Counting Keith Numbers

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Abstract
A Keith number is a positive integer $N$ with the decimal representation $a_1a_2\cdots a_n$ such that $n \geq 2$ and $N$ appears in the sequence $(K_m)_{m \geq 1}$ given by the recurrence $K_1 = a_1, \ldots, K_n = a_n$ and $K_m = K_{m-1} + K_{m-2} + \cdots + K_{m-n}$ for $m > n$. We prove that there are only finitely many Keith numbers using only one decimal digit (i.e., $a_1 = a_2 = \cdots = a_n$), and that the set of Keith numbers is of asymptotic density zero.

1 Introduction
With the number 197, let $(K_m)_{m \geq 1}$ be the sequence whose first three terms $K_1 = 1$, $K_2 = 9$ and $K_3 = 7$ are the digits of 197 and that satisfies the recurrence $K_m = K_{m-1} + K_{m-2} + K_{m-3}$
for all \( m > 3 \). Its initial terms are

\[
1, 9, 7, 17, 33, 57, 107, 197, 361, 665, \ldots
\]

Note that 197 itself is a member of this sequence. This phenomenon was first noticed by Mike Keith and such numbers are now called \textit{Keith numbers}. More precisely, a number \( N \) with decimal representation \( a_1a_2\cdots a_n \) is a Keith number if \( n \geq 2 \) and \( N \) appears in the sequence \( K_N = (K_m^N)_{m \geq 1} \) whose \( n \) initial terms are the digits of \( N \) read from left to right and satisfying \( K_m^N = K_{m-1}^N + K_{m-2}^N + \cdots + K_{m-n}^N \) for all \( m > n \). These numbers appear in Keith’s papers [3, 4] and they are the subject of entry A007629 in Neil Sloane’s Encyclopedia of Integer Sequences [11] (see also [7, 8, 9]).

Let \( \mathcal{K} \) be the set of all Keith numbers. It is not known if \( \mathcal{K} \) is infinite or not. The sequence \( \mathcal{K} \) begins

\[
14, 19, 28, 47, 61, 75, 197, 742, 1104, 1537, 2208, 2580, 3684, 4788, \ldots
\]

M. Keith and D. Lichtblau found all 94 Keith numbers smaller than \( 10^{29} \) [4]. D. Lichtblau found the first \textit{pandigital} Keith number (containing each of the digits 0 to 9 at least once): 27847652577905793413.

Recall that a rep-digit is a positive integer \( N \) of the form \( a(10^n - 1)/9 \) for some \( a \in \{1, \ldots, 9\} \) and \( n \geq 1 \); i.e., a number which is a string of the same digit \( a \) when written in base 10. Our first result shows that there are only finitely many Keith numbers which are rep-digits.

\textbf{Theorem 1.1.} \textit{There are only finitely many Keith numbers that are rep-digits and their set can be effectively determined.}

We point out that some authors refer to the Keith numbers as \textit{replicating Fibonacci digits} in analogy with the Fibonacci sequence \( (F_n)_{n \geq 1} \) given by \( F_1 = 1, F_2 = 1 \) and \( F_{n+2} = F_{n+1} + F_n \) for all \( n \geq 1 \). F. Luca showed [5] that the largest rep-digit Fibonacci number is 55.

The proof of Theorem 1.1 uses Baker-type estimates for linear forms in logarithms. It will be clear from the proof that it applies to all \textit{base b Keith numbers} for any fixed integer \( b \geq 3 \), where these numbers are defined analogously starting with their base \( b \) expansion (see the remark after the proof of Theorem 1.1).

For a positive integer \( x \) we write \( \mathcal{K}(x) = \mathcal{K} \cap [1, x] \). As we mentioned before, \( \mathcal{K}(10^{29}) = 94 \). A heuristic argument [4] suggests that \( \# \mathcal{K}(x) \gg \log x \), and, in particular, that \( \mathcal{K} \) should be infinite. Going in the opposite way, we show that \( \mathcal{K} \) is of asymptotic density zero.

\textbf{Theorem 1.2.} \textit{The estimate}

\[
\# \mathcal{K}(x) \ll \frac{x}{\sqrt{\log x}}
\]

\textit{holds for all positive integers} \( x \geq 2 \).
The above estimate is very weak. It does not even imply that that sum of the reciprocals of the members of $\mathcal{K}$ is convergent. We leave to the reader the task of finding a better upper bound on $\# \mathcal{K}(x)$. Typographical changes (see the remark after the proof of Theorem 1.2) show that Theorem 1.2 also is valid for the set of base $b$ Keith numbers if $b \geq 4$. Perhaps it can be extended also to the case $b = 3$. For $b = 2$, Kenneth Fan has an unpublished manuscript (mentioned by Keith [4]) showing how to construct all Keith numbers and that, in particular, there are infinitely many of them. For example, any power of 2 is a binary Keith number.

Throughout this paper, we use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols $O$ and $o$ with their usual meaning. Recall that for functions $A$ and $B$ the inequalities $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent to the fact that there exists a positive constant $c$ such that the inequality $|A| \leq cB$ holds. The constants in the inequalities implied by these symbols may occasionally depend on other parameters. For a real number $x$ we use $\log x$ for the natural logarithm of $x$. For a set $\mathcal{A}$, we use $\#\mathcal{A}$ and $|\mathcal{A}|$ to denote its cardinality.

2 Preliminary Results

For an integer $N > 0$, recall the definition of the sequence $K^N = (K_m^N)_{m \geq 1}$ given in the Introduction. In $K^N$ we allow $N$ to be any string of the digits $0, 1, \ldots, 9$, so $N$ may have initial zeros. So, for example, $K^{020} = (0, 2, 0, 2, 4, 6, 12, 22, \ldots)$. For $n \geq 1$ we define the sequence $L^m_n$ as $L^m_n = K^M$ where $M = 11 \cdots 1$ with $n$ digits 1. In particular, $L^1_n = (1, 1, 1, \ldots)$ and $L^2_n = (1, 1, 2, 3, 5, 8, \ldots)$, the Fibonacci numbers. In the following lemma, which will be used in the proofs of both Theorems 1 and 2, we establish some properties of the sequences $K^N$ and $L^m_n$.

**Lemma 2.1.** Let $N$ be a string of the digits $0, 1, \ldots, 9$ with length $n \geq 1$. If $N$ does not start with 0, we understand it also as the decimal representation of a positive integer.

(a) If $N$ has at least $k \geq 1$ nonzero entries, then $K^N_m \geq L^k_{m-n}$ holds for every $m \geq n + 1$.

(b) If $N$ has at least one nonzero entry, then $K^N_m \geq L^m_{m-n}$ holds for every $m \geq n + 1$. We have $K^N_m \ll 9L^m_{m-n}$ for every $m \geq 1$.

(c) If $n \geq 3$ and $N = K^N_m$ for some $m \geq 1$ (so $N$ is a Keith number), then $2n < m < 7n$.

(d) For fixed $n \geq 2$ and growing $m \geq n + 1$,

$$L^m_n = 2^{m-n-1}(n-1)(1 + O(m/2^n)) + 1$$

where the constant in $O$ is absolute.

**Proof.** (a). By the recurrences defining $K^N$ and $L^k$, the inequality clearly holds for the first $k$ indices $m = n + 1, n + 2, \ldots, n + k$. For $m > n + k$ it holds by induction.

(b). We have $K^N_m \geq 1 = L^m_{m-n}$ for $m = n + 1, n + 2, \ldots, 2n$ and the inequality holds. For $m > 2n$ it holds by induction. The second inequality follows easily by induction.
(c). The lower bound \( m > 2n \) follows from the fact that \( K^N \) is nondecreasing and that
\[
K^N_{2n} \leq 9L^n_{2n} = 9 \cdot 2^{n-1}(n-1) + 9 < 10^{n-1} \leq N
\]
for \( n \geq 3 \). To obtain the upper bound, note that for \( m \geq n \) we have by induction that
\[
L^m_n \geq L^2_{m-n+2} \geq \phi^{m-n}
\]
where \( \phi = 1.61803 \cdots \) is the golden ratio. Thus, by part (b),
\[
10^n > N = K^N_m \geq L^m_{m-n} \geq \phi^{m-2n}
\]
and \( m < (2 + \log 10/\log \phi)n < 7n \).

(d). We write \( L^m_n \) in the form \( L^m_n = (2^{m-n-1} - d(m))(n-1) + 1 \) and prove by induction on \( m \) that for \( m \geq n+1 \),
\[
0 \leq d(m) < m2^{m-2n}.
\]
This will prove the claim.

It is easy to see by the recurrence that \( L^n_{n+1}, L^n_{n+2}, \ldots, L^n_{2n+1} \) are equal, respectively, to
\( 2^0(n-1) + 1, 2^1(n-1) + 1, \ldots, 2^n(n-1) + 1 \). So \( d(m) = 0 \) for \( n+1 \leq m \leq 2n+1 \) and the claim holds. For \( m \geq 2n+1 \),
\[
L^m_n = L^m_{m-1} + L^m_{m-2} + \cdots + L^m_{m-n}
\]
\[
= \sum_{k=1}^{n} \left( (2^{m-n-1-k} - d(m-k))(n-1) + 1 \right)
\]
\[
= \left( 2^{m-n-1} - 2^{m-2n-1} + 1 - \sum_{k=1}^{n} d(m-k) \right)(n-1) + 1
\]
and the induction hypothesis gives
\[
0 \leq d(m) = 2^{m-2n-1} - 1 + \sum_{k=1}^{n} d(m-k)
\]
\[
< 2^{m-2n-1} + (m-1) \sum_{k=1}^{n} 2^{m-2n-k}
\]
\[
< m2^{m-2n}.
\]

In part (d), if \( m \) is roughly of size \( 2^n \) or larger then the error term swallows the main term and the asymptotic estimate is useless. Indeed, the actual asymptotic behavior of \( L^m_n \) when \( m \to \infty \) is \( ca^m \) where \( c > 0 \) is a constant and \( a < 2 \) is the only positive root of the polynomial \( x^n - x^{n-1} - \cdots - x - 1 \). But for \( m \) small relative to \( 2^n \), say \( m = O(n) \) (ensured for Keith numbers by part (c)), this “incorrect” asymptotic estimate of \( L^m_n \) is very precise and useful, as we shall demonstrate in the proofs of Theorems 1.1 and 1.2.

In the proof of Theorem 1.1 we will apply also a lower bound for a linear form in logarithms. The following bound can be deduced from a result due to Matveev [6, Corollary 2.3].
Lemma 2.2. Let $A_1, \ldots, A_k$, $A_i > 1$, and $n_1, \ldots, n_k$ be integers, and let $N = \max\{|n_1|, \ldots, |n_k|, 2\}$. There exist positive absolute constants $c_1$ and $c_2$ (which are effective), such that if

$$\Lambda = n_1 \log A_1 + n_2 \log A_2 + \cdots + n_k \log A_k \neq 0,$$

then

$$\log |\Lambda| > -c_1 c_2^k (\log A_1) \cdots (\log A_k) \log N.$$

For the proof of Theorem 2 we will need an upper bound on sizes of antichains (sets of mutually incomparable elements) in the poset (partially ordered set)

$$P(k, n) = (\{1, 2, \ldots, k\}^n, \leq_p)$$

where $\leq_p$ is the product ordering

$$a = (a_1, a_2, \ldots, a_n) \leq_p b = (b_1, b_2, \ldots, b_n) \iff a_i \leq b_i \text{ for } i = 1, 2, \ldots, n.$$

We have $|P(k, n)| = k^n$ and for $k = 2$ the poset $P(2, n)$ is the Boolean poset of subsets of an $n$-element set ordered by inclusion. The classical theorem of Sperner [1, 2] asserts that the maximum size of an antichain in $P(2, n)$ equals the middle binomial coefficient $\binom{n}{n/2}$.

In the next lemma we obtain an upper bound for any $k \geq 2$.

Lemma 2.3. If $k \geq 2$, $n \geq 1$ and $X \subset P(k, n)$ is an antichain to $\leq_p$, then

$$|X| < \frac{(k/2) \cdot k^n}{n^{1/2}}.$$ 

Proof. We proceed by induction on $k$. For $k = 2$ this bound holds by Sperner’s theorem because

$$\binom{n}{n/2} < \frac{2^n}{n^{1/2}}$$

for every $n \geq 1$. Let $k \geq 3$ and $X \subset P(k, n)$ be an antichain. For $A$ running through the subsets of $[n] = \{1, 2, \ldots, n\}$, we partition $X$ in the sets $X_A$ where $X_A$ consists of the $u \in X$ satisfying $u_i = k \iff i \in A$. If we delete from all $u \in X_A$ all appearances of $k$, we obtain (after appropriate relabelling of coordinates) a set of $|X_A|$ distinct $(n - |A|)$-tuples from $P(k - 1, n - |A|)$ that must be an antichain to $\leq_p$. Thus, by induction, for $|A| < n$ we have

$$|X_A| < \frac{(k - 1/2) \cdot (k - 1)^{n-|A|}}{(n - |A|)^{1/2}}$$

and $|X_{[n]}| \leq 1$. Summing over all $A$s and using the inequality $\sqrt{n/m} \leq (n+1)/(m+1)$
(which holds for $1 \leq m \leq n$) and standard properties of binomial coefficients, we get

$$|X| = \sum_{A \subseteq [n]} |X_A|$$

$$< 1 + \sum_{i=0}^{n-1} \binom{n}{i} \frac{((k-1)/2) \cdot (k-1)^{n-i}}{(n-i)^{1/2}}$$

$$= \frac{1}{\sqrt{n}} \left( \sqrt{n} + \frac{1}{2} \sum_{i=0}^{n-1} \binom{n}{i} \sqrt{n/(n-i)} \cdot (k-1)^{n-i+1} \right)$$

$$\leq \frac{1}{\sqrt{n}} \left( \sqrt{n} + \frac{1}{2} \sum_{i=0}^{n-1} \frac{n+1}{n-i+1} (k-1)^{n-i+1} \right)$$

$$< \frac{k^{n+1}}{2\sqrt{n}}.$$

\[\square\]

We conclude this section with three remarks as to the last lemma.

1. Various generalizations and strengthenings of Sperner’s theorem were intensively studied, see, e.g., the book of Engel and Gronau [2]. Therefore, we do not expect much originality in our bound.

2. It is clear that for $k = 2$ the exponent $1/2$ of $n$ in the bound of Lemma 2.3 cannot be increased. The same is true for any $k \geq 3$. We briefly sketch a construction of a large antichain when $k = 3$; for $k > 3$ similar constructions can be given. For $k = 3$ and $n = 3m \geq 3$ consider the set $X \subseteq P(3, n)$ consisting of all $u$ which have $i$ 1s, $n-2i$ 2s and $i$ 3s, where $i = 1, 2, \ldots, m = n/3$. It follows that $X$ is an antichain and that

$$|X| = \sum_{i=1}^{m} \binom{n}{i, i, n-2i} = \sum_{i=1}^{m} \frac{n!}{(i!)^2(n-2i)!}.$$ 

By the usual estimates of factorials, if $m - \sqrt{n} < i \leq m$ then

$$\binom{n}{i, i, n-2i} \gg \binom{n}{m, m, m} \gg \frac{3^n}{n}.$$ 

Hence $X$ is an antichain in $P(3, n)$ with size

$$|X| \gg \sqrt{n} \cdot \frac{3^n}{n} = \frac{3^n}{\sqrt{n}}.$$ 

3. For composite $k$ we can decrease the factor $k/2$ in the bound of Lemma 2.3. Suppose that $k = lm$ where $l \geq m \geq 2$ are integers and let $X \subseteq P(k, n)$ be an antichain. We associate with every $u \in X$ the pair of $n$-tuples $(v^u, w^u) \in P(m, n) \times P(l, n)$ defined by $v^u_i = u_i - m \lceil u_i/m \rceil + m$ and $w^u_i = \lceil u_i/m \rceil$, $1 \leq i \leq n$. Note that the pair $(v^u, w^u)$ uniquely determines $u$ and that if $w^u = w^{u'}$ then $v^u$ and $v^{u'}$ are incomparable by $\leq_p$. Thus, by
Lemma 2.3, for fixed $w \in P(l, n)$ there are less than $(m/2)m^n/\sqrt{n}$ elements $u \in X$ with $w^u = w$. The number of $w$s is at most $|P(l, n)| = l^n$. Hence

$$|X| < \frac{(m/2) \cdot m^n}{n^{1/2}} \cdot l^n = \frac{(m/2) \cdot k^n}{n^{1/2}}.$$  

In particular, if $k$ is a power of 2 then $|X| < k^n/\sqrt{n}$ for every antichain $X \subset P(k, n)$.

3 The proof of Theorem 1.1

Let $N = a(10^n - 1)/9 = aa \cdots a$, $1 \leq a \leq 9$, be a rep-digit. Since $K^N = aL^n$, $N$ is a Keith number if and only if the rep-unit $M = (10^n - 1)/9 = 11 \cdots 1$ is a Keith number. Suppose that $M$ is a Keith number: for some $m$ we have

$$M = \frac{10^n - 1}{9} = L_m^n = 2^{m-n-1}(n-1) \left(1 + O \left(\frac{m}{2^n}\right)\right),$$

where the asymptotic relation was proved in part (d) of Lemma 2.1. We rewrite this relation as

$$\frac{2^{2n+1-m}5^n}{9(n-1)} - 1 = \frac{1}{9(n-1)2^{m-n-1}} + O \left(\frac{m}{2^n}\right).$$

Since $2n < m < 7n$ by part (c) of Lemma 2.1, we get

$$\frac{2^{2n+1-m}5^n}{9(n-1)} - 1 = O \left(\frac{n}{2^n}\right).$$

Because $5^n > 9(n-1)$ for every $n \geq 1$, the left side is always non-zero (the power of 5 cannot be canceled). Writing it in the form $e^\Lambda - 1$ and using that $e^\Lambda - 1 = O(\Lambda)$ (as $\Lambda \to 0$), we get

$$0 \neq \Lambda = (2n + 1 - m) \log 2 + n \log 5 - \log(9(n-1)) \ll \frac{n}{2^n}.$$  

Taking logarithms and applying Lemma 2.2, we finally obtain

$$-d(\log n)^2 < \log |\Lambda| < c(\log n - n \log 2)$$

where $c, d > 0$ are effectively computable constants. This implies that $n$ is effectively bounded and completes the proof of Theorem 1.1. \hfill \Box

Remark. The same argument shows that for every integer $b \geq 3$ there are only effectively finitely many base $b$ rep-digits, i.e., positive integers of the form $a(b^n - 1)/(b - 1)$ with $a \in \{1, \ldots, b - 1\}$, which are base $b$ Keith numbers. Indeed, we argue as for $b = 10$ and derive the equation

$$\frac{b^n}{(b-1)(n-1)2^{m-n-1}} - 1 = O(n/2^n).$$

In order to apply Lemma 2.2, we need to justify that the left side is not zero. If $b$ is not a power of 2, it has an odd prime divisor $p$, and $p^n$ cannot be cancelled, for big enough $n$, by $(b-1)(n-1)$. If $b \geq 3$ is a power of 2, then $b-1$ is odd and has an odd prime divisor, which cannot be cancelled by the rest of the expression.

7
4 The proof of Theorem 1.2

For an integer $N > 0$, we denote by $n$ the number of its digits: $10^{n-1} \leq N < 10^n$. We shall prove that there are $\ll 10^n/\sqrt{n}$ Keith numbers with $n$ digits; it is easy to see that this implies Theorem 2. There are only few numbers with $n$ digits and $\geq n/2$ zero digits: their number is bounded by

$$\sum_{i \geq n/2} \binom{n}{i} 9^{n-i} \leq 2^n 9^{n/2} = 6^n < (10^n)^{0.8}.$$ 

Hence it suffices to count only the Keith numbers with $n$ digits, of which at least half are nonzero.

Let $N$ be a Keith number with $n \geq 3$ digits, at least half of them nonzero. So, $N = K_m^N$ for some index $m \geq 1$. By part (c) of Lemma 2.1, $2n < m < 7n$ and we may use the asymptotic estimate in part (d). Setting $k = \lfloor n/2 \rfloor$ and using the inequality in part (a) of Lemma 2.1, we get

$$10^n > N = K_m^N \geq L_{k+m-n}^k.$$ 

Part (d) of Lemma 2.1 gives that for big $n$,

$$L_{k+m-n}^k > \frac{2^{m-n-1}(k-1)}{2} > \frac{2^{m-n}n}{12}.$$ 

On the other hand, the second inequality in part (b) of Lemma 2.1 and part (d) give, for big $n$,

$$10^{n-1} \leq N = K_m^N \leq 9L_m^n < 9 \cdot 2^{m-n}n.$$ 

Combining the previous inequalities, we get

$$\frac{10^n}{90} < 2^{m-n}n < 12 \cdot 10^n.$$ 

This implies that, for $n > n_0$, the index $m$ attains at most 12 distinct values and

$$m = (1 + \log 10/\log 2 + o(1))n = (\kappa + o(1))n.$$ 

Now we partition the set $S$ of considered Keith numbers (with $n$ digits, at least half of them nonzero) in blocks of numbers $N$ having the same value of the index $m$ and the same string of the first (most significant) $k = \lfloor n/2 \rfloor$ digits. So, we have at most $12 \cdot 10^k$ blocks. We show in a moment that the numbers in one block $B$, when regarded as $(n-k)$-tuples from $\mathcal{P}(10, n-k)$, form an antichain to $\leq_p$. Assuming this, Lemma 2.3 implies that $|B| < 10^{n-k+1}/(2\sqrt{n-k})$. Summing over all blocks, we get

$$|S| < 12 \cdot 10^k \cdot \frac{10^{n-k+1}}{2\sqrt{n-k}} \ll \frac{10^n}{\sqrt{n}},$$

which proves Theorem 2.
To show that $B$ is an antichain, we suppose for the contradiction that $N_1$ and $N_2$ are two Keith numbers from $B$ with $N_1 <_p N_2$. Let $M = N_2 - N_1$ and $M^* = 00 \cdots 0M \in P(10, n)$ (we complete $M$ to a string of length $n$ by adding initial zeros). It follows that $M$ has at most $n - k$ digits and $M < 10^{n-k}$. On the other hand, by the linearity of recurrence and by $N_1 <_p N_2$, we have

$$M = N_2 - N_1 = K^{N_2}_m - K^{N_1}_m = K^{M^*}_m.$$  

Since $M^*$ has some nonzero entry, the first inequality in part (b) of Lemma 2.1 and part (d) give, for big $n$,

$$K^{M^*}_m \geq L^{n}_{m-n} > 2^{m-2n-2}.$$  

Thus

$$10^{n-k} = 10^{n-\lfloor n/2 \rfloor} > M > 2^{m-2n-2}.$$  

Using the above asymptotic estimate of $m$ in terms of $n$, we arrive at the inequality

$$\exp((\frac{1}{2} \log 10 + o(1))n) > \exp((\kappa \log 2 - 2 \log 2 + o(1))n) = \exp((\log 5 + o(1))n)$$

that is contradictory for big $n$ because $10^{1/2} < 5 = 10/2$. This finishes the proof of Theorem 2.

**Remark.** The above proof generalizes, with small modifications, to all bases $b \geq 4$. We replace base 10 by $b$, modify the proof accordingly, and have to satisfy two conditions. First, in the beginning of the proof we delete from the numbers with $n$ base $b$ digits those with $\geq \alpha n$ zero digits, for some constant $0 < \alpha < 1$. In order that we delete negligibly many, compared to $b^n$, numbers, we must have $2 \cdot (b-1)^{1-\alpha} < b$. Second, for the final contradiction we need that $b^n < b/2$. For $b \geq 5$, both conditions are satisfied with $\alpha = 1/2$, as in case $b = 10$. For $b = 4$ they are satisfied with $\alpha = 0.49$, say. However, for $b = 3$ they cannot be satisfied by any $\alpha$. Thus, the case $b = 3$ seems to require more substantial changes.

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