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Sums of Squares and Products of Jacobsthal Numbers

Zvonko Čerin
Department of Mathematics
University of Zagreb
Bijenička 30
Zagreb 10 000
Croatia
cerin@math.hr

Abstract

We consider sums of squares of odd and even terms of the Jacobsthal sequence and sums of their products. We also study the analogous alternating sums. These sums are related to products of appropriate Jacobsthal numbers and several integer sequences. The formulas that we discover show that a certain translation property for these sums holds, so that in practice, only sums of initial values and the information where the summation begins are necessary.

1 Introduction

The Jacobsthal and Jacobsthal-Lucas sequences J_n and j_n are defined by the recurrence relations

$$J_0 = 0, \qquad J_1 = 1, \qquad J_n = J_{n-1} + 2\,J_{n-2} \ \text{ for } n \geqslant 2,$$

and

$$j_0 = 2, \qquad j_1 = 1, \qquad j_n = j_{n-1} + 2\,j_{n-2} \ \text{ for } n \geqslant 2.$$

In Sections 2–4 we consider sums of squares of odd and even terms of the Jacobsthal sequence and sums of their products. These sums have nice representations as products of appropriate Jacobsthal and Jacobsthal-Lucas numbers.

The numbers J_k appear as the integer sequence $A001045$ from [9] while the numbers j_k is $A014551$. The properties of these numbers are summarized in [7]. For the convenience of the reader we shall now explicitly define these sequences.

The first eleven terms of the sequence J_k are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171 and 341. It is given by the formula $J_n = \frac{2^n-(-1)^n}{3}$.

The first eleven terms of the sequence j_k are 2, 1, 5, 7, 17, 31, 65, 127, 257, 511 and 1025. It is given by the formula $j_n = 2^n + (-1)^n$.

In the last three sections we look into the alternating sums of squares of odd and even terms of the Jacobsthal sequence and the alternating sums of products of two consecutive Jacobsthal numbers. These sums also have nice representations as products of appropriate Jacobsthal and Jacobsthal-Lucas numbers.

These formulas for ordinary sums and for alternating sums have been discovered with the help of a PC computer and all algebraic identities needed for the verification of our theorems can be easily checked in either Derive, Mathematica or Maple V. Running times of all these calculations are in the range of a few seconds.

Similar results for Fibonacci, Lucas, Pell, and Pell-Lucas numbers have recently been discovered by G. M. Gianella and the author in papers [1, 2, 3, 4, 5, 6]. They improved some results in [8].

2 Jacobsthal even squares

The following lemma is needed to accomplish the inductive step in the proof of the first part of our first theorem.

For any $n = 0, 1, 2, \ldots$ let $\beta_n = \frac{2^{4n+2}-1}{3} = J_{4n+2}$, $\gamma_n = \frac{2^{4n+2}+1}{5} = \frac{j_{4n+2}}{5}$ and $\tau_n = \frac{2^{4n}-1}{15} = \frac{J_{4n}}{5}$.

Lemma 1. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds*

$$\begin{aligned}(\beta_{n+1}\,\gamma_{n+1} - \beta_n\,\gamma_n)\,J_{2k}^2 + 8\,(\beta_{n+1}\,\tau_{n+1} - \beta_n\,\tau_n)\,J_{2k} \\ = J_{2k+4n+4}^2 + J_{2k+4n+2}^2 - J_{4n+4}^2 - J_{4n+2}^2.\end{aligned} \tag{2.1}$$

Proof. (**P1**). Let $A = 2^{2k}$, $B = 2^{8n}$ and $C = 2^{4n}$. The difference of the left hand side and the right hand side of the relation (2.1) is equal to

$$\begin{aligned}\frac{544}{9}\left((-1)^{1+2\,k}+1\right)AB + \frac{40}{9}\left((-1)^{4\,n+2\,k}+(-1)^{1+4\,n}\right)(A+1)C \\ + \frac{272}{9}\left((-1)^{4\,k}+2\,(-1)^{1+2\,k}+1\right)B + \frac{2}{9}\left((-1)^{8\,n}+(-1)^{1+8\,n+4\,k}\right).\end{aligned}$$

It is obvious that all coefficients in the above expression are zero and the proof is complete. □

There are essentially three types of proofs in this paper that we indicate as **P1**, **P2** and **P3**.

In **P1** and **P2** we use the substitutions $A = 2^{2k}$, $B = 2^{8n}$, $C = 2^{4n}$ and $M = 2^k$, $P = (-1)^k$, respectively. We prove that the difference of the left hand side and the right hand side of the relation from the statement is equal to zero with algebraic simplification to an expression that is obviously vanishing. This part is easily done with the help of a computer (for example in Maple V) so that we shall only indicate the final expression.

In **P3** we argue by induction on n taking care first of the initial value $n = 0$ and then showing that it holds for $n = r + 1$ under the assumption that it is true for $n = r$.

The following lemma is needed to accomplish the inductive step in the proof of the second part of our first theorem.

For any $n = 0, 1, 2, \dots$ let $\eta_n = 2^{4n+4} + 1 = j_{4n+4}$.

Lemma 2. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds*

$$\begin{aligned}(\tau_{n+2}\,\eta_{n+1} - \tau_{n+1}\,\eta_n)\,J_{2k}^2 + 8\,(\beta_{n+1}\,\tau_{n+2} - \beta_n\,\tau_{n+1})\,J_{2k} \\ = J_{2k+4n+6}^2 + J_{2k+4n+4}^2 - J_{4n+6}^2 - J_{4n+4}^2.\end{aligned} \tag{2.2}$$

Proof. (**P1**).

$$\begin{aligned}\frac{8704}{9}\left((-1)^{1+2\,k}+1\right)AB + \frac{160}{9}\left((-1)^{4\,n+2\,k}+(-1)^{1+4\,n}\right)(A+1)C \\ + \frac{4352}{9}\left((-1)^{4\,k}+2\,(-1)^{1+2\,k}+1\right)B + \frac{2}{9}\left((-1)^{8\,n}+(-1)^{1+8\,n+4\,k}\right).\end{aligned}$$

□

Theorem 1. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equalities hold*

$$\sum_{i=0}^{m} J_{2k+2i}^2 = \sum_{i=0}^{m} J_{2i}^2 + \beta_n\,J_{2\,k}\left[\gamma_n\,J_{2k} + 8\,\tau_n\right], \tag{2.3}$$

if $m = 2n$ and $n = 0, 1, 2, \dots$ and

$$\sum_{i=0}^{m} J_{2k+2i}^2 = \sum_{i=0}^{m} J_{2i}^2 + \tau_{n+1}\,J_{2\,k}\left[\eta_n\,J_{2k} + 8\,\beta_n\right], \tag{2.4}$$

if $m = 2n + 1$ and $n = 0, 1, 2, \dots$.

Proof of (2.3). (**P3**). The proof is by induction on n. When $n = 0$ we obtain

$$J_{2k}^2 = J_0^2 + \beta_0\,J_{2k}\left[\gamma_0\,J_{2k} + 8\,\tau_0\right] = J_{2k}\,J_{2k} = J_{2k}^2,$$

because $J_0 = 0$, $\beta_0 = 1$, $\gamma_0 = 1$ and $\tau_0 = 0$.

Assume that the relation (2.3) is true for $n = r$. Then

$$\begin{aligned}\sum_{i=0}^{2(r+1)} J_{2k+2i}^2 = J_{2k+4r+2}^2 + J_{2k+4r+4}^2 + \sum_{i=0}^{2r} J_{2k+2i}^2 = J_{2k+4r+2}^2 + J_{2k+4r+4}^2 + \\ \sum_{i=0}^{2r} J_{2i}^2 + \beta_r\,J_{2\,k}\left[\gamma_r\,J_{2k} + 8\,\tau_r\right] = \sum_{i=0}^{2(r+1)} J_{2i}^2 + \beta_{r+1}\,J_{2\,k}\left[\gamma_{r+1}\,J_{2k} + 8\,\tau_{r+1}\right],\end{aligned}$$

where the last step uses Lemma 1 for $n = r + 1$. Hence, (2.3) is true for $n = r + 1$ and the proof is completed. □

Proof of (2.4). (**P3**). When $n = 0$ we obtain

$$J_{2k}^2 + J_{2k+2}^2 = J_0^2 + J_2^2 + \tau_1 \, J_{2k} \, [\eta_0 \, J_{2k} + 8 \, \beta_0] = 17 \, J_{2k}^2 + 8 \, J_{2k} + 1,$$

since $J_0 = 0$, $J_1 = 1$, $\beta_0 = 1$, $\eta_0 = 17$ and $\tau_1 = 1$. The above equality is equivalent to $J_{2k+2}^2 = 16 \, J_{2k}^2 + 8 \, J_{2k} + 1 = (4 \, J_{2k} + 1)^2$ which is true because it follows from the relation $J_{n+2} = 4 \, J_n + 1$.

Assume that the relation (2.4) is true for $n = r$. Then

$$\sum_{i=0}^{2(r+1)+1} J_{2k+2i}^2 = J_{2k+4r+4}^2 + J_{2k+4r+6}^2 + \sum_{i=0}^{2r+1} J_{2k+2i}^2 = J_{2k+4r+4}^2 + J_{2k+4r+6}^2 +$$
$$\sum_{i=0}^{2r+1} J_{2i}^2 + \tau_{r+1} \, J_{2\,k} \, [\eta_r \, J_{2k} + 8 \, \beta_r] = \sum_{i=0}^{2(r+1)+1} J_{2i}^2 + \tau_{r+2} \, J_{2\,k} \, [\eta_{r+1} \, J_{2k} + 8 \, \beta_{r+1}] \, ,$$

where the last step uses Lemma 2 for $n = r + 1$. □

3 Jacobsthal odd squares

The initial step in an inductive proof of the first part of our second theorem uses the following lemma.

Lemma 3. *For every $k \geqslant 0$ the following identity holds*

$$J_{2k+1}^2 = 16 \, J_{2k} \, J_{2k-2} + 8 \, J_{2k} + 1.$$

Proof. (**P2**). Let $M = 2^k$ and $P = (-1)^k$. The difference of the left and the right hand side is equal to $\frac{(P-1)(P+1)}{3} \, (8 \, M^2 - 5 \, P^2 + 3) = 0$. □

The initial step in an inductive proof of the second part of our second theorem uses the following lemma.

Lemma 4. *For every $k \geqslant 0$ the following identity holds*

$$J_{2k+1}^2 + J_{2k+3}^2 = 10 + 8 \, J_{2k} \, (34 \, J_{2k-2} + 15).$$

Proof. (**P2**). $10 \, (P - 1) \, (P + 1) \, (4 \, M^2 - 3 \, P^2 + 1)$. □

The following lemma is needed to accomplish the inductive step in the proof of the first part of our second theorem.

For any $n = 0, \, 1, \, 2, \ldots$ let $\pi_n = \frac{2^{4n+1}-1}{3} = J_{4n+1}$.

Lemma 5. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds*

$$8 \, J_{2k} \, \left[2 \, (\beta_{n+1} \, \gamma_{n+1} - \beta_n \, \gamma_n) \, J_{2k-2} + \beta_{n+1} \, \pi_{n+1} - \beta_n \, \pi_n\right]$$
$$= J_{2k+4n+5}^2 + J_{2k+4n+3}^2 - J_{4n+5}^2 - J_{4n+3}^2. \tag{3.1}$$

Proof. (**P1**).

$$\frac{5440}{9}\,\left((-1)^{1+2\,k}+1\right)AB+\frac{80}{9}\,\left((-1)^{4\,n+2\,k+1}+(-1)^{4\,n}\right)(A+1)C$$
$$+\,\frac{1088}{9}\,\left(4\,(-1)^{4\,k}+5\,(-1)^{1+2\,k}+1\right)B+\frac{2}{9}\,\left((-1)^{8\,n}+(-1)^{8\,n+4\,k+1}\right).$$

□

The following lemma is needed to accomplish the inductive step in the proof of the second part of our second theorem.

For any $n = 0, 1, 2, \ldots$ let $\sigma_n = 5\,J_{4n+3}$.

Lemma 6. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds*

$$8\,J_{2k}\,\left[2\,(\tau_{n+2}\,\eta_{n+1}-\tau_n\,\eta_n)\,J_{2k-2}+\tau_{n+2}\,\sigma_{n+1}-\tau_{n+1}\,\sigma_n\right]$$
$$=J^2_{2k+4n+7}+J^2_{2k+4n+5}-J^2_{4n+7}-J^2_{4n+5}. \tag{3.2}$$

Proof. (**P1**).

$$\frac{87040}{9}\,\left((-1)^{1+2\,k}+1\right)AB+\frac{320}{9}\,\left((-1)^{4\,n+2\,k+1}+(-1)^{4\,n}\right)(A+1)C$$
$$+\,\frac{17408}{9}\,\left(4\,(-1)^{4\,k}+5\,(-1)^{1+2\,k}+1\right)B+\frac{2}{9}\,\left((-1)^{8\,n}+(-1)^{8\,n+4\,k+1}\right).$$

□

Theorem 2. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equalities hold*

$$\sum_{i=0}^{m} J^2_{2k+2i+1}=\sum_{i=0}^{m} J^2_{2i+1}+8\,\beta_n\,J_{2\,k}\,[2\,\gamma_n\,J_{2k-2}+\pi_n]\,, \tag{3.3}$$

if $m = 2n$ and $n = 0, 1, 2, \ldots$ and

$$\sum_{i=0}^{m} J^2_{2k+2i+1}=\sum_{i=0}^{m} J^2_{2i+1}+8\,\tau_{n+1}\,J_{2\,k}\,[2\,\eta_n\,J_{2k-2}+\sigma_n]\,, \tag{3.4}$$

if $m = 2n+1$ and $n = 0, 1, 2, \ldots$.

Proof of (3.3). (**P3**). When $n = 0$ we obtain

$$J^2_{2k+1}=J^2_1+8\,\beta_0\,J_{2k}\,[2\,\gamma_0\,J_{2k-2}+\pi_0]=1+8\,J_{2k}\,[2\,J_{2k-2}+1],$$

because $J_1 = 1$, $\beta_0 = 1$, $\gamma_0 = 1$ and $\pi_0 = 1$. But, this equality is true by Lemma 3.

Assume that the relation (3.3) is true for $n = r$. Then

$$\sum_{i=0}^{2(r+1)} J^2_{2k+2i+1}=J^2_{2k+4r+5}+J^2_{2k+4r+3}+\sum_{i=0}^{2r} J^2_{2k+2i+1}=J^2_{2k+4r+5}+J^2_{2k+4r+3}+\sum_{i=0}^{2r} J^2_{2i+1}$$
$$+\,8\,\beta_n\,J_{2\,k}\,[2\,\gamma_n\,J_{2k-2}+\pi_n]=\sum_{i=0}^{2(r+1)} J^2_{2i+1}+8\,\beta_{n+1}\,J_{2\,k}\,[2\,\gamma_{n+1}\,J_{2k-2}+\pi_{n+1}]\,,$$

where the last step uses Lemma 5 for $n = r + 1$. □

Proof of (3.4). (**P3**). The proof is by induction on n. When $n = 0$ we obtain

$$J_{2k+1}^2 + J_{2k+3}^2 = J_1^2 + J_3^2 + 8\,\tau_1\,J_{2\,k}\,[2\,\eta_0\,J_{2k-2} + \sigma_0] = 10 + 8\,J_{2\,k}\,[34\,J_{2k-2} + 15]\,,$$

since $J_1 = 1$, $J_3 = 3$, $\tau_1 = 1$, $\eta_0 = 17$ and $\sigma_0 = 15$. The above equality is true by Lemma 4.

Assume that the relation (3.4) is true for $n = r$. Then

$$\begin{aligned}\sum_{i=0}^{2(r+1)+1} J_{2k+2i+1}^2 = J_{2k+4r+7}^2 + J_{2k+4r+5}^2 + \sum_{i=0}^{2r+1} J_{2k+2i+1}^2 = \\ J_{2k+4r+7}^2 + J_{2k+4r+5}^2 + \sum_{i=0}^{2r+1} J_{2i+1}^2 + 8\,\tau_{r+1}\,J_{2\,k}\,[2\,\eta_r\,J_{2k-2} + \sigma_r] \\ = \sum_{i=0}^{2(r+1)+1} J_{2i+1}^2 + 8\,\tau_{r+2}\,J_{2\,k}\,[2\,\eta_{r+1}\,J_{2k-2} + \sigma_{r+1}]\,,\end{aligned}$$

where the last step uses Lemma 6 for $n = r + 1$. □

4 Jacobsthal products

For the first two steps in a proof by induction of our next theorem we require the following lemma.

Lemma 7. *For every $k \geqslant 0$ the following equalities hold*

$$J_{2k+1} = 8\,J_{2k-2} + 3. \tag{4.1}$$

$$J_{2k}\,J_{2k+1} + J_{2k+2}\,J_{2k+3} = 3 + J_{2k}\,(136\,J_{2k-2} + 55). \tag{4.2}$$

Proof of (4.1). By the formula $J_k = \frac{2^k - (-1)^k}{3}$ we have

$$J_{2k+1} \;=\; \frac{2^{2k+1} - (-1)^{2k+1}}{3} \;=\; \frac{2^{2k+1} + 1}{3} \;=\; 8 \cdot \frac{2^{2k-2} - (-1)^{2k-2}}{3} + 3 \;=\; 8\,J_{2k-2} + 3.$$

□

Proof of (4.2). (**P2**). $\frac{(P-1)(P+1)}{3}\,(55\,M^2 - 46\,P^2 + 9)\,.$ □

With the following lemma we shall make the inductive step in the proof of the first part of our third theorem.

Lemma 8. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds*

$$\begin{aligned}&J_{2k}\,\left[8\,(\beta_{n+1}\,\gamma_{n+1} - \beta_n\,\gamma_n)\,J_{2k-2} + \beta_{n+1}\,\mu_{n+1} - \beta_n\,\mu_n\right] = \\ &\quad J_{2k+4n+5}\,J_{2k+4n+4} + J_{2k+4n+3}\,J_{2k+4n+2} - J_{4n+5}\,J_{4n+4} - J_{4n+3}\,J_{4n+2}.\end{aligned} \tag{4.3}$$

Proof. (**P1**).

$$\frac{2720}{9}\,\left((-1)^{1+2\,k}+1\right)AB+\frac{20}{9}\,\left((-1)^{4\,n+2\,k}+(-1)^{4\,n+1}\right)(A+1)C+$$
$$\frac{544}{9}\,\left(4\,(-1)^{4\,k}+5\,(-1)^{1+2\,k}+1\right)B+\frac{2}{9}\,\left((-1)^{8\,n+1}+(-1)^{8\,n+4\,k}\right).$$

□

With the following lemma we shall make the inductive step in the proof of the second part of our third theorem.

Lemma 9. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds*

$$\begin{aligned} &J_{2k}\,[8\,(\tau_{n+2}\,\eta_{n+1}-\tau_{n+1}\,\eta_n)\,J_{2k-2}+\tau_{n+2}\,\nu_{n+1}-\tau_{n+1}\,\nu_n]=\\ &\quad J_{2k+4n+5}\,J_{2k+4n+4}+J_{2k+4n+7}\,J_{2k+4n+6}-J_{4n+5}\,J_{4n+4}-J_{4n+7}\,J_{4n+6}. \end{aligned} \tag{4.4}$$

Proof. (**P1**).

$$\frac{43520}{9}\,\left((-1)^{1+2\,k}+1\right)AB+\frac{80}{9}\,\left((-1)^{4\,n+2\,k}+(-1)^{4\,n+1}\right)(A+1)\,C$$
$$+\,\frac{8704}{9}\,\left(4\,(-1)^{4\,k}+5\,(-1)^{1+2\,k}+1\right)B+\frac{2}{9}\,\left((-1)^{8\,n+1}+(-1)^{8\,n+4\,k}\right).$$

□

For any $n=0,\,1,\,2,\dots$ let $\mu_n=\frac{2^{4n+3}+1}{3}=J_{4n+3}$ and $\nu_n=\frac{5\left(2^{4n+5}+1\right)}{3}=5\,J_{4n+5}$.

Theorem 3. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equalities hold*

$$\sum_{i=0}^{m} J_{2k+2i}\,J_{2k+2i+1}=\sum_{i=0}^{m} J_{2i}\,J_{2i+1}+\beta_n\,J_{2\,k}\,[8\,\gamma_n\,J_{2k-2}+\mu_n]\,, \tag{4.5}$$

if $m=2n$ and $n=0,\,1,\,2,\dots$ and

$$\sum_{i=0}^{m} J_{2k+2i}\,J_{2k+2i+1}=\sum_{i=0}^{m} J_{2i}\,J_{2i+1}+\tau_{n+1}\,J_{2\,k}\,[8\,\eta_n\,J_{2k-2}+\nu_n]\,, \tag{4.6}$$

if $m=2n+1$ and $n=0,\,1,\,2,\dots$.

Proof of (4.5). (**P3**). For $n=0$ the relation (4.5) is

$$J_{2k}\,J_{2k+1}=J_0\,J_1+\beta_0\,J_{2k}(8\,\gamma_0\,J_{2k-2}+\mu_0)=J_{2k}(8\,J_{2k-2}+3)$$

which is true since $J_{2k+1}=8\,J_{2k-2}+3$ by the relation (4.1) in Lemma 7.

Assume that the relation (4.5) is true for $n = r$. Then

$$\sum_{i=0}^{2(r+1)} J_{2k+2i}\, J_{2k+2i+1} = J_{2k+4r+4}\, J_{2k+4r+5} + J_{2k+4r+2}\, J_{2k+4r+3} + \sum_{i=0}^{2r} J_{2k+2i}\, J_{2k+2i+1} =$$

$$J_{2k+4r+4}\, J_{2k+4r+5} + J_{2k+4r+2}\, J_{2k+4r+3} + \sum_{i=0}^{2r} J_{2i}\, J_{2i+1} + \beta_n\, J_{2\,k}\, [8\, \gamma_n\, J_{2k-2} + \mu_n]$$

$$= \sum_{i=0}^{2(r+1)} J_{2i}\, J_{2i+1} + \beta_{n+1}\, J_{2\,k}\, [8\, \gamma_{n+1}\, J_{2k-2} + \mu_{n+1}]\,,$$

where the last step uses Lemma 8 for $n = r + 1$. □

Proof of (4.6). (**P3**). For $n = 0$ the relation (4.6) is

$$J_{2k}\, J_{2k+1} + J_{2k+2}\, J_{2k+3} = J_0\, J_1 + J_2\, J_3 + \tau_1\, J_{2k}(8\, \eta_0\, J_{2k-2} + \nu_0) = 3 + J_{2k}(136\, J_{2k-2} + 55)$$

which is true by (4.2) in Lemma 7.

Assume that the relation (4.6) is true for $n = r$. Then

$$\sum_{i=0}^{2(r+1)+1} J_{2k+2i}\, J_{2k+2i+1} = J_{2k+4r+4}\, J_{2k+4r+5} + J_{2k+4r+6}\, J_{2k+4r+7} + \sum_{i=0}^{2r+1} J_{2k+2i}\, J_{2k+2i+1} =$$

$$J_{2k+4r+4}\, J_{2k+4r+5} + J_{2k+4r+6}\, J_{2k+4r+7} + \sum_{i=0}^{2r+1} J_{2i}\, J_{2i+1} + \tau_{n+1}\, J_{2\,k}\, [8\, \eta_n\, J_{2k-2} + \nu_n]$$

$$= \sum_{i=0}^{2(r+1)+1} J_{2i}\, J_{2i+1} + \tau_{n+2}\, J_{2\,k}\, [8\, \eta_{n+1}\, J_{2k-2} + \nu_{n+1}]\,,$$

where the last step uses Lemma 9 for $n = r + 1$. □

5 Alternating Jacobsthal even squares

In this section we look for formulas that give closed forms for alternating sums of squares of Jacobsthal numbers with even indices.

Lemma 10. *For every $k \geqslant 0$ we have*

$$J_{2k} = 4\, J_{2k-2} + 1. \tag{5.1}$$

Proof. By the formula $J_k = \frac{2^k-(-1)^k}{3}$ we get

$$4\, J_{2k-2} + 1 = 4 \cdot \frac{2^{2k-2} - (-1)^{2k-2}}{3} + 1 = \frac{2^{2k} - (-1)^{2k}}{3} = J_{2k}.$$

□

Lemma 11. *For every $k \geqslant 0$ we have*

$$J_{2k+2}^2 - J_{2k}^2 = 1 + J_{2k}\,(60\,J_{2k-2} + 23). \tag{5.2}$$

Proof. (**P2**). $\frac{(P-1)(P+1)}{3}\,(23\,M^2 - 20\,P^2 + 3)\,.$ $\square$

Let $\tau_0^* = 1$ and $\tau_{n+1}^* - \tau_n^* = 2^{4n+3}\,(50\cdot 2^{4n} - 1)$, for $n = 0,\ 1,\ 2,\ldots$. For any $n = 0,\ 1,\ 2,\ldots$ let $\gamma_n^* = \frac{2^{8n+4}+1}{17} = \frac{j_{8n+4}}{17}$.

Lemma 12. *For every $k \geqslant 0$ and every $n \geqslant 0$ we have*

$$J_{2k}\,(4\,(\gamma_{n+1}^* - \gamma_n^*)\,J_{2k-2} + \tau_{n+1}^* - \tau_n^*) = J_{2k+4n+4}^2 - J_{2k+4n+2}^2 - J_{4n+4}^2 + J_{4n+2}^2. \tag{5.3}$$

Proof. (**P1**).

$$\frac{400}{3}\left((-1)^{1+2\,k}+1\right)AB + \frac{8}{3}\left((-1)^{4\,n+2\,k}-1\right)AC + \\ \frac{80}{3}\left(4\,(-1)^{4\,k} + 5\,(-1)^{1+2\,k}+1\right)B + \frac{8}{3}\left((-1)^{2\,k} + (-1)^{4\,n+1}\right)C.$$

$\square$

Let $\beta_0^* = 23$ and $\beta_{n+1}^* - \beta_n^* = 2^{4n+5}\,(200\cdot 2^{4n} - 1)$ for any $n = 0,\ 1,\ 2,\ldots$. For the same values of n, let $\eta_n^* = \frac{2^{8n+8}-1}{17} = \frac{3}{17}\,J_{8n+8}$.

Lemma 13. *For every $k \geqslant 0$ and every $n \geqslant 0$ we have*

$$J_{2k}\,(4\,(\eta_{n+1}^* - \eta_n^*)\,J_{2k-2} + \beta_{n+1}^* - \beta_n^*) = J_{2k+4n+6}^2 - J_{2k+4n+4}^2 + J_{4n+4}^2 - J_{4n+6}^2. \tag{5.4}$$

Proof. (**P1**).

$$\frac{6400}{3}\left((-1)^{1+2\,k}+1\right)AB + \frac{32}{3}\left((-1)^{4\,n+2\,k}-1\right)AC + \\ \frac{1280}{3}\left(4\,(-1)^{4\,k} + 5\,(-1)^{1+2\,k}+1\right)B + \frac{32}{3}\left((-1)^{2\,k} + (-1)^{4\,n+1}\right)C.$$

$\square$

Theorem 4. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equalities hold*

$$\sum_{i=0}^{m}(-1)^i\,J_{2k+2i}^2 = \sum_{i=0}^{m}(-1)^i\,J_{2i}^2 + J_{2\,k}\,[4\,\gamma_n^*\,J_{2k-2} + \tau_n^*]\,, \tag{5.5}$$

if $m = 2n$ and $n = 0,\ 1,\ 2,\ldots$ and

$$\sum_{i=0}^{m}(-1)^i\,J_{2k+2i}^2 = \sum_{i=0}^{m}(-1)^i\,J_{2i}^2 - J_{2\,k}\,[4\,\eta_n^*\,J_{2k-2} + \beta_n^*]\,, \tag{5.6}$$

if $m = 2n+1$ and $n = 0,\ 1,\ 2,\ldots$.

Proof of (5.5). (**P3**). For $n = 0$ the relation (5.5) is

$$J_{2k}^2 = J_0^2 + J_{2k}\,(4\,\gamma_0^*\,J_{2k-2} + \tau_0^*) = J_{2k}\,(4\,J_{2k-2} + 1)$$

(i. e., the relation (5.1) multiplied by J_{2k}) which is true by Lemma 10.

Assume that the relation (5.5) is true for $n = r$. Then

$$\sum_{i=0}^{2(r+1)} (-1)^i \cdot J_{2k+2i}^2 = Q_{2k+4r+4}^2 - Q_{2k+4r+2}^2 + \sum_{i=0}^{2r} (-1)^i \cdot J_{2k+2i}^2$$
$$= Q_{2k+4r+4}^2 - Q_{2k+4r+2}^2 + \sum_{i=0}^{2r} (-1)^i \cdot J_{2i}^2 + J_{2\,k}\,[4\,\gamma_n^*\,J_{2k-2} + \tau_n^*] =$$
$$\sum_{i=0}^{2(r+1)} (-1)^i \cdot J_{2i}^2 + J_{2\,k}\left[4\,\gamma_{n+1}^*\,J_{2k-2} + \tau_{n+1}^*\right],$$

where the last step uses Lemma 12. □

Proof of (5.6). (**P3**). For $n = 0$ the relation (5.6) is

$$J_{2k}^2 - J_{2k+2}^2 = (J_0^2 - J_2^2) - J_{2\,k}\,[4\,\eta_0^*\,J_{2k-2} + \beta_0^*] = -1 - J_{2\,k}\,[60\,J_{2k-2} + 23]\,,$$

which is true by Lemma 11.

Assume that the relation (5.6) is true for $n = r$. Then

$$\sum_{i=0}^{2(r+1)+1} (-1)^i \cdot J_{2k+2i}^2 = J_{2k+4r+4}^2 - J_{2k+4r+6}^2 + \sum_{i=0}^{2r+1} (-1)^i \cdot J_{2k+2i}^2$$
$$= J_{2k+4r+4}^2 - J_{2k+4r+6}^2 + \sum_{i=0}^{2r+1} (-1)^i \cdot J_{2i}^2 - J_{2\,k}\,[4\,\eta_n^*\,J_{2k-2} + \beta_n^*]$$
$$= \sum_{i=0}^{2(r+1)+1} (-1)^i \cdot J_{2i}^2 - J_{2\,k}\left[4\,\eta_{n+1}^*\,J_{2k-2} + \beta_{n+1}^*\right],$$

where the last step uses Lemma 13. □

6 Alternating Jacobsthal odd squares

In this section we look for formulas that give closed forms for alternating sums of squares of Jacobsthal numbers with odd indices.

Lemma 14. *For every $k \geqslant 0$ we have*

$$J_{2k+1}^2 = 1 + 8\,J_{2k}\,[2\,J_{2k-2} + 1]\,. \tag{6.1}$$

Proof. (**P2**). $\frac{(P-1)(P+1)}{3}\,(8\,M^2 - 5\,P^2 + 3)\,.$ □

Lemma 15. *For every $k \geqslant 0$ we have*

$$J_{2k+3}^2 - J_{2k+1}^2 = 8 + 8\, J_{2k}\,(30\, J_{2k-2} + 13). \tag{6.2}$$

Proof. (**P2**). $\frac{(P-1)(P+1)}{3}\,(13\, M^2 - 10\, P^2 + 3)\,.$ □

Let $\tau_0^{**} = 1$ and $\tau_{n+1}^{**} - \tau_n^{**} = 101 \cdot 2^{4n+1} + 25 \cdot 2^{4n+3}\,(2^{4n} - 1)$ for $n = 0,\, 1,\, 2, \ldots$.

Lemma 16. *For every $k \geqslant 0$ and every $n \geqslant 0$ we have*

$$8\, J_{2k}\,(2\,(\gamma_{n+1}^* - \gamma_n^*)\, J_{2k-2} + \tau_{n+1}^{**} - \tau_n^{**}) \;=\; J_{2k+4n+5}^2 - J_{2k+4n+3}^2 - J_{4n+5}^2 + J_{4n+3}^2. \tag{6.3}$$

Proof. (**P1**).

$$\frac{1600}{3}\,\left((-1)^{1+2\,k} + 1\right) AB + \frac{16}{3}\,\left((-1)^{4\,n+2\,k+1} + 1\right) AC +$$
$$\frac{320}{3}\,\left(4\,(-1)^{4\,k} + 5\,(-1)^{1+2\,k} + 1\right) B + \frac{16}{3}\,\left((-1)^{2\,k+1} + (-1)^{4\,n}\right) C.$$

□

Let $\beta_0^{**} = 13$ and $\beta_{n+1}^{**} - \beta_n^{**} = 401 \cdot 2^{4n+3} + 100 \cdot 2^{4n+5}\,(2^{4n} - 1)$ for any $n = 0,\, 1,\, 2, \ldots$.

Lemma 17. *For every $k \geqslant 0$ and every $n \geqslant 0$ we have*

$$8\, J_{2k}\,(2\,(\eta_{n+1}^* - \eta_n^*)\, J_{2k-2} + \beta_{n+1}^{**} - \beta_n^{**}) = J_{2k+4n+7}^2 - J_{2k+4r+5}^2 + J_{4n+5}^2 - J_{4r+7}^2. \tag{6.4}$$

Proof. (**P1**).

$$\frac{25600}{3}\,\left((-1)^{1+2\,k} + 1\right) AB + \frac{64}{3}\,\left((-1)^{4\,n+2\,k+1} + 1\right) AC +$$
$$\frac{5120}{3}\,\left(4\,(-1)^{4\,k} + 5\,(-1)^{1+2\,k} + 1\right) B + \frac{64}{3}\,\left((-1)^{2\,k+1} + (-1)^{4\,n}\right) C.$$

□

Theorem 5. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equalities hold*

$$\sum_{i=0}^{m} (-1)^i\, J_{2k+2i+1}^2 = \sum_{i=0}^{m} (-1)^i\, J_{2i+1}^2 + 8\, J_{2\,k}\,[2\,\gamma_n^*\, J_{2k-2} + \tau_n^{**}]\,, \tag{6.5}$$

if $m = 2n$ and $n = 0,\, 1,\, 2, \ldots$ and

$$\sum_{i=0}^{m} (-1)^i\, J_{2k+2i+1}^2 = \sum_{i=0}^{m} (-1)^i\, J_{2i+1}^2 - 8\, J_{2\,k}\,[2\,\eta_n^*\, J_{2k-2} + \beta_n^{**}]\,, \tag{6.6}$$

if $m = 2n + 1$ and $n = 0,\, 1,\, 2, \ldots$.

Proof of (6.5). (**P3**). For $n=0$ the relation (6.5) is

$$J_{2k+1}^2 = J_1^2 + 8\,J_{2k}(2\,\gamma_0^*\,J_{2k-2} + \tau_0^{**})$$

(i. e., the relation (6.1)) which is true by Lemma 14.

Assume that the relation (6.5) is true for $n=r$. Then

$$\begin{aligned}\sum_{i=0}^{2(r+1)} (-1)^i\,J_{2k+2i+1}^2 &= J_{2k+4r+5}^2 - J_{2k+4r+3}^2 + \sum_{i=0}^{2r} (-1)^i\,J_{2k+2i+1}^2 \\ &= J_{2k+4r+5}^2 - J_{2k+4r+3}^2 + \sum_{i=0}^{2r} (-1)^i\,J_{2i+1}^2 + 8\,J_{2\,k}\,[2\,\gamma_n^*\,J_{2k-2} + \tau_n^{**}] \\ &= \sum_{i=0}^{2(r+1)} (-1)^i\,J_{2i+1}^2 + 8\,J_{2\,k}\left[2\,\gamma_{n+1}^*\,J_{2k-2} + \tau_{n+1}^{**}\right],\end{aligned}$$

where the last step uses Lemma 16. □

Proof of (6.6). (**P3**). The proof is again by induction on n. For $n=0$ the relation (6.6) is $J_{2k+1}^2 - J_{2k+3}^2 = J_1^2 - J_3^2 - 8\,J_{2\,k}\,[2\,\eta_0^*\,J_{2k-2} + \beta_0^{**}]$ which is true by Lemma 15 since $\eta_0^* = 15$ and $\beta_0^{**} = 13$.

Assume that the relation (6.6) is true for $n=r$. Then

$$\begin{aligned}\sum_{i=0}^{2(r+1)+1} (-1)^i\,J_{2k+2i+1}^2 &= J_{2k+4r+5}^2 - J_{2k+4r+7}^2 + \sum_{i=0}^{2r+1} (-1)^i\,J_{2k+2i+1}^2 \\ &= J_{2k+4r+5}^2 - J_{2k+4r+7}^2 + \sum_{i=0}^{2r+1} (-1)^i\,J_{2i+1}^2 - 8\,J_{2\,k}\,[2\,\eta_n^*\,J_{2k-2} + \beta_n^{**}] \\ &= \sum_{i=0}^{2(r+1)+1} (-1)^i\,J_{2i+1}^2 - 8\,J_{2\,k}\left[2\,\eta_{n+1}^*\,J_{2k-2} + \beta_{n+1}^{**}\right],\end{aligned}$$

where the last step uses Lemma 17. □

7 Alternating Jacobsthal products

Lemma 18. *For every $k \geqslant 0$ we have*

$$J_{2k+3}\,J_{2k+2} - J_{2k+1}\,J_{2k} = 3 + J_{2k}\,\left[120\,J_{2k-2} + 49\right]. \tag{7.1}$$

Proof. (**P2**). $\frac{(P-1)(P+1)}{3}\,(49\,M^2 - 40\,P^2 + 9)$. □

Lemma 19. *For every $k \geqslant 0$ and every $n \geqslant 0$ we have*

$$\begin{aligned}&J_{2k}\,\left(8\,(\gamma_{n+1}^* - \gamma_n^*)\,J_{2k-2} + \tau_{n+1}^{***} - \tau_n^{***}\right) = \\ &\qquad J_{2k+4n+5}\,J_{2k+4n+4} - J_{2k+4n+3}\,J_{2k+4n+2} - J_{4n+5}\,J_{4n+4} + J_{4n+3}\,J_{4n+2}.\end{aligned} \tag{7.2}$$

Proof. (**P1**).

$$\frac{800}{3}\left((-1)^{1+2k}+1\right)AB+\frac{4}{3}\left((-1)^{4n+2k}-1\right)AC+ \\ \frac{160}{3}\left(4\,(-1)^{4k}+5\,(-1)^{1+2k}+1\right)B+\frac{4}{3}\left((-1)^{2k}+(-1)^{4n+1}\right)C.$$

□

Let $\beta_0^{***}=49$ and $\beta_{n+1}^{***}-\beta_n^{***}=799\cdot 2^{4n+4}+25\cdot 2^{4n+9}\,(2^{4n}-1)$ for any $n=0,\,1,\,2,\ldots$.

Lemma 20. *For every $k\geqslant 0$ and every $n\geqslant 0$ we have*

$$J_{2k}\,(8\,(\eta_{n+1}^{*}-\eta_n^{*})\,J_{2k-2}+\beta_{n+1}^{***}-\beta_n^{***})= \\ J_{2k+4n+6}\,J_{2k+4n+7}-J_{2k+4n+4}\,J_{2k+4n+5}+J_{4n+5}\,J_{4n+4}-J_{4n+6}\,J_{4n+7}. \tag{7.3}$$

Proof. (**P1**).

$$\frac{12800}{3}\left((-1)^{1+2k}+1\right)AB+\frac{16}{3}\left((-1)^{4n+2k}-1\right)AC+ \\ \frac{2560}{3}\left(4\,(-1)^{4k}+5\,(-1)^{1+2k}+1\right)B+\frac{16}{3}\left((-1)^{2k}+(-1)^{4n+1}\right)C.$$

□

Theorem 6. *For every $m\geqslant 0$ and $k\geqslant 0$ the following equalities hold*

$$\sum_{i=0}^{m}(-1)^i\,J_{2k+2i}\,J_{2k+2i+1}=\sum_{i=0}^{m}(-1)^i\,J_{2i}\,J_{2i+1}+J_{2\,k}\,[8\,\gamma_n^{*}\,J_{2k-2}+\tau_n^{***}]\,, \tag{7.4}$$

if $m=2n$ and $n=0,\,1,\,2,\ldots$ and

$$\sum_{i=0}^{m}(-1)^i\,J_{2k+2i}\,J_{2k+2i+1}=\sum_{i=0}^{m}(-1)^i\,J_{2i}\,J_{2i+1}-J_{2\,k}\,[8\,\eta_n^{*}\,J_{2k-2}+\beta_n^{***}]\,, \tag{7.5}$$

if $m=2n+1$ and $n=0,\,1,\,2,\ldots$.

Proof of (7.4). (**P3**). For $n=0$ the relation (7.4) is

$$J_{2k}\,J_{2k+1}=J_0\,J_1+J_{2\,k}\,[8\,\gamma_0^{*}\,J_{2k-2}+\tau_0^{***}]=J_{2\,k}\,[8\,J_{2k-2}+3]$$

which is true by the relation (4.1) in Lemma 7 since $J_0=0$, $J_1=1$, $\gamma_0^{*}=1$ and $\tau_0^{***}=3$.

Assume that the relation (7.4) is true for $n=r$. Then

$$\sum_{i=0}^{2(r+1)}(-1)^i J_{2k+2i}J_{2k+2i+1}=\sum_{i=0}^{2r}(-1)^i J_{2k+2i}J_{2k+2i+1}+J_{2k+4r+4}J_{2k+4r+5}-J_{2k+4r+2}J_{2k+4r+3}$$

$$=J_{2k+4r+4}J_{2k+4r+5}-J_{2k+4r+2}J_{2k+4r+3}+\sum_{i=0}^{2r}(-1)^i J_{2i}J_{2i+1}+J_{2\,k}\,[8\,\gamma_n^{*}\,J_{2k-2}+\tau_n^{***}]$$

$$=\sum_{i=0}^{2(r+1)}(-1)^i J_{2i}J_{2i+1}+J_{2\,k}\left[8\,\gamma_{n+1}^{*}\,J_{2k-2}+\tau_{n+1}^{***}\right],$$

where the last step uses Lemma 19. □

Proof of (7.5). (**P3**). For $n = 0$ the relation (7.5) is

$$J_{2k}\,J_{2k+1} - J_{2k+2}\,J_{2k+3} = J_0\,J_1 - J_2\,J_3 - \\ J_{2k}\,[8\,\eta_0^*\,J_{2k-2} + \beta_0^{***}] = -3 - J_{2k}\,[120\,J_{2k-2} + 49]\,,$$

(i. e., the relation (7.1)) which is true by Lemma 18.

Assume that the relation (7.5) is true for $n = r$. Then

$$\sum_{i=0}^{2(r+1)+1} (-1)^i\,J_{2k+2i}\,J_{2k+2i+1} = \sum_{i=0}^{2r+1} (-1)^i\,J_{2k+2i}\,J_{2k+2i+1} + J_{2k+4r+4}\,J_{2k+4r+5} - \\ J_{2k+4r+6}\,J_{2k+4r+7} = J_{2k+4r+4}\,J_{2k+4r+5} - J_{2k+4r+6}\,J_{2k+4r+7} + \sum_{i=0}^{2r+1} (-1)^i\,J_{2i}\,J_{2i+1} - \\ J_{2\,k}\,[8\,\eta_n^*\,J_{2k-2} + \beta_n^{***}] = \sum_{i=0}^{2(r+1)+1} (-1)^i\,J_{2i}\,J_{2i+1} - J_{2\,k}\left[8\,\eta_{n+1}^*\,J_{2k-2} + \beta_{n+1}^{***}\right],$$

where the last step uses Lemma 20. $\square$

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