Journal of Integer Sequences, Vol. 10 (2007),
Article 07.4.1

# Special Multi-Poly-Bernoulli Numbers

Y. Hamahata and H. Masubuchi
Department of Mathematics
Tokyo University of Science
Noda, Chiba, 278-8510
Japan
hamahata_yoshinori@ma.noda.tus.ac.jp

**Abstract**

In this paper we investigate generalized poly-Bernoulli numbers. We call them multi-poly-Bernoulli numbers, and we establish a closed formula and a duality property for them.

## 1 Introduction and background

Kaneko [5] introduced poly-Bernoulli numbers $B_n^{(k)}$ ($k \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$) which are generalizations of Bernoulli numbers. One knows that special values of certain zeta functions at non-positive integers can be described in terms of poly-Bernoulli numbers. Kaneko [6] suggests to study multi-poly-Bernoulli numbers, which are generalizations of poly-Bernoulli numbers, as an open problem. Kim and Kim [7] consider them and give a relationship with special values of certain zeta functions. We consider special multi-poly-Bernoulli numbers. It seems for the authors that they are more natural than multi-poly-Bernoulli numbers when one tries to generalize the results of poly-Bernoulli numbers. The purpose of the present paper is to establish some results for them. To be more precise, we prove the closed formula and the duality for them.

We briefly recall poly-Bernoulli numbers. For an integer $k \in \mathbb{Z}$, put

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

The formal power series $Li_k(z)$ is the $k$-th polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. When $k = 1$, $Li_1(z) = -\log(1 - z)$. Using $Li_k(z)$, one can introduce poly-Bernoulli numbers. The *poly-Bernoulli numbers* $B_n^{(k)}$ $(n = 0, 1, 2, \ldots)$ are defined by the generating series

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

We find that for any $n \geq 0$, $B_n^{(1)} = B_n$, the classical Bernoulli number.

For nonnegative integers $n$, $m$, put

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{(-1)^m}{m!} \sum_{l=0}^{m} (-1)^l \binom{m}{l} l^n.$$

We call it the *Stirling number of the second kind.* Kaneko obtained in [5] an explicit formula for $B_n^{(k)}$:

**Theorem 1** ([5])**.** *For a nonnegative integer $n$ and an integer $k$, we have*

$$B_n^{(k)} = (-1)^n \sum_{m=1}^{n+1} \frac{(-1)^{m-1}(m-1)! \left\{ \begin{matrix} n \\ m-1 \end{matrix} \right\}}{m^k}.$$

Using it, the following formula can be shown:

**Theorem 2** (Closed formula [1])**.** *For any $n, k \geq 0$, we have*

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\}.$$

By this theorem, we get

**Theorem 3** (Duality [1], [5])**.** *For $n, k \geq 0$, $B_n^{(-k)} = B_k^{(-n)}$ holds.*

The last theorem can be proved in another way. Namely, using Theorem 1, we have

**Theorem 4** (Symmetric formula [5])**.**

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

As corollary to this theorem, we have the duality theorem.

We would like to extend these results to our generalized poly-Bernoulli numbers.

## 2 Multi-poly-Bernoulli numbers

In this section, we investigate generalized poly-Bernoulli numbers. First, we define a generalization of $Li_k(z)$.

**Definition 5.** For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$, define

$$Li_{k_1,k_2,\ldots,k_r}(z) = \sum_{\substack{m_1,\ldots,m_r \in \mathbb{Z} \\ 0<m_1<m_2<\cdots<m_r}} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

Now let us introduce a generalization of poly-Bernoulli numbers with the use of $L_{k_1,k_2,\ldots,k_r}(z)$.

**Definition 6.** *Multi-poly-Bernoulli numbers* $B_n^{(k_1,k_2,\ldots,k_r)}$ *($n = 0, 1, 2, \ldots$)* are defined for each integer $k_1, k_2, \ldots, k_r$ by the generating series

$$\frac{Li_{k_1,k_2,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!}.$$

We generalize Theorem 1:

**Theorem 7.** *For a nonnegative integer $n$ and integers $k_1, \ldots, k_r$, we have*

$$B_n^{(k_1,k_2,\ldots,k_r)} = (-1)^n \sum_{m_r=r}^{n+r} \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-r}(m_r-r)! \left\{ {n \atop m_r-r} \right\}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

*Proof.* By definition,

$$\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!} = \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-r}(e^{-t}-1)^{m_r-r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

Here we apply the formula

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{t^n}{n!} \quad (n \geq m \geq 0)$$

(see (4) in [5]) to the right hand side of the above equation.

$$\begin{aligned}
\text{R.H.S.} &= \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-r}(m_r-r)!}{m_1^{k_1} \cdots m_r^{k_r}} \sum_{n=m_r-r}^{\infty} \left\{ {n \atop m_r-r} \right\} \frac{(-t)^n}{n!} \\
&= \sum_{0<m_1<\cdots<m_r} \sum_{n=m_r-r}^{\infty} \frac{(-1)^{m_r-r}(m_r-r)! \left\{ {n \atop m_r-r} \right\}}{m_1^{k_1} \cdots m_r^{k_r}} \cdot \frac{(-1)^n t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m_r=r}^{n+r} \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-r}(m_r-r)! \left\{ {n \atop m_r-r} \right\}}{m_1^{k_1} \cdots m_r^{k_r}} \cdot \frac{(-1)^n t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( (-1)^n \sum_{m_r=r}^{n+r} \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-r}(m_r-r)! \left\{ {n \atop m_r-r} \right\}}{m_1^{k_1} \cdots m_r^{k_r}} \right) \frac{t^n}{n!}.
\end{aligned}$$

This shows the claim. □

By this theorem, for the smaller $n$, we can compute $B_n^{(k_1,k_2,\dots,k_r)}$ more specifically. For example,

$$
\begin{aligned}
B_0^{(k_1,k_2,\dots,k_r)} &= \frac{1}{1^{k_1}2^{k_2}\cdots r^{k_r}}, \\
B_1^{(k_1,k_2,\dots,k_r)} &= \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(r+1)^{k_r}}.
\end{aligned}
$$

# 3 Closed formula and duality

Let $n$ be a nonnegative integer, $r$ a positive integer, and $k \in \mathbb{Z}$. We define

$$
B[r]_n^{(k)} = B_n^{(\overbrace{0,\dots,0}^{r-1},k)}.
$$

We say that $B[r]_n^{(k)}$ is a *special multi-Bernoulli number* of order $r$. By definition, it is clear that $B[1]_n^{(k)} = B_n^{(k)}$. Hence the notion of special multi-poly-Bernoulli number is a generalization of that of poly-Bernoulli number. For $B[r]_n^{(k)}$, we establish closed formula:

**Theorem 8** (Closed formula)**.** *For $n, k \geq 0$, we have*

$$
\begin{aligned}
&B[r]_n^{(-k)} \\
&= \sum_{\substack{n=n_1+\cdots+n_r \\ n_1,\dots,n_r\geq 0}} \sum_{\substack{k=k_1+\cdots+k_r \\ k_1,\dots,k_r\geq 0}} \frac{n!k!}{n_1!\cdots n_r!k_1!\cdots k_r!} \\
&\times \left( \sum_{j_1=0}^{\min(n_1,k_1)} \cdots \sum_{j_r=0}^{\min(n_r,k_r)} (j_1!\cdots j_r!)^2 \left\{ \begin{matrix} n_1+1 \\ j_1+1 \end{matrix} \right\} \cdots \left\{ \begin{matrix} n_r+1 \\ j_r+1 \end{matrix} \right\} \left\{ \begin{matrix} k_1+1 \\ j_1+1 \end{matrix} \right\} \cdots \left\{ \begin{matrix} k_r+1 \\ j_r+1 \end{matrix} \right\} \right).
\end{aligned}
$$

To prove the theorem, we need the following lemma:

**Lemma 9.**

$$
\sum_{n=m+1}^{\infty} (e^y - e^{y-x})^n = \frac{e^{x+y}(1-e^{-x})}{e^x + e^y - e^{x+y}} (e^y - e^{y-x})^m.
$$

*Proof.*

$$
\begin{aligned}
\text{L.H.S.} &= \frac{1}{1-(e^y - e^{y-x})}(e^y - e^{y-x})^{m+1} \\
&= \frac{e^y - e^{y-x}}{1-(e^y - e^{y-x})}(e^y - e^{y-x})^m \\
&= \text{R.H.S.}.
\end{aligned}
$$

□

*Proof of Theorem 8.*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B[r]_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left( (-1)^n \sum_{m_r=r}^{n+r} \sum_{0<m_1<\cdots<m_r} (-1)^{m_r-r}(m_r-r)! \left\{ {n \atop m_r-r} \right\} m_r^k \right) \frac{x^n}{n!}\frac{y^k}{k!}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Theorem 7)}\\
&= \sum_{n=0}^{\infty}\sum_{m_r=r}^{n+r}\sum_{m_{r-1}=r-1}^{m_r-1}\cdots\sum_{m_1=1}^{m_2-1} \left\{ {n \atop m_r-r} \right\} \frac{(-x)^n}{n!}(m_r-r)!(-1)^{m_r-r}\sum_{k=0}^{\infty}\frac{(m_r y)^k}{k!}\\
&= \sum_{n=0}^{\infty}\sum_{m_1=1}^{n+1}\sum_{m_2=m_1+1}^{n+2}\cdots\sum_{m_r=m_{r-1}+1}^{n+r} \left\{ {n \atop m_r-r} \right\} \frac{(-x)^n}{n!}(m_r-r)!(-1)^{m_r-r}\sum_{k=0}^{\infty}\frac{(m_r y)^k}{k!}\\
&= \sum_{m_1=1}^{\infty}\sum_{m_2=m_1+1}^{\infty}\cdots\sum_{m_r=m_{r-1}+1}^{\infty}\sum_{n=m_r-r}^{\infty} \left\{ {n \atop m_r-r} \right\} \frac{(-x)^n}{n!}(m_r-r)!(-1)^{m_r-r}\sum_{k=0}^{\infty}\frac{(m_r y)^k}{k!}\\
&= \sum_{m_1=1}^{\infty}\sum_{m_2=m_1+1}^{\infty}\cdots\sum_{m_r=m_{r-1}+1}^{\infty} (1-e^{-x})^{m_r-r} e^{m_r y}\\
&= \frac{1}{(1-e^{-x})^r}\sum_{m_1=1}^{\infty}\sum_{m_2=m_1+1}^{\infty}\cdots\sum_{m_r=m_{r-1}+1}^{\infty} (e^y-e^{y-x})^{m_r}.
\end{aligned}$$

We use Lemma 9 repeatedly to the right hand side of the last equation. Then

$$\begin{aligned}
\text{R.H.S} &= \frac{1}{(1-e^{-x})^r}\left(\frac{e^{x+y}(1-e^{-x})}{e^x+e^y-e^{x+y}}\right)^{r-1}\sum_{m_1=1}^{\infty}(e^y-e^{y-x})^{m_1}\\
&= \frac{1}{(1-e^{-x})^r}\left(\frac{e^{x+y}(1-e^{-x})}{e^x+e^y-e^{x+y}}\right)^{r}\\
&= \left(\frac{e^{x+y}}{e^x+e^y-e^{x+y}}\right)^{r}.
\end{aligned}$$

By Theorem 2, the right hand side of the last equation is equal to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}\right)^r\\
&= \left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ {n+1 \atop j+1} \right\}\left\{ {k+1 \atop j+1} \right\}\frac{x^n}{n!}\frac{y^k}{k!}\right)^r.
\end{aligned}$$

This implies the theorem. □

A generalization of Theorem 3 follows from the last theorem:

**Corollary 10** (Duality)**.** *For $n, k \geq 0$, we have*

$$B[r]_n^{(-k)} = B[r]_k^{(-n)}.$$

We note that in the process of proof of the last theorem, we have obtained two formulae:

**Proposition 11** (Symmetric formula)**.** *For $n, k \geq 0$,*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B[r]_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!} = \left(\frac{e^{x+y}}{e^x + e^y - e^{x+y}}\right)^r.$$

**Proposition 12.** *For $n, k \geq 0$,*

$$B[r]_n^{(-k)} = \sum_{\substack{n=n_1+\cdots+n_r \\ n_1,\ldots,n_r \geq 0}} \sum_{\substack{k=k_1+\cdots+k_r \\ k_1,\ldots,k_r \geq 0}} \frac{n!k!}{n_1!\cdots n_r! k_1! \cdots k_r!} B_{n_1}^{(-k_1)} \cdots B_{n_r}^{(-k_r)}.$$

Proposition 11 is a generalization of Theorem 4.

## References


[1] T. Arakawa and M. Kaneko, On poly-Bernoulli numbers. *Comment. Math. Univ. St. Pauli* **48** (1999), 159–167.

[2] L. Carlitz, Some theorems on Bernoulli numbers of higher order. *Pacific J. Math.* **2** (1952), 127–139.

[3] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics.* Addison-Wesley, 1989.

[4] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory.* Second Edition, Graduate Texts in Mathematics **84**, Springer-Verlag, 1990.

[5] M. Kaneko, Poly-Bernoulli numbers. *J. Théorie de Nombres* **9** (1997), 221–228.

[6] M. Kaneko, *Multiple Zeta Values and Poly-Bernoulli Numbers.* Tokyo Metropolitan University Seminar Report, 1997.

[7] M.-S. Kim and T. Kim, An explicit formula on the generalized Bernoulli number with order $n$. *Indian J. Pure Appl. Math.* **31** (2000), 1455–1461.

[8] R. Sánchez-Peregrino, Closed formula for poly-Bernoulli numbers. *Fibonacci Quart.* **40** (2002), 362–364.


---


2000 *Mathematics Subject Classification*: Primary 11B68; Secondary 11B73.

*Keywords*: poly-Bernoulli numbers, Stirling numbers.


---


Received February 24 2007; revised version received April 13 2007. Published in *Journal of Integer Sequences*, April 13 2007.


---