}^{s} \mathcal{H}_j^{(w_i)}(n+1)$. Note also that any word w from $\mathcal{M}_j$ is determined uniquely by the prefix $w[1 : n - \lfloor 3j/4 \rfloor]$ and satisfies the conditions $w[n+2-m-j] = w[n+2-m] = 0$ and $w[n+3-m-j] = w[n+3-m] = 1$. Thus $|\mathcal{M}_j| \leq |\mathcal{M}'_j|$ where $\mathcal{M}'_j$ is the set of all words w from $\mathcal{F}_m(n - \lfloor 3j/4 \rfloor)$ such that $w[n+2-m-j] = 0$ and $w[n+3-m-j] = 1$. Consider also the set $\mathcal{M}''_j$ of all words w from $\mathcal{F}_m(n - \lfloor 3j/4 \rfloor)$ such that $w[n+1-\lfloor 3j/4 \rfloor - m] = 0$ and $w[n+2-\lfloor 3j/4 \rfloor - m] = 1$. There is an evident bijection between the sets $\mathcal{M}'_j$ and $\mathcal{M}''_j$, so $|\mathcal{M}'_j| = |\mathcal{M}''_j|$. Note also that the set $\mathcal{M}''_j$ is the union of the non-overlapping sets $\mathcal{H}_j^{(w_i)}(n - \lfloor 3j/4 \rfloor)$ for $i = 1, \dots, s$, i.e.,

$$|\mathcal{M}''_j| \leq \sum_{i=1}^{s} |\mathcal{H}_j^{(w_i)}(n - \lfloor 3j/4 \rfloor)| \leq S_m^{\langle \mathrm{lf} \rangle}(n - \lfloor 3j/4 \rfloor)/(\min_{i=1,\dots,s} x_i).$$

Therefore, it follows from (19) that

$$\sum_{i=1}^{s} x_i \cdot |\mathcal{H}_j^{(w_i)}(n+1)| \leq |\mathcal{M}'_j| \cdot \max_{i=1,\dots,s} x_i = |\mathcal{M}''_j| \cdot \max_{i=1,\dots,s} x_i \leq \mu \cdot S_m^{\langle \mathrm{lf} \rangle}(n - \lfloor 3j/4 \rfloor).$$

Thus,

$$\sum_{p < j \leq \lfloor 4n/7 \rfloor} \left(\sum_{i=1}^{s} x_i \cdot |\mathcal{H}_j^{(w_i)}(n+1)| \right) \leq \mu \cdot \sum_{p < j \leq \lfloor 4n/7 \rfloor} S_m^{\langle \mathrm{lf} \rangle}(n - \lfloor 3j/4 \rfloor). \tag{20}$$

Let $\eta_k(j) = \sum_{i=1}^{s} x_i \cdot \zeta_j^{(k)}(w_i)$ where $\zeta_j^{(k)}(w_i)$ is the number of words w_k in the set $U_j(w_i)$, $k = 1, \dots, s$. Then, analogously to equality (11),

$$\sum_{i=1}^{s} x_i \cdot |A_p^{(w_i)}(n+1)| = \sum_{\lfloor (4m+3)/7 \rfloor < j \leq p} \sum_{k=1}^{s} \eta_k(j) \cdot |\mathcal{F}_m^{(w_k)}(n + m - \lfloor 7j/4 \rfloor - 1)|. \tag{21}$$

Take some integer $q \geq \lfloor 7p/4 \rfloor + 1 - m$ and assume that $n \geq q + m$. Analogously to the case of square-free words, sum (21) can be majorized by some sum $\sum_{d=d_0}^{q} \rho_d \cdot S_m(n-d)$ where $d_0 = \lfloor 7j_0/4 \rfloor + 1 - m$ for $j_0 = \lfloor (4m+3)/7 \rfloor + 1$.

Let for some $\alpha > 1$ and each $i = m, m+1, \dots, n-1$ the inequality $S_m^{\langle \mathrm{lf} \rangle}(i+1) \geq \alpha S_m^{\langle \mathrm{lf} \rangle}(i)$ be valid, i.e., $S_m^{\langle \mathrm{lf} \rangle}(i) \leq S_m^{\langle \mathrm{lf} \rangle}(n)/\alpha^{n-i}$. Then

$$\sum_{i=1}^{s} x_i \cdot A_p^{(w_i)}(n+1) \leq \sum_{d=d_0}^{q} \rho_d \cdot (S_m^{\langle \mathrm{lf} \rangle}(n)/\alpha^d) = \mathcal{P}_m^{(p,q)}(1/\alpha) \cdot S_m^{\langle \mathrm{lf} \rangle}(n) \tag{22}$$

where $\mathcal{P}_m^{(p,q)}(z) = \sum_{d=d_0}^{q} \rho_d \cdot z^d$. Moreover, it follows from (20) that

$$\begin{aligned}\sum_{p<j\leq\lfloor 4n/7\rfloor} \left(\sum_{i=1}^{s} x_i \cdot |\mathcal{H}_j^{(w_i)}(n+1)|\right) &\leq \mu \cdot \sum_{p<j\leq\lfloor 4n/7\rfloor} S_m^{\langle\mathrm{lf}\rangle}(n)/\alpha^{\lfloor 3j/4\rfloor} \\ &< \mu S_m^{\langle\mathrm{lf}\rangle}(n) \cdot \sum_{p<j} 1/\alpha^{\lfloor 3j/4\rfloor} \\ &= \mu S_m^{\langle\mathrm{lf}\rangle}(n) \cdot \left(\sum_{j\geq\lfloor 3(p+1)/4\rfloor} 1/\alpha^j + \sum_{j\geq\lceil (p+1)/4\rceil} 1/\alpha^{3j}\right) \\ &= \mu S_m^{\langle\mathrm{lf}\rangle}(n) \cdot \Big(1/\big(\alpha^{\lfloor(3p-1)/4\rfloor}(\alpha-1)\big) + 1/\big(\alpha^{3\lfloor p/4\rfloor}(\alpha^3-1)\big)\Big).\end{aligned}$$

Thus, from inequality (17) together with relations (18) and (22) we obtain that

$$\begin{aligned}S_m^{\langle\mathrm{lf}\rangle}(n+1) \;>\; S_m^{\langle\mathrm{lf}\rangle}(n) \cdot \Bigg(& r - \mathcal{P}_m^{(p,q)}(1/\alpha) - \\ & \mu \cdot \Big(1/\big(\alpha^{\lfloor(3p-1)/4\rfloor}(\alpha-1)\big) + 1/\big(\alpha^{3\lfloor p/4\rfloor}(\alpha^3-1)\big)\Big)\Bigg).\end{aligned}$$

Therefore, if

$$r - \mathcal{P}_m^{(p,q)}(1/\alpha) - \mu \cdot \Big(1/\big(\alpha^{\lfloor(3p-1)/4\rfloor}(\alpha-1)\big) + 1/\big(\alpha^{3\lfloor p/4\rfloor}(\alpha^3-1)\big)\Big) \geq \alpha$$

then $S_m^{\langle\mathrm{lf}\rangle}(n+1) \geq \alpha S_m^{\langle\mathrm{lf}\rangle}(n)$ for any n, i.e., $S_m^{\langle\mathrm{lf}\rangle}(n) = \Omega(\alpha^n)$. Since the order of growth of $S^{\langle\mathrm{lf}\rangle}(n)$ is not less than $S_m^{\langle\mathrm{lf}\rangle}(n)$, in this case we have $S^{\langle\mathrm{lf}\rangle}(n) = \Omega(\alpha^n)$, i.e., $\gamma^{\langle\mathrm{lf}\rangle} \geq \alpha$.

Using computer computations with the parameters $m = 42$, $p = 72$, $q = 85$, we obtained that $|\mathcal{F}''(42)| = 36141$, $r = 1.247500$, all components of the eigenvector corresponding to r were positive, and $\mathcal{P}_{42}^{(72,85)}(z)$ was

$$\begin{aligned}&1.976268 \cdot z^{42} + 1.148062 \cdot z^{44} + 3.519576 \cdot z^{45} + 1.741046 \cdot z^{47} + \\ &9.687624 \cdot z^{49} + 0.126312 \cdot z^{50} + 31.479339 \cdot z^{52} + 12.284335 \cdot z^{53} + \\ &21.010557 \cdot z^{54} + 24.183001 \cdot z^{56} + 96.529327 \cdot z^{61} + 129.216325 \cdot z^{64} + \\ &256.213310 \cdot z^{66} + 14.826731 \cdot z^{67} + 64.163103 \cdot z^{68} + 6.862805 \cdot z^{69} + \\ &84.819931 \cdot z^{70} + 2.337610 \cdot z^{72} + 175.026144 \cdot z^{73} + 41.068102 \cdot z^{74} + \\ &335.714818 \cdot z^{75} + 341.576384 \cdot z^{78} + 329.970329 \cdot z^{80} + 693.282157 \cdot z^{81} + \\ &763.104210 \cdot z^{82} + 303.272754 \cdot z^{83} + 583.157071 \cdot z^{84} + 10510.070498 \cdot z^{85}.\end{aligned}$$

Let $\alpha = 1.245$. It is immediately checked that

$$r - \mathcal{P}_{42}^{(72,85)}(1/\alpha) - \frac{1}{\alpha^{53}(\alpha-1)} - \frac{1}{\alpha^{54}(\alpha^3-1)} \geq \alpha.$$

Moreover, we estimate $S_{42}^{\langle\mathrm{lf}\rangle}(n+1) \geq \alpha S_{42}^{\langle\mathrm{lf}\rangle}(n)$ for each $n = 42, 43, \ldots, q+m-1 = 126$ in the same inductive way with evident modifications following from the restriction $n < q+m$. Thus we obtain that $\gamma^{\langle\mathrm{lf}\rangle} \geq 1.245$.

4 Estimation for the number of binary cube-free words

To obtain a lower bound on $\gamma^{\langle \mathrm{cf} \rangle}$, we consider the alphabet $\Sigma_2 = \{0, 1\}$. Denote by $\mathcal{F}$ the set of all cube-free words from Σ_2^*. Analogously to the case of square-free words, for any natural number $m > 2$ we can introduce the notions of descendant, ancestor, and closed word for cube-free words. Denote also by $\mathcal{L}_m$ the set of all words from Σ_2^* which do not contain closed words from $\mathcal{F}(m)$ as factors, and by $\mathcal{F}_m$ the set of all cube-free words from $\mathcal{L}_m$. By $\mathcal{F}'(m)$ we denote the set of all words w from $\mathcal{F}(m)$ such that $w[1] = 0$. Note that for any word w from $\mathcal{F}(m)$ there exists a single word from $\mathcal{F}'(m)$ which is isomorphic to w. By $\mathcal{F}''(m)$ we denote the set of all words from $\mathcal{F}'(m)$ which are not closed. We introduce also the notions of quasi-descendant and quasi-ancestor, define for $\mathcal{F}''(m)$ the matrix Δ_m of size $s \times s$ where $s = |\mathcal{F}''(m)|$, and compute the maximal in modulus eigenvalue r of this matrix. If $r > 1$ and all components of the eigenvector $\tilde{x} = (x_1; \dots; x_s)$ corresponding to r are positive, then for $n \geq m$ we consider $S_m^{\langle \mathrm{cf} \rangle}(n) = \sum_{i=1}^{s} x_i \cdot |\mathcal{F}_m^{(w_i)}(n)|$ where w_i is i-th word of the set $\mathcal{F}''(m)$, $i = 1, \dots, s$.

As in the case of square-free words, for $i = 1, \dots, s$ we have

$$|\mathcal{F}_m^{(w_i)}(n+1)| = |\mathcal{G}^{(w_i)}(n+1)| - |\mathcal{H}^{(w_i)}(n+1)| \tag{23}$$

where $\mathcal{G}^{(w_i)}(n+1)$ is the set of all words w from $\mathcal{L}_m^{(w_i)}(n+1)$ such that the words $w[1:n]$ and $w[n-m+1:n+1]$ are cube-free, and $\mathcal{H}^{(w_i)}(n+1)$ is the set of all words from $\mathcal{G}^{(w_i)}(n+1)$ which contain some cube as a suffix. Analogously to equality (2), we obtain

$$|\mathcal{G}^{(w_i)}(n+1)| = \sum_{w \in \pi(i)} |\mathcal{F}_m^{(w)}(n)|$$

where $\pi(i)$ is the set of all quasi-ancestors of w_i. For any word w from $\mathcal{H}^{(w_i)}(n+1)$ denote by $\lambda(w)$ the period of the minimal cube which is a suffix of w. Then, analogously to equality (3),

$$|\mathcal{H}^{(w_i)}(n+1)| = \sum_{\lfloor (m+1)/3 \rfloor < j \leq \lfloor (n+1)/3 \rfloor} |\mathcal{H}_j^{(w_i)}(n+1)|, \tag{24}$$

where $\mathcal{H}_j^{(w_i)}(n+1)$ is the set of all words w from $\mathcal{H}^{(w_i)}(n+1)$ such that $\lambda(w) = j$. Take some integer $p \geq m$ and assume that $n \geq 3p$. Let w be an arbitrary word from $\mathcal{H}_j^{(w_i)}(n+1)$ where $j \leq p$. Then the suffix $w[n-3j+2 : n+1]$ is a cube which contains neither closed words from $\mathcal{F}(m)$ nor other cubes as factors and contains the word w_i as a suffix. Let $v_1, \dots, v_t$ be all possible cubes of period j which satisfy the given conditions. Denote by $\mathcal{H}_{j,k}^{(w_i)}(n+1)$ the set of all words from $\mathcal{H}_j^{(w_i)}(n+1)$ which contain the cube v_k as a suffix, $k = 1, \dots, t$. Let $w \in \mathcal{H}_{j,k}^{(w_i)}(n+1)$. Since the prefix $w[1:n]$ is cube-free, in this case we have $w[n-3j+1] \neq w[n-2j+1] = v_k[j]$. Thus, the symbol $w[n-3j+1]$ is determined uniquely by v_k as the symbol from Σ_2 which is different from $v_k[j]$. Denoting this symbol by b_k, we conclude that v_k determines uniquely the factor $w[n-3j+1:n]$ as the word $b_k v_k[1:3j-1]$. If this word is cube-free and does not contain closed words from $\mathcal{F}(m)$ as factors then we denote by u'_k the word from $\mathcal{F}'(m)$ which is isomorphic to $w[n-3j+1 : n-3j+m]$.

Analogously to inequality (4), one can obtain the inequality

$$|\mathcal{H}_j^{(w_i)}(n+1)| \le \sum_{u \in U_j(w_i)} |\mathcal{F}_m^{(u)}(n-3j+m)| \tag{25}$$

where $U_j(w_i)$ is the set of all words[7] u'_k. For $j > p$, analogously to inequality (5), we have

$$|\mathcal{H}_j^{(w_i)}(n+1)| \le |\mathcal{F}_m^{(w_i)}(n-2j+1)|. \tag{26}$$

Thus, from (24), (25) and (26) we obtain that

$$|\mathcal{H}^{(w_i)}(n+1)| \le A_p^{(w_i)}(n+1) + B_p^{(w_i)}(n+1) \tag{27}$$

where

$$\begin{aligned} A_p^{(w_i)}(n+1) &= \sum_{\lfloor (m+1)/3 \rfloor < j \le p} \left(\sum_{u \in U_j(w_i)} |\mathcal{F}_m^{(u)}(n-3j+m)| \right), \\ B_p^{(w_i)}(n+1) &= \sum_{p < j \le \lfloor (n+1)/3 \rfloor} |\mathcal{F}_m^{(w_i)}(n-2j+1)|. \end{aligned}$$

Moreover, analogously to equalities (8) and (10), the equalities

$$\sum_{i=1}^{s} x_i \cdot |\mathcal{G}^{(w_i)}(n+1)| = r \cdot S_m^{\langle \mathrm{cf} \rangle}(n) \tag{28}$$

and

$$\sum_{i=1}^{s} x_i \cdot B_p^{(w_i)}(n+1) = \sum_{p < j \le \lfloor (n+1)/3 \rfloor} S_m^{\langle \mathrm{cf} \rangle}(n-2j+1) \tag{29}$$

hold. Let $\eta_k(j) = \sum_{i=1}^{s} x_i \cdot \zeta_j^{(k)}(w_i)$ where $\zeta_j^{(k)}(w_i)$ is the number of words w_k in the set $U_j(w_i)$, $k = 1, \ldots, s$. Then, analogously to equality (11),

$$\sum_{i=1}^{s} x_i \cdot A_p^{(w_i)}(n+1) = \sum_{\lfloor (m+1)/3 \rfloor < j \le p} \sum_{k=1}^{s} \eta_k(j) \cdot |\mathcal{F}_m^{(w_k)}(n-3j+m)|. \tag{30}$$

Take some integer $q \ge 3p - m$ and assume that $n \ge q + m$. Analogously to the case of square-free words, sum (30) can be majorized by some sum $\sum_{d=d_0}^{q} \rho_d \cdot S_m^{\langle \mathrm{cf} \rangle}(n-d)$ where $d_0 = 3 \cdot \lfloor (m+4)/3 \rfloor - m$.

Let for some $\alpha > 1$ and each $i = m, m+1, \ldots, n-1$ the inequalities $S_m^{\langle \mathrm{cf} \rangle}(i+1) \ge \alpha S_m^{\langle \mathrm{cf} \rangle}(i)$ be valid. Then

$$\sum_{i=1}^{s} x_i \cdot A_p^{(w_i)}(n+1) \le \sum_{d=d_0}^{q} \rho_d \cdot (S_m^{\langle \mathrm{cf} \rangle}(n)/\alpha^d) = \mathcal{P}_m^{(p,q)}(1/\alpha) \cdot S_m^{\langle \mathrm{cf} \rangle}(n),$$

[7] Note that, as in the case of square-free words, the same word can be counted several times in $U_j(w_i)$.

where $\mathcal{P}_m^{(p,q)}(z) = \sum_{d=d_0}^{q} \rho_d \cdot z^d$. Moreover, it follows from (29) that

$$\sum_{i=1}^{s} x_i \cdot B_p^{(w_i)}(n+1) \le \sum_{p<j\le\lfloor(n+1)/3\rfloor} \frac{S_m^{\langle\mathrm{cf}\rangle}(n)}{\alpha^{2j-1}} < \sum_{j=p}^{\infty} \frac{S_m^{\langle\mathrm{cf}\rangle}(n)}{\alpha^{2j+1}} = \frac{S_m^{\langle\mathrm{cf}\rangle}(n)}{\alpha^{2p-1}(\alpha^2-1)}.$$

Thus, from (27) we have

$$\begin{aligned}\sum_{i=1}^{s} x_i \cdot |\mathcal{H}^{(w_i)}(n+1)| &\le \sum_{i=1}^{s} x_i \cdot A_p^{(w_i)}(n+1) + \sum_{i=1}^{s} x_i \cdot B_p^{(w_i)}(n+1) \\ &< S_m^{\langle\mathrm{cf}\rangle}(n) \cdot \left(\mathcal{P}_m^{(p,q)}(1/\alpha) + \frac{1}{\alpha^{2p-1}(\alpha^2-1)}\right).\end{aligned}$$

Using this inequality and equalities (23) and (28), we obtain

$$\begin{aligned}S_m^{\langle\mathrm{cf}\rangle}(n+1) &= \sum_{i=1}^{s} x_i \cdot |\mathcal{G}^{(w_i)}(n+1)| - \sum_{i=1}^{s} x_i \cdot |\mathcal{H}^{(w_i)}(n+1)| \\ &> S_m^{\langle\mathrm{cf}\rangle}(n) \cdot \left(r - \mathcal{P}_m^{(p,q)}(1/\alpha) - \frac{1}{\alpha^{2p-1}(\alpha^2-1)}\right).\end{aligned}$$

Therefore, if

$$r - \mathcal{P}_m^{(p,q)}(1/\alpha) - \frac{1}{\alpha^{2p-1}(\alpha^2-1)} \ge \alpha,$$

then $S_m^{\langle\mathrm{cf}\rangle}(n+1) \ge \alpha S_m^{\langle\mathrm{cf}\rangle}(n)$ for any n, i. e. $S_m^{\langle\mathrm{cf}\rangle}(n) = \Omega(\alpha^n)$. Since the order of growth of $S^{\langle\mathrm{cf}\rangle}(n)$ is not less than $S_m^{\langle\mathrm{cf}\rangle}(n)$, we obtain in this case that $S^{\langle\mathrm{cf}\rangle}(n) = \Omega(\alpha^n)$, i. e. $\gamma^{\langle\mathrm{cf}\rangle} \ge \alpha$.

Using computer computations with the parameters $m = 35$, $p = 35$, $q = 70$, we obtained that $|\mathcal{F}''(35)| = 732274$, $r = 1.457599$, all components of the eigenvector corresponding to r were positive, and $\mathcal{P}_{35}^{(35,70)}(z)$ was

$$\begin{aligned}&0.890340 \cdot z^{35} + 1.398382 \cdot z^{37} + 1.096456 \cdot z^{38} + 30.292784 \cdot z^{40} + \\ &2.533687 \cdot z^{41} + 1.296919 \cdot z^{42} + 28.893958 \cdot z^{43} + 22.780262 \cdot z^{44} + \\ &10.699704 \cdot z^{45} + 64.314464 \cdot z^{47} + 92.853910 \cdot z^{49} + 91.743094 \cdot z^{50} + \\ &67.688387 \cdot z^{51} + 48.613345 \cdot z^{52} + 68.285930 \cdot z^{53} + 113.239316 \cdot z^{54} + \\ &144.612325 \cdot z^{56} + 346.136318 \cdot z^{58} + 173.468149 \cdot z^{59} + 465.000388 \cdot z^{60} + \\ &134.993653 \cdot z^{61} + 224.831969 \cdot z^{62} + 585.928351 \cdot z^{63} + 355.591901 \cdot z^{65} + \\ &1335.518621 \cdot z^{67} + 343.074473 \cdot z^{68} + 2202.468159 \cdot z^{69} + 11098.126369 \cdot z^{70}.\end{aligned}$$

It is immediately checked that for $\alpha = 1.457567$ the inequality

$$r - \mathcal{P}_{35}^{(35,70)}(1/\alpha) - \frac{1}{\alpha^{69}(\alpha^2-1)} \ge \alpha$$

is valid. Moreover, the inequalities $S_{35}^{\langle\mathrm{cf}\rangle}(n+1) \ge \alpha S_{35}^{\langle\mathrm{cf}\rangle}(n)$ for $n = 35, 36, \ldots, q+m-1 = 104$ are also verified in the same inductive way with evident modifications following from the restriction $n < q + m$. Thus $\gamma^{\langle\mathrm{cf}\rangle} \ge 1.457567$.

5 Conclusions

Basing on results of computer experiments, we believe that by increasing the parameter m, one can estimate $\gamma^{\langle sf \rangle}$, $\gamma^{\langle lf \rangle}$, and $\gamma^{\langle cf \rangle}$ with an arbitrarily high precision. Note also that the proposed method for estimation of growth rates of repetition-free words is quite general: it can be applyed for estimating the growth rate of words over any finite alphabet with any (including fractional) minimal threshold for exponents of prohibited factors (provided that the growth is exponential). Moreover, this method can be easily modified for the case when additional restrictions are imposed on the minimal value of periods of prohibited factors (see [11]). We suppose that this method can be also generalized for estimation of growth rates of words avoiding patterns (see, e.g., [6]).

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