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Polynomials Generated by the Fibonacci Sequence

David Garth[1]
Division of Mathematics & Computer Science
Truman State University
Kirksville, MO 63501
USA
dgarth@truman.edu

Donald Mills
Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803-3999
USA
mills@rose-hulman.edu

Patrick Mitchell
Department of Mathematics
Midwestern State University
Wichita Falls, TX 76308
USA
patrick.mitchell@mwsu.edu

Abstract

The Fibonacci sequence's initial terms are $F_0 = 0$ and $F_1 = 1$, with $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. We define the polynomial sequence $\mathbf{p}$ by setting $p_0(x) = 1$ and

[1]Corresponding author.

$p_n(x) = xp_{n-1}(x) + F_{n+1}$ for $n \geq 1$, with $p_n(x) = \sum_{k=0}^{n} F_{k+1}x^{n-k}$. We call $p_n(x)$ the *Fibonacci-coefficient polynomial (FCP) of order n.* The FCP sequence is distinct from the well-known Fibonacci polynomial sequence.

We answer several questions regarding these polynomials. Specifically, we show that each even-degree FCP has no real zeros, while each odd-degree FCP has a unique, and (for degree at least 3) irrational, real zero. Further, we show that this sequence of unique real zeros converges monotonically to the negative of the golden ratio. Using Rouché's theorem, we prove that the zeros of the FCP's approach the golden ratio in modulus. We also prove a general result that gives the Mahler measures of an infinite subsequence of the FCP sequence whose coefficients are reduced modulo an integer $m \geq 2$. We then apply this to the case that $m = L_n$, the n^{th} Lucas number, showing that the Mahler measure of the subsequence is ϕ^{n-1}, where $\phi = \frac{1+\sqrt{5}}{2}$.

1 Introduction

The Fibonacci sequence's initial terms are $F_0 = 0$ and $F_1 = 1$, with $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. We define the polynomial sequence $\{p_n(x)\}$ by setting $p_0(x) = 1$ and $p_n(x) = xp_{n-1}(x) + F_{n+1}$ for $n \geq 1$. Equivalently, we have that

$$p_n(x) = \sum_{k=0}^{n} F_{k+1}x^{n-k}.$$

Thus, $p_1(x) = x + 1$, $p_2(x) = x^2 + x + 2$, $p_3(x) = x^3 + x^2 + 2x + 3$, and so forth. We call $p_n(x)$ the *Fibonacci-coefficient polynomial (FCP) of order n.*

The FCP sequence is distinct from the well-known Fibonacci polynomial sequence, given by $P_1(x) = 1$ and $P_2(x) = x$ with $P_{n+1}(x) = xP_n(x) + P_{n-1}(x)$ for $n \geq 2$. For, while $P_1(x) = p_0(x)$ and $P_2(x) = p_1(x-1)$, there does not exist $a \in \mathbb{C}$ such that $x^2+1 = P_3(x) = p_2(x-a)$.

Next, for $m \geq 2$, let $\{p_n^{(m)}(x)\}$ be the sequence obtained from the Fibonacci coefficient polynomials by reducing the coefficients modulo m and adjusting the residue class, using $(F_k \bmod m) - m$ if $F_k \bmod m > \frac{m}{2}$. Thus, for odd m the residue class is centered about 0, and for even m the residue class ranges from $-m/2+1$ to $m/2$. As an example, $p_5^{(2)}(x) = x^5 + x^4 + x^2 + x$, $p_5^{(3)}(x) = x^5 + x^4 - x^2 - x - 1$, and $p_5^{(4)}(x) = x^5 + x^4 + 2x^3 - x^2 + x$.

In this paper, we answer several questions regarding these sequences of polynomials. In Section 3 we show that the number of real zeros of $p_n(x)$ is either 0 or 1, according to whether n is even or odd, respectively. Further, we prove that the sequence of unique real zeros for the odd-degree FCP's converges monotonically to the negative of the golden ratio, and that the zeros of the FCP's approach the golden ratio in modulus. In Section 4 we consider the polynomial sequence $\{p_n^{(m)}(x)\}$, showing that there is an infinite subsequence of these polynomials whose Mahler measure equals the Mahler measure of $p_{t-2}^{(m)}(x)$, where t is the period of the Fibonacci sequence reduced modulo m. We then consider the specific case when $m = L_n$, the n^{th} Lucas number. In this case the coefficients of the polynomials in the aforementioned infinite subsequence are Fibonacci numbers, and we show that the Mahler measure of the polynomials is φ^{n-1}, where $\varphi = \frac{1+\sqrt{5}}{2}$. We start, however, by proving some necessary Lemmas.

2 Preliminary Lemmas

In this section we prove some technical lemmas that will be needed later. We will make use of the close relationship between the Fibonacci numbers, the Lucas numbers, and the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. Recall that the Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. We first remind the reader of Binet's formula ([1, Chapter 5]).

Lemma 2.1 (Binet's Formula)**.** *Let* $\varphi = (1+\sqrt{5})/2$ *and* $\tau = (1-\sqrt{5})/2$*. For all natural numbers* n*, we have*

$$F_n = \frac{\varphi^n - \tau^n}{\varphi - \tau} = \frac{1}{\sqrt{5}}\varphi^n + \frac{1}{\sqrt{5}} \cdot \left(-\frac{1}{\varphi}\right)^n = \left\lfloor \frac{\varphi^n}{\sqrt{5}} \right\rceil$$

and

$$L_n = \varphi^n + \tau^n = \varphi^n + \left(-\frac{1}{\varphi}\right)^n.$$

Recall that the sequence of Fibonacci numbers, reduced modulo an integer $m \geq 2$, is purely periodic. Renault [4] gave a nice discussion of the properties of the reduced Fibonacci sequence, including a list of the periods for each m. We will be interested in the period when reducing modulo a Lucas number. The following lemmas show that in this case the period has a particularly nice form.

Lemma 2.2. *If* n *is odd, then the period of the Fibonacci sequence modulo* L_n*, with residue classes adjusted to range between* $-\frac{L_n-1}{2}$ *to* $\frac{L_n-1}{2}$ *for odd* L_n *and* $-\frac{L_n}{2}+1$ *to* $\frac{L_n}{2}$ *for even* L_n*, is*

$$F_0,\ F_1,\ F_2,\ F_3,\ \ldots,\ F_n,\ (-F_{n-1}),\ F_{n-2},\ (-F_{n-3}), F_{n-4},\ \ldots,\ (-F_2),\ F_1. \tag{1}$$

Proof. Recall the well-known identity $L_n = F_{n-1} + F_{n+1}$ ([1], page 80). Notice that for $0 \leq i \leq n$

$$\begin{aligned} L_n &= F_{n+1} + F_{n-1} \\ &= F_n + F_{n-1} + F_{n-1} \\ &\geq F_n + F_{n-1} + F_{n-2} \\ &= 2F_n \\ &\geq 2F_i \quad \text{for} \quad 1 \leq i \leq n \end{aligned}$$

Thus, the first $n+1$ elements of the adjusted period mod L_n are as in the list (1), since $F_i = 0 \cdot L_n + F_i$ and $0 \leq F_i \leq \frac{L_n}{2}$ for $0 \leq i \leq n$. The remaining entries in the list follow from the identity

$$F_{n+k} = F_k L_n + (-1)^k F_{n-k} \quad \text{for} \quad 1 \leq k \leq n. \tag{2}$$

To verify this identity, notice that by Binet's Theorem we have

$$\begin{aligned} F_{n+k} - F_k l_n &= \frac{\varphi^{n+k} - \tau^{n+k}}{\varphi - \tau} - \frac{(\varphi^k - \tau^k)(\varphi^n + \tau^n)}{\varphi - \tau} \\ &= \frac{\varphi^{n+k} - \tau^{n+k} - \varphi^{n+k} - \varphi^k \tau^n + \varphi^n \tau^k + \tau^{n+k}}{\varphi - \tau} \\ &= \frac{\varphi^k \tau^k (\varphi^{n-k} - \tau^{n-k})}{\varphi - \tau} \\ &= (-1)^k F_{n-k} \end{aligned}$$

The last equality follows from Binet's formula and the fact that $\varphi\tau = -1$. This establishes the remaining entries in the list (1). To see that there are no further entries in the period, it is easy to see from Binet's formulas that $F_{2n} = F_n L_n$, and so the next entry in the Fibonacci sequence reduced modulo m is 0. Also, since n is odd, it can be shown in using Binet's formulas that $F_{2n+1} = F_{n+1} L_n + 1$. Thus, the next entry in the list is 1. Since the next pair of entries in the sequence after those in the list (1) are 0 and 1, and the list (1) repeats. □

Lemma 2.3. *If n is even, then the period of the Fibonacci sequence modulo L_n, with residue classes adjusted as in Lemma 2.2 is*

$$F_0\ F_1\ F_2\ F_3\ \cdots\ F_n\ (-F_{n-1})\ F_{n-2}\ (-F_{n-3})\ \cdots\ F_2\ (-F_1)\ F_0$$

$$(-F_1)\ (-F_2)\ (-F_3)\ \cdots\ (-F_n)\ F_{n-1}\ (-F_{n-2}) F_{n-3}\ \cdots\ (-F_2)\ (F_1).$$

Proof. The first $2n$ terms of the period are established in the same manner as in the proof of Lemma 2.3. The rest of the period follows from the formulas

$$F_{2n+k} = F_{n+k} L_n - F_k \quad \text{for} \quad 1 \leq k \leq n$$

and

$$F_{3n+k} = (F_{2n+k} - F_k) L_n + (-1)^{k+1} F_{n-k},$$

both of which can be proved using Binet's formulas in a manner similar to equation (2). After this it is easy to show that the pair $0,\ 1$ are the next terms of the sequence, and this proves the Lemma. □

We close this section with the following technical lemma which will prove crucial in the proof of Theorem 4.2.

Lemma 2.4. *For $n \geq 13$, the sequence $\varphi - \frac{1}{n^2} - \left(\frac{2}{\sqrt{5}}\right)^{1/n} \varphi$ is monotone decreasing and approaches 0 as $n \to \infty$.*

Proof. It is clear that the sequence approaches 0 as $n \to \infty$. We now show it is monotone decreasing. Define $d : (0, \infty) \to \mathbb{R}$ by

$$d(x) = \varphi - \frac{1}{x^2} - \left(\frac{2}{\sqrt{5}}\right)^{1/x} \varphi.$$

Notice that

$$\begin{aligned} d'(x) &= \frac{2}{x^3} - \left(\frac{2}{\sqrt{5}}\right)^{1/x} \log\left(\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{x^2}\right)\varphi \\ &= \frac{1}{x^2}\left(\frac{2}{x} + \left(\frac{2}{\sqrt{5}}\right)^{1/x} \log\left(\frac{2}{\sqrt{5}}\right)\varphi\right) \end{aligned}$$

Now, when $x = 13$, $d'(x) < 0$. Also, notice that

$$\frac{2}{x} + \left(\frac{2}{\sqrt{5}}\right)^{1/x} \log\left(\frac{2}{\sqrt{5}}\right)\varphi \to \log\left(\frac{2}{\sqrt{5}}\right)\varphi < 0$$

as $x \to \infty$. Now, since $\frac{2}{x}$ decreases to 0, and $\left(\frac{2}{\sqrt{5}}\right)^{1/x}$ increases to 1, it follows that $d'(x) < d'(13) < 0$ for all $x \geq 13$. Thus, $d(x)$ is decreasing for $x \geq 13$. □

3 Zeros of FCPs

In this section we investigate the zeros of the FCP sequence. For the remainder of this paper let $g_n(x) = p_n(x)(x^2 - x - 1)$ for $n \in \mathbb{Z}^+$. We will make use of the following identities, which are straightforward to verify. For each identity, $n \in \mathbb{Z}^+$.

$$g_n(x) := p_n(x)(x^2 - x - 1) = x^{n+2} - F_{n+2}x - F_{n+1} \tag{3}$$

$$x^{n+1} - F_{n+1} = (x-1)p_n(x) - xp_{n-2}(x) \tag{4}$$

$$x^n + F_n = p_n(x) - (x+1)p_{n-2}(x) \tag{5}$$

We start by characterizing the real zeros of the polynomials in the FCP sequence.

Theorem 3.1. *For even n, $p_n(x)$ has no real zeros. For odd n, $p_n(x)$ has exactly one real zero, which lies in the interval $(-\varphi, -1]$. Moreover, the sequence of real zeros of the $p_n(x)$, with n odd, decreases monotonically to zero.*

Proof. First assume n is even. By Descartes' rule of signs $g_n(x)$ has exactly one positive real zero and one negative real zero. Since $\varphi > 0$ and $\tau < 0$ are zeros of $x^2 - x - 1$, a factor of $g_n(x)$, it follows that $p_n(x)$ has no real zeros.

Now assume n is odd. By similar reasoning it follows from Descartes' rule of signs applied to $g_n(x)$ that $p_n(x)$ has exactly one real zero, which is negative. Notice that $g_1(-1) = 0$, and $g_n(-1) = (-1)^{n+2} + F_{n+2} - F_{n+1} = F_n - 1 > 0$ for $n > 1$. Also, since n is odd, for $n > 1$ we have

$$g_n(-\varphi) = -\varphi(\varphi^{n+1} - F_{n+2}) - F_{n+1}. \tag{6}$$

By Binet's formulas, for odd $n > 1$ we have that

$$\begin{aligned}\varphi^{n+1} - F_{n+2} &= \varphi^{n+1} - \frac{\varphi^{n+2}}{\sqrt{5}} - \frac{1}{\sqrt{5}\varphi^{n+2}} \\ &= \varphi^{n+2}\left(\frac{1}{\varphi} - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}\varphi^{2n+4}}\right) > \varphi^{n+2}\left(\frac{1}{\varphi} - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}\varphi^{10}}\right) > 0.\end{aligned}$$

It follows that the right hand side of (6) is negative, and thus $g_n(x)$ has a zero in the interval $(-\varphi, -1]$ for odd $n \geq 1$. Since $x^2 - x - 1$ has no zero in this interval, it follows that this zero must be the negative zero of $p_n(x)$.

Now let α_n be the real zero of $p_n(x)$, for odd n. Taking the derivative of $g_n(x)$ gives $g_n'(x) = (n+2)x^{n+1} - F_{n+2}$. As n is odd, the critical points of $g_n(x)$ are

$$x_{n,\pm} = \pm\left(\frac{F_{n+2}}{n+2}\right)^{1/(n+1)}.$$

Since α_n, φ and τ are the zeros of g_n, with $\alpha_n \in (-\varphi, -1)$ for $n > 1$, it follows that $g_n(x)$ achieves a local maximum at $x_{n,-}$, with $-\varphi < \alpha_n \leq x_{n,-}$. As $F_{n+2}^{1/(n+1)} \to \varphi$ as $n \to \infty$, it follows that $\alpha_n \to -\varphi$ as $n \to \infty$ through odd values of n.

All that remains is to show that the convergence is monotonic. That is, we need to show that $\alpha_n \leq \alpha_{n-2}$ for all odd n. We assume $n \geq 3$, as an easy check shows that $\alpha_3 < \alpha_1 = -1$. We saw earlier that $p_n(-1) > 0$ and $p_n(-\varphi) < 0$ for all odd $n > 1$. The result will follow if we can show that $p_n(\alpha_{n-2}) > 0$ for all odd n, since each $p_n(x)$ has a unique real zero.

For convenience of notation, let $a = \alpha_{n-2}$ in the remainder of the proof. By identity (5), we have that $p_n(a) = a^n + F_n$. So we need to show that $a^n > -F_n$, i.e. that $a > -\sqrt[n]{F_n}$. This will follow if we can show that $p_{n-2}(-\sqrt[n]{F_n}) < 0$, since $p_{n-2}(-1) > 0$. It will follow then that $a \in (-\sqrt[n]{F_n}, -1)$.

Using identity (3), we have that

$$p_{n-2}(-\sqrt[n]{F_n}) = \frac{(-\sqrt[n]{F_n})^n + F_n(\sqrt[n]{F_n}) - F_{n-1}}{(-\sqrt[n]{F_n})^2 + \sqrt[n]{F_n} - 1}$$

Notice that the denominator of this last expression is positive, since $-\sqrt[n]{F_n} < -1$, and $x^2 - x - 1 > 0$ whenever $x \leq -1$. So $p_{n-2}(-\sqrt[n]{F_n})$ has the same sign as the numerator of the previous expression.

Now notice that

$$\begin{aligned}(-\sqrt[n]{F_n})^n + F_n(\sqrt[n]{F_n}) - F_{n-1} &= -F_n + F_n(\sqrt[n]{F_n}) - F_{n-1} \\ &= -F_{n+1} + F_n(\sqrt[n]{F_n})\end{aligned}$$

We want to show that $-F_{n+1} + F_n(\sqrt[n]{F_n}) < 0$, or equivalently, that

$$\sqrt[n]{F_n} < \frac{F_{n+1}}{F_n}.$$

One verifies this numerically for $3 \leq n \leq 11$. For $n \geq 13$, we approach the matter as follows.

It is well-known from the theory of continued fractions that for odd n, $\frac{F_{n+1}}{F_n} < \varphi$ (see Theorem 7.10 of [3]). We also have that

$$\left| \varphi - \frac{F_{n+1}}{F_n} \right| < \frac{1}{F_n F_{n+1}}, \tag{7}$$

(see Theorem 7.11 of [3]). It follows from (7) that

$$\frac{F_{n+1}}{F_n} > \varphi - \frac{1}{F_n F_{n+1}} > \varphi - \frac{1}{n^2}.$$

On the other hand, by Binet's formula, since n is odd, we have that

$$F_n = \frac{1}{\sqrt{5}}\varphi^n + \frac{1}{\sqrt{5}} \cdot \frac{1}{\varphi^n}.$$

Thus,

$$\begin{aligned} \sqrt[n]{F_n} &= \sqrt[n]{\frac{1}{\sqrt{5}}\varphi^n + \frac{1}{\sqrt{5}} \cdot \frac{1}{\varphi^n}} \\ &\leq \sqrt[n]{\frac{1}{\sqrt{5}}\varphi^n + \frac{1}{\sqrt{5}}\varphi^n} \\ &= \sqrt[n]{\frac{2}{\sqrt{5}}} \cdot \varphi \end{aligned}$$

By Lemma 2.4, we then have that for $n \geq 13$,

$$\sqrt[n]{F_n} \leq \sqrt[n]{\frac{2}{\sqrt{5}}} \cdot \varphi < \varphi - \frac{1}{n^2} < \frac{F_{n+1}}{F_n}.$$

This completes the proof of the theorem. □

We remark that it follows immediately from Theorem 3.1 and Gauss' Polynomial Theorem that $p_n(x)$ has no rational zeros for $n \geq 2$. Based on numerical tests we are led to conjecture that $p_n(x)$ is irreducible for all n.

Theorem 3.2. *As n increases without bound, the roots of $p_n(x)$ approach φ in modulus.*

Before proving this, we recall some facts. First, we remind the reader of Rouché's Theorem. This version of the theorem is found page 2 of [2].

Theorem 3.3 (Rouché's Theorem)**.** *If $P(z)$ and $Q(z)$ are analytic interior to a simple closed Jordan curve C, and if they are continuous on C and*

$$|P(z)| < |Q(z)|, \quad z \in C, \tag{8}$$

then the function $H(z) = P(z) + Q(z)$ has the same number of zeros interior to C as does $Q(z)$.

We also need the following Lemma, which follows easily from what we have shown of the function $x^{n+2} - F_{n+2}x - F_{n+1}$.

Lemma 3.4. *For all $n \in \mathbb{Z}^+$, we have $\varphi^n = \varphi F_n + F_{n-1}$. Moreover, if $c > \varphi$ then $c^{n+2} > F_{n+2}c + F_{n+1}$, and if $0 < c < \varphi$ then $c^{n+2} < F_{n+2}c + F_{n+1}$.*

Proof of Theorem 3.2. Consider the circle $|z| = c$ in the complex plane, where $c > \varphi$. Then for $|z| = c$ we have that

$$\begin{aligned} |F_{n+2}z + F_{n+1}| &\leq F_{n+2}|z| + F_{n+1} \\ &= F_{n+2}c + F_{n+1} \\ &< c^{n+2} \\ &= |z^{n+2}|. \end{aligned}$$

Notice that the inequality in the third line above follows from Lemma 3.4. It then follows from Rouche's theorem, with $P_n(z) = -F_{n+2}z - F_{n+1}$ and $Q_n(z) = z^{n+2}$, that $p_n(z)(z^2 - z - 1) = z^{n+2} - F_{n+2}z - F_{n+1}$ has $n+2$ zeros inside $|z| = c$. Since $c > \varphi$ was artibrary, it follows that $p_n(z)(z^2 - z - 1)$ has $n+2$ zeros inside *or on* $|z| = \varphi$.

Now consider the circle $|z| = c$, where $\frac{1}{\varphi} < c < \varphi$. Then for $|z| = c$ we have that

$$\begin{aligned} |F_{n+2}z + F_{n+1}|^2 &= (F_{n+2}z + F_{n+1})(F_{n+2}\overline{z} + F_{n+1}) \\ &= F_{n+2}^2|z|^2 + F_{n+2}F_{n+1}(z + \overline{z}) + F_{n+1}^2 \\ &= F_{n+2}^2c^2 + 2F_{n+2}F_{n+1}Re(z) + F_{n+1}^2 \\ &\geq F_{n+2}^2c^2 - 2F_{n+2}F_{n+1}c + F_{n+1}^2 \\ &= (F_{n+2}c - F_{n+1})^2. \end{aligned}$$

Therefore

$$|F_{n+2}z + F_{n+1}| \geq |F_{n+2}c - F_{n+1}| \quad \text{for} \quad |z| = c, \quad \frac{1}{\varphi} < c < \varphi. \tag{9}$$

We now apply Binet's formula to get a lower bound for $|F_{n+2}c - F_{n+1}|$. Recall that Binet's formula is

$$F_n = \frac{\varphi^n}{\sqrt{5}} - \frac{1}{\sqrt{5}(-\varphi)^n}.$$

Thus, we have that

$$\begin{aligned} F_{n+2}c - F_{n+1} &= \left(\frac{\varphi^{n+2}}{\sqrt{5}} - \frac{(-1)^{n+2}}{\sqrt{5}\varphi^{n+2}}\right)c - \left(\frac{\varphi^{n+1}}{\sqrt{5}} - \frac{(-1)^{n+1}}{\sqrt{5}\varphi^{n+1}}\right) \\ &= \varphi^{n+2}\left(\frac{c}{\sqrt{5}} - \frac{1}{\varphi\sqrt{5}}\right) + \frac{(-1)^{n+1}}{\sqrt{5}\varphi^{n+1}}\left(\frac{c}{\varphi} + 1\right). \end{aligned}$$

Since $\frac{1}{\varphi} < c < \varphi$, it then follows that

$$F_{n+2}c - F_{n+1} > c^{n+2} \quad \text{for} \quad n \quad \text{large enough.} \tag{10}$$

Therefore, by (9) and (10), for $|z| = c$, with $\frac{1}{\varphi} < c < \varphi$,

$$|F_{n+2}z + F_{n+1}| \geq |F_{n+2}c - F_{n+1}| > c^{n+2} \text{ for large } n.$$

By Rouche's theorem then applied to $P_n(z) = z^{n+2}$ and $Q_n(z) = -F_{n+2}z - F_{n+1}$, it follows that if $\frac{1}{\varphi} < c < \varphi$ then for n large enough $p_n(z)(z^2 - z - 1)$ has one zero in $|z| = c$. Since $-\frac{1}{\varphi}$ is one of these zeros, it follows that $p_n(z)$ has no zeros inside $|z| = c$. Since c was arbitrary, it follows that as $n \to \infty$, the zeros of $p_n(z)$ approach the circle $|z| = \varphi$. □

4 Mahler Measures of Reduced-Coefficient FCPs

Theorem 3.2 shows that the Mahler measure of the FCP sequence is asymptotic to φ^n. In this section we consider is the Mahler measure of the reduced coefficient FCP's $p_k^{(m)}(x)$ mentioned in the introduction. Recall that the *Mahler measure* of an nth-degree polynomial $g(x) \in \mathbb{R}[x]$ with leading coefficient a_n and complex roots α_1, ..., α_n, is given by

$$M(g(x)) = |a_n| \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

Theorem 4.1. *Let $m \geq 2$, and let t be the number of terms in one period of the Fibonacci sequence reduced modulo m. If $k \equiv -2$ or $-1 \bmod t$, then*

$$M\left(p_k^{(m)}(x)\right) = M\left(p_{t-2}^{(m)}(x)\right).$$

Proof. Let $a_1, \ldots, a_t$ be the period of the Fibonacci sequence modulo m, adjusted as described in the introduction. Since $a_1 = 0$ and $a_2 = 1$, notice that

$$p_{t-2}^{(m)}(x) = x^{t-2} + a_3x^{t-3} + \cdots + a_{t-1}x + a_t.$$

Now let $C_j(x) = x^j + x^{j-1} + \cdots + x + 1$, where $j \geq 1$. If $k \equiv -2 \bmod t$, then

$$p_k^{(m)}(x) = p_{t-2}^{(m)}(x)C_l(x^t),$$

where $l = (k - t + 2) \bmod t$. Now suppose $k \equiv -1 \bmod t$. Then $a_{k+1} = a_1 = 0$, and so

$$p_k^{(m)}(x) = xp_{t-2}^{(m)}(x)C_l(x^t).$$

Notice that for every j, all the the roots of $C_j(x^t)$ have modulus 1. Since the Mahler measure is multiplicative the Theorem follows. □

We now turn our attention to the particular case in which m is a Lucas number. As Lemmas 2.2 and 2.3 show, the period of the Fibonacci sequence reduced modulo L_n is $2n$ if n is odd and $4n$ if n is even. Moreover, for odd n,

$$\begin{aligned} p_{2n-2}^{(L_n)}(x) &= x^{2n-2} + F_2x^{2n-3} + \cdots + F_nx^{n-1} \\ &\quad -F_{n-1}x^{n-2} + F_{n-2}x^n - 3 - \cdots + F_3x^2 - F_2x + F_1 \\ &= x^{n-1}p_{n-1}(x) + x^{n-2}p_{n-2}\left(-\frac{1}{x}\right), \end{aligned}$$

and by Theorem 4.1, if $k \equiv -2$ or $-1 \bmod 2n$, then $M\left(p_k^{(L_n)}(x)\right) = M\left(p_{2n-2}^{(L_n)}(x)\right)$. If n is even, then notice that Lemma 2.3 gives that the period mod L_n is divided into two halves, the second of which is the negation of the first. By an argument similar to the proof of Theorem 4.1, we can show that if n is even and $k \equiv -2$ or $-1 \bmod 2n$, then $M\left(p_k^{(L_n)}(x)\right) = M\left(p_{2n-2}^{(L_n)}(x)\right)$.

For the next Theorem, let $\Phi_k(x)$ be the k^{th} cyclotomic polynomial. Recall the following well-known fact about cyclotomic polynomials:

$$\prod_{d|n} \Phi_d(x) = x^n - 1. \tag{11}$$

Theorem 4.2. *If $k \equiv -2$ or $-1 \bmod 2n$ then $M\left(p_k^{(L_n)}(x)\right) = \varphi^{n-1}$.*

Proof. By the preceding discussion, it follows that all we need to show is that

$$M\left(p_{2n-2}^{(L_n)}(x)\right) = \varphi^{n-1}. \tag{12}$$

We first consider the case when n is odd. It is straightforward to show that

$$p_{2n-2}^{(L_n)}(x)(x^2 - x - 1) = x^{2n} - L_n x^n - 1, \quad \text{for} \quad n \quad \text{odd}. \tag{13}$$

Now, using identity (11), notice that

$$\begin{aligned} \prod_{\substack{d|n \\ d\neq 1}} \Phi_d(-\varphi x)\Phi_d(x/\varphi) &= \frac{-\varphi^n x^n - 1}{-\varphi x - 1} \cdot \frac{\frac{x^n}{\varphi^n} - 1}{\frac{x}{\varphi} - 1} \\ &= \frac{x^{2n} - \left(\varphi^n - \frac{1}{\varphi^n}\right) x - 1}{x^2 - \left(\varphi - \frac{1}{\varphi}\right) x - 1} \\ &= \frac{x^{2n} - L_n x - 1}{x^2 - x - 1}. \end{aligned} \tag{14}$$

Notice that the last equality above follows from Binet's formula, since n is odd. It now follows from (13) and (14)

$$p_{2n-2}^{(L_n)}(x)(x^2 - x - 1) = (x^2 - x - 1) \prod_{\substack{d|n \\ d\neq 1}} \Phi_d(-\varphi x)\Phi_d(x/\varphi),$$

and therefore

$$p_{2n-2}^{(L_n)}(x) = \prod_{\substack{d|n \\ d\neq 1}} \Phi_d(-\varphi x)\Phi_d(x/\varphi).$$

Now, it is clear that the roots of $\Phi_d(\varphi x)$ all have modulus φ, and those of $\Phi_d(x\varphi)$ all have modulus $\frac{1}{\varphi}$. Thus, it follows that $M\left(p_{2n-2}(x)\right) = \varphi^{n-1}$.

Now suppose $n \geq 2$ is even. Let

$$r_n(x) = x^{2n-4} + F_4x^{2n-6} + F_6x^{2n-8} + \cdots + F_{n-2}x^n + F_nx^{n-2} + F_{n-2}x^{n-4} + \cdots + F_4x^2 + 1.$$

A straightforward calculation shows that

$$p_{2n-2}^{(L_n)}(x) = (x^2 + x - 1)r_n(x). \tag{15}$$

It is also straightforward to show that

$$r_n(x)(x^4 - 3x^2 + 1) = x^{2n} - L_nx^n + 1. \tag{16}$$

In a manner similar to that which was performed in the case when n is odd, we can show that

$$\prod_{\substack{d|n \\ d\neq 1,2}} \Phi_d(-\varphi x)\Phi_d(x/\varphi) = \frac{x^{2n} - L_nx^n + 1}{x^4 - 3x^2 + 1}.$$

(This identity made use of the fact that n is even). Combining this last result with (15) and (16) we see that

$$\begin{aligned} p_{2n-2}^{(L_n)}(x) &= (x^2 + x - 1)r_n(x) \\ &= (x^2 + x - 1)\left(\frac{x^{2n} - L_nx^n + 1}{x^4 - 3x^2 + 1}\right) \\ &= (x^2 + x - 1)\prod_{\substack{d|n \\ d\neq 1,2}} \Phi_d(-\varphi x)\Phi_d(x/\varphi). \end{aligned}$$

From this it is clear that $M\left(p_{2n-2}(x)\right) = \varphi^{n-1}$. $\square$

It is worth mentioning that the proof of Theorem 4.2 shows that for $k \equiv -1$ or $-2 \bmod L_n$ the zeros of $p_k^{(L_n)}(x)$ which lie outside the unit circle are $\varphi e^{2j\pi i/n}$ for $1 \leq j \leq n$ if n is odd, and for $1 \leq j \leq n$ and $j \neq n/2$ if n is even.

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