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Mean Values of Generalized gcd-sum and lcm-sum Functions

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Abstract

We consider a generalization of the gcd-sum function, and obtain its average order with a quasi-optimal error term. We also study the reciprocals of the gcd-sum and lcm-sum functions.

1 Introduction and notation

The so-called gcd-sum function, defined by

$$g(n) = \sum_{j=1}^{n} (n, j)$$

where (a, b) denotes the greatest common divisor of a and b, was first introduced by Broughan ([3, 4]) who studied its main properties, and showed among other things that g satisfies the convolution identity (see also the beginning of the proof of Lemma 3.1)

$$g = \varphi * \mathrm{Id}$$

where $F * G$ is the usual Dirichlet convolution product. By using the following alternative convolution identity

$$g = \mu * (\mathrm{Id} \cdot \tau),$$

where μ is the Möbius function and τ is the divisor function, we were able in [1] to get the average order of g. Our result can be stated as follow. If θ is the exponent in the Dirichlet divisor problem, then the following asymptotic formula

$$\sum_{n \leqslant x} g(n) = \frac{x^2 \log x}{2\zeta(2)} + \frac{x^2}{2\zeta(2)} \left(\gamma - \frac{1}{2} + \log \left(\frac{\mathcal{A}^{12}}{2\pi} \right) \right) + O_\varepsilon \left(x^{1+\theta+\varepsilon} \right) \tag{1}$$

holds for any real number $\varepsilon > 0$, where $\mathcal{A} \approx 1.282\ 427\ 129\dots$ is the Glaisher-Kinkelin constant. The inequality $\theta \geqslant 1/4$ is well-known, and, from the work of Huxley [5] we know that $\theta \leqslant 131/416 \approx 0.3149$.

The aim of this paper is first to work with a function generalizing the function g and prove an asymptotic formula for its average order similarly as in (1). In sections 5, 6 and 7 we will establish estimates for the lcm-sum function, and for reciprocals of the gcd-sum and lcm-sum functions. We begin with classical notation.

1. *Multiplicative functions*. The following arithmetic functions are well-known.

$$\begin{aligned} \mathrm{Id}^a(n) &= n^a \quad (a \in \mathbb{Z}^*) \\ \mathbf{1}(n) &= 1 \end{aligned}$$

and μ, φ, σ_k and τ_k are respectively the Möbius function, the Euler totient function, the sum of kth powers of divisors function and the kth Piltz divisor function. Recall that τ_k can be defined by $\tau_k = \underbrace{\mathbf{1} * \cdots * \mathbf{1}}_{k \text{ times}}$ for any integer $k \geqslant 1$ and that $\tau_2 = \tau$. We also have $\sigma_k = \sum_{d|n} d^k$ and $\sigma_0 = \tau$.

2. *Exponent in the Dirichlet-Piltz divisor problem*. For any integer $k \geqslant 2$, θ_k is defined to be the smallest positive real number such that the asymptotic formula

$$\sum_{n \leqslant x} \tau_k(n) = x\mathcal{P}_{k-1}(\log x) + O_{\varepsilon,k}\left(x^{\theta_k+\varepsilon} \right) \tag{2}$$

holds for any real number $\varepsilon > 0$. Here $\mathcal{P}_{k-1}$ is a polynomial of degree $k-1$ with real coefficients, the leading coefficient being $\frac{1}{(k-1)!}$. It is now well-known that $\frac{1}{3} \leqslant \theta_3 \leqslant \frac{43}{96}$ and that $\frac{k-1}{2k} \leqslant \theta_k \leqslant \frac{k-1}{k+2}$ for $k \geqslant 4$ (see [6], for example).

By convention, we set

$$\tau_0(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{otherwise;} \end{cases}$$

and $\theta_1 = 0$.

2 A generalization of the gcd-sum function

Definition 2.1. *We define the sequence of arithmetic functions* $f_{k,j}(n)$ *in the following way.*

(i) For any integers $j, n \geqslant 1$*, we set*

$$f_{1,j}(n) = \begin{cases} 1, & \text{if } (n,j) = 1; \\ 0, & \text{otherwise;} \end{cases}$$

$$f_{2,j}(n) = \begin{cases} (n,j), & \text{if } j \leqslant n; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) For any integers $j \geqslant 1$ *and* $k \geqslant 3$*, we set*

$$f_{k,j} = f_{2,j} * (Id \cdot \tau_{k-2}).$$

Definition 2.2. *For any integers* $n, k \geqslant 1$*, we define the sequence of arithmetic functions* $g_k(n)$ *by*

$$g_k(n) = \sum_{j=1}^{n} f_{k,j}(n).$$

Examples.

$$g_1(n) = \sum_{\substack{j=1 \\ (n,j)=1}}^{n} 1 = \varphi(n),$$

$$g_2(n) = \sum_{j=1}^{n} (j,n) = g(n),$$

$$g_3(n) = \sum_{j=1}^{n} \sum_{\substack{d \mid n \\ d \geqslant j}} \frac{n}{d}(j,d),$$

$$\vdots$$

$$g_k(n) = \sum_{j=1}^{n} \sum_{d_{k-2} \mid n} \sum_{d_{k-3} \mid d_{k-2}} \cdots \sum_{\substack{d_1 \mid d_2 \\ d_1 \geqslant j}} \frac{n}{d_1}(j, d_1).$$

Now we are able to state the following result.

Theorem 2.3. *Let* $\varepsilon > 0$ *be any real number and* $k \geqslant 1$ *any integer. Then, for any real number* $x \geqslant 1$ *sufficiently large, we have*

$$\sum_{n \leqslant x} g_k(n) = \frac{x^2}{2\zeta(2)} \mathcal{R}_{k-1}(\log x) + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right)$$

where $\mathcal{R}_{k-1}$ *is a polynomial of degree* $k-1$ *and leading coefficient* $\frac{1}{(k-1)!}$. *The following table gives* $\mathcal{R}_{k-1}$ *for* $k \in \{1,2,3\}$

k	1	2	3
$\mathcal{R}_{k-1}$	1	$X+\gamma-\frac{1}{2}+\log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)$	$\frac{X^2}{2}+\alpha X+\beta$

where

$$\begin{aligned}\alpha &= 2\gamma-\frac{1}{2}+\log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)\\ \beta &= -\frac{\zeta''(2)}{2\zeta(2)}+\left(\gamma-\log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)\right)^2-\left(3\gamma-\frac{1}{2}\right)\left(\gamma-\log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)\right)\\ &\quad -\frac{1}{4}\left(12\gamma_1-12\gamma^2+6\gamma-1\right)\end{aligned}$$

and

Constant	Name
$\gamma \approx 0.577\ 215\ 664\dots$	Euler − Mascheroni
$\gamma_1 \approx -0.072\ 815\ 845\dots$	Stieltjes
$\mathcal{A} \approx 1.282\ 427\ 129\dots$	Glaisher − Kinkelin

3 Main properties of the function g_k

The following lemma lists the main tools used in the proof of Theorem 2.3.

Lemma 3.1. *For any integer* $k \geqslant 1$, *we have*

$$g_{k+1} = g_k * \mathrm{Id}$$

and then

$$g_k = \varphi * (\mathrm{Id}\cdot\tau_{k-1}).$$

Moreover, we have

$$g_k = \mu * (\mathrm{Id}\cdot\tau_k). \tag{3}$$

Thus, the Dirichlet series $G_k(s)$ *of* g_k *is absolutely convergent in the half-plane* $\Re s > 2$, *and has an analytic continuation to a meromorphic function defined on the whole complex plane with value*

$$G_k(s) = \frac{\zeta(s-1)^k}{\zeta(s)}.$$

Proof. Broughan already proved the first relation for $k=1$ (see [3, Thm. 4.7]), but, for the sake of completeness, we give here another proof.

$$\begin{aligned}
(g_1 * \mathrm{Id})(n) &= (\varphi * \mathrm{Id})(n) = \sum_{d \mid n} d\varphi\left(\frac{n}{d}\right) \\
&= \sum_{d \mid n} d \sum_{\substack{k \leqslant n/d \\ (k,n/d)=1}} 1 = \sum_{d \mid n} d \sum_{\substack{j=1 \\ (j,n)=d}}^{n} 1 \\
&= \sum_{j=1}^{n} (j,n) = \sum_{j=1}^{n} f_{2,j}(n) = g_2(n).
\end{aligned}$$

For $k=2$, we get

$$(g_2 * \mathrm{Id})(n) = \sum_{d \mid n} \frac{n}{d} \sum_{j=1}^{d} (j,d) = \sum_{j=1}^{n} \sum_{\substack{d \mid n \\ d \geqslant j}} \frac{n}{d}(j,d) = \sum_{j=1}^{n} f_{3,j}(n) = g_3(n).$$

Now let us suppose $k \geqslant 3$. We have

$$\begin{aligned}
g_{k+1}(n) &= \sum_{j=1}^{n} f_{k+1,j}(n) = \sum_{j=1}^{n} \left(f_{2,j} * (\mathrm{Id} \cdot \tau_{k-1})\right)(n) \\
&= \sum_{j=1}^{n} \left(f_{2,j} * \mathrm{Id} \cdot \tau_{k-2} * \mathrm{Id}\right)(n) \\
&= \sum_{d \mid n} \frac{n}{d} \sum_{j=1}^{d} \left(f_{2,j} * (\mathrm{Id} \cdot \tau_{k-2})\right)(d) = (g_k * \mathrm{Id})(n).
\end{aligned}$$

The second relation is easily shown by induction. For the third, we have using $\varphi = \mu * \mathrm{Id}$

$$\begin{aligned}
g_k &= \varphi * (\mathrm{Id} \cdot \tau_{k-1}) = \mu * (\mathrm{Id} * (\mathrm{Id} \cdot \tau_{k-1})) \\
&= \mu * (\mathrm{Id} \cdot (\mathbf{1} * \tau_{k-1})) = \mu * (\mathrm{Id} \cdot \tau_k).
\end{aligned}$$

The last proposition comes from the equality (3)

$$g_k = \mu * (\mathrm{Id} \cdot \tau_k) = \mu * \underbrace{\mathrm{Id} * \cdots * \mathrm{Id}}_{k \text{ times}}$$

and the Dirichlet series of μ and Id. □

4 Proof of Theorem 1

Lemma 4.1. *For any integer $k \geqslant 1$ and any real numbers $x > 1$ and $\varepsilon > 0$, we have*

$$\sum_{n \leqslant x} n\tau_k(n) = x^2 \mathcal{Q}_{k-1}(\log x) + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right)$$

where $\mathcal{Q}_{k-1}$ is a polynomial of degree $k-1$ and leading coefficient $\frac{1}{2(k-1)!}$.

Proof. Using summation by parts and (2), we get

$$\begin{aligned}\sum_{n\leqslant x} n\tau_k(n) &= x\sum_{n\leqslant x}\tau_k(n) - \int_1^x \left(\sum_{n\leqslant t}\tau_k(n)\right) dt \\ &= x^2\mathcal{P}_{k-1}(\log x) + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right) - \int_1^x \left(t\mathcal{P}_{k-1}(\log t) + O_{\varepsilon,k}\left(t^{\theta_k+\varepsilon}\right)\right) dt.\end{aligned}$$

Writing

$$\mathcal{P}_{k-1}(X) = \sum_{j=0}^{k-1} a_j X^j$$

with $a_{k-1} = \frac{1}{(k-1)!}$, we obtain

$$\sum_{n\leqslant x} n\tau_k(n) = x^2\sum_{j=0}^{k-1} a_j(\log x)^j - \sum_{j=0}^{k-1} a_j\int_1^x t(\log t)^j\, dt + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right)$$

and the formula

$$\int_1^x t(\log t)^j\, dt = x^2\sum_{i=0}^{j}(-1)^{j-i}\frac{j!}{2^{j+1-i}\times i!}(\log x)^i - (-1)^j\frac{j!}{2^{j+1}}$$

(easily proved by induction) gives

$$\begin{aligned}\sum_{n\leqslant x} n\tau_k(n) &= x^2\sum_{j=0}^{k-1} a_j\left\{(\log x)^j - \sum_{i=0}^{j}(-1)^{j-i}\frac{j!}{2^{j+1-i}\times i!}(\log x)^i\right\} \\ &\quad + \sum_{i=0}^{j}(-1)^j\frac{j!\,a_j}{2^{j+1}} + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right) \\ &= x^2\sum_{j=0}^{k-1} a_j\left\{\frac{(\log x)^j}{2} - \sum_{i=0}^{j-1}(-1)^{j-i}\frac{j!}{2^{j+1-i}\times i!}(\log x)^i\right\} + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right)\end{aligned}$$

which completes the proof of the lemma. □

Remark. For $k=3$, the following result is well-known (see [7, Exer. II.3.4], for example)

$$\sum_{n\leqslant x}\tau_3(n) = x\left\{\frac{(\log x)^2}{2} + (3\gamma-1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1\right\} + O_\varepsilon\left(x^{\theta_3+\varepsilon}\right)$$

and gives

$$\sum_{n\leqslant x} n\tau_3(n) = x^2\left\{\frac{(\log x)^2}{4} + \left(\frac{6\gamma-1}{4}\right)\log x - \frac{12\gamma_1 - 12\gamma^2 + 6\gamma - 1}{8}\right\} + O_\varepsilon\left(x^{1+\theta_3+\varepsilon}\right). \tag{4}$$

Now we are able to prove Theorem 2.3.

Using (3) we get

$$\sum_{n\leqslant x} g_k\left(n\right) = \sum_{d\leqslant x} \mu\left(d\right) \sum_{m\leqslant x/d} m\tau_k\left(m\right),$$

and lemma 3.1 gives

$$\begin{aligned}\sum_{n\leqslant x} g_k\left(n\right) &= \sum_{d\leqslant x} \mu\left(d\right) \left\{ \left(\frac{x}{d}\right)^2 \mathcal{Q}_{k-1}\left(\log\frac{x}{d}\right) + O_{\varepsilon,k}\left(\left(\frac{x}{d}\right)^{1+\theta_k+\varepsilon}\right)\right\} \\ &= x^2 \sum_{d\leqslant x} \frac{\mu\left(d\right)}{d^2} \mathcal{Q}_{k-1}\left(\log\frac{x}{d}\right) + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right).\end{aligned}$$

Writing

$$\mathcal{Q}_{k-1}\left(X\right) = \sum_{j=0}^{k-1} b_j X^j$$

with $b_{k-1} = \frac{1}{2(k-1)!}$, we get

$$\begin{aligned}\sum_{n\leqslant x} g_k\left(n\right) &= x^2 \sum_{d\leqslant x} \frac{\mu\left(d\right)}{d^2} \sum_{j=0}^{k-1} b_j \left(\log\frac{x}{d}\right)^j + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right) \\ &= x^2 \sum_{j=0}^{k-1}\sum_{h=0}^{j} \binom{j}{h} b_j \left(\log x\right)^{j-h} \sum_{d\leqslant x} \left(-1\right)^h \frac{\mu\left(d\right)}{d^2} \left(\log d\right)^h + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right)\end{aligned}$$

and the equality

$$\begin{aligned}\sum_{d\leqslant x} \left(-1\right)^h \frac{\mu\left(d\right)}{d^2} \left(\log d\right)^h &= \sum_{d=1}^{\infty} \left(-1\right)^h \frac{\mu\left(d\right)}{d^2} \left(\log d\right)^h - \sum_{d>x} \left(-1\right)^h \frac{\mu\left(d\right)}{d^2} \left(\log d\right)^h \\ &= \left[\frac{d^h}{ds^h}\left(\frac{1}{\zeta\left(s\right)}\right)\right]_{[s=2]} + O\left(\frac{\left(\log x\right)^h}{x}\right),\end{aligned}$$

implies

$$\begin{aligned}\sum_{n\leqslant x} g_k\left(n\right) &= x^2 \sum_{j=0}^{k-1}\sum_{h=0}^{j} \binom{j}{h} \left(\left[\frac{d^h}{ds^h}\left(\frac{1}{\zeta\left(s\right)}\right)\right]_{[s=2]}\right) b_j \left(\log x\right)^{j-h} \\ &\quad + O\left(x\left(\log x\right)^{k-1}\right) + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right) \\ &= x^2 \sum_{j=0}^{k-1}\sum_{h=0}^{j} \binom{j}{h} \left(\left[\frac{d^h}{ds^h}\left(\frac{1}{\zeta\left(s\right)}\right)\right]_{[s=2]}\right) b_j \left(\log x\right)^{j-h} + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right),\end{aligned}$$

and writing

$$\left[\frac{d^h}{ds^h}\left(\frac{1}{\zeta\left(s\right)}\right)\right]_{[s=2]} = \frac{A_h}{2\zeta\left(2\right)^{h+1}}$$

with $A_h \in \mathbb{R}$ (and $A_0 = 2$), we obtain

$$\sum_{n \leqslant x} g_k(n) = \frac{x^2}{2\zeta(2)} \sum_{j=0}^{k-1} \sum_{h=0}^{j} \binom{j}{h} \frac{A_h b_j (\log x)^{j-h}}{\zeta(2)^h} + O_{\varepsilon,k}\left(x^{1+\theta_k+\varepsilon}\right)$$

which is the desired result. The leading coefficient is $\binom{k-1}{0} A_0 b_{k-1} = \frac{1}{(k-1)!}$. The particular cases are easy to check.

(i) For $k = 1$, the result is well-known (see [2, Exer. 4.14])

$$\sum_{n \leqslant x} g_1(n) = \sum_{n \leqslant x} \varphi(n) = \frac{x^2}{2\zeta(2)} + O(x \log x).$$

(ii) For $k = 2$, see [1].

(iii) For $k = 3$, we use (4) and the computations made above. The proof of Theorem 2.3 is now complete.

5 Sums of reciprocals of the gcd

The purpose of this section is to prove the following estimate.

Theorem 5.1. *For any real number $x > e$ sufficiently large, we have*

$$\sum_{n \leqslant x} \left(\sum_{j=1}^{n} \frac{1}{(j,n)} \right) = \frac{\zeta(3)}{2\zeta(2)} x^2 + O\left(x (\log x)^{2/3} (\log \log x)^{4/3}\right).$$

Proof. For any integer $n \geqslant 1$, we set

$$\mathcal{G}(n) = \sum_{j=1}^{n} \frac{1}{(j,n)}.$$

With a similar argument used in the proof of the identity $g = \varphi * \mathrm{Id}$ (see lemma 3.1), it is easy to check that

$$\mathcal{G} = \varphi * \mathrm{Id}^{-1},$$

and thus

$$\sum_{n \leqslant x} \mathcal{G}(n) = \sum_{d \leqslant x} \frac{1}{d} \sum_{m \leqslant x/d} \varphi(m).$$

The well-known result (see [8], for example)

$$\sum_{n \leqslant x} \varphi(n) = \frac{x^2}{2\zeta(2)} + O\left(x (\log x)^{2/3} (\log \log x)^{4/3}\right),$$

combined with some classical computations, allows us to conclude the proof of Theorem 5.1. □

6 The lcm-sum function

Definition 6.1. *For any integer $n \geqslant 1$, we define*

$$l(n) = \sum_{j=1}^{n} [n, j]$$

where $[a, b]$ is the least common multiple of a and b.

Lemma 6.2. *We have the following convolution identity*

$$l = \frac{1}{2}\left(\left(\mathrm{Id}^2 \cdot (\varphi + \tau_0)\right) * \mathrm{Id}\right).$$

Proof. We have

$$\sum_{j=1}^{n} \frac{j}{(n,j)} = \sum_{d|n} \frac{1}{d} \sum_{\substack{j=1 \\ (n,j)=d}}^{n} j = \sum_{d|n} \frac{1}{d} \sum_{\substack{k \leqslant n/d \\ (k,n/d)=1}} kd = \sum_{d|n} \sum_{\substack{k \leqslant n/d \\ (k,n/d)=1}} k,$$

with

$$\begin{aligned}
\sum_{\substack{k \leqslant N \\ (k,N)=1}} k &= \sum_{d|N} d\mu(d) \sum_{m \leqslant N/d} m \\
&= \frac{1}{2} \sum_{d|N} d\mu(d) \left\{ \frac{N}{d} \left(\frac{N}{d} + 1 \right) \right\} \\
&= \frac{N}{2} \sum_{d|N} \mu(d) \left(\frac{N}{d} + 1 \right) = \frac{N}{2} (\varphi + \tau_0)(N),
\end{aligned}$$

and hence

$$\sum_{j=1}^{n} \frac{j}{(n,j)} = \frac{1}{2} \sum_{d|n} \frac{n}{d} (\varphi + \tau_0)\left(\frac{n}{d}\right) = \frac{1}{2} \left((\mathrm{Id} \cdot (\varphi + \tau_0)) * \mathbf{1} \right)(n),$$

and we conclude by noting that

$$l(n) = n \sum_{j=1}^{n} \frac{j}{(n,j)}$$

which completes the proof, since Id is completely multiplicative. □

Theorem 6.3. *For any real number $x > e$ sufficiently large, we have the following estimate*

$$\sum_{n \leqslant x} \left(\sum_{j=1}^{n} [n, j] \right) = \frac{\zeta(3)}{8\zeta(2)} x^4 + O\left(x^3 (\log x)^{2/3} (\log\log x)^{4/3} \right).$$

Proof. Using lemma 6.2, we get

$$\begin{aligned}\sum_{n\leqslant x} l(n) &= \frac{1}{2}\sum_{d\leqslant x} d \sum_{m\leqslant x/d} m^2 (\varphi+\tau_0)(m)\\ &= \frac{1}{2}\sum_{d\leqslant x} d \sum_{m\leqslant x/d} m^2 \varphi(m) + O\left(x^2\right)\end{aligned}$$

and the estimation (see [8])

$$\sum_{n\leqslant x} n^2\varphi(n) = \frac{x^4}{4\zeta(2)} + O\left(x^3 (\log x)^{2/3} (\log\log x)^{4/3}\right)$$

implies

$$\begin{aligned}\sum_{n\leqslant x} l(n) &= \frac{1}{2}\sum_{d\leqslant x} d\left\{\frac{1}{4\zeta(2)}\left(\frac{x}{d}\right)^4 + O\left(\left(\frac{x}{d}\right)^3 (\log x)^{2/3}(\log\log x)^{4/3}\right)\right\} + O\left(x^2\right)\\ &= \frac{x^4}{8\zeta(2)}\sum_{d=1}^{\infty}\frac{1}{d^3} + O\left(x^3(\log x)^{2/3}(\log\log x)^{4/3}\right) + O\left(x^2\right),\end{aligned}$$

which is the desired result. □

7 Sum of reciprocals of the lcm

We will prove the following result.

Theorem 7.1. *For any real number $x > 1$ sufficiently large, we have*

$$\sum_{n\leqslant x}\left(\sum_{j=1}^{n}\frac{1}{[n,j]}\right) = \frac{(\log x)^3}{6\zeta(2)} + \frac{(\log x)^2}{2\zeta(2)}\left(\gamma + \log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)\right) + O(\log x).$$

Some useful estimates are needed.

Lemma 7.2. *Set* $C_\varphi = \log\left(\frac{\mathcal{A}^{12}}{2\pi}\right) \approx 1.147\,176\ldots$ *For any real number $x \geqslant 1$, we have*

$$(i): \sum_{n\leqslant x}\frac{\varphi(n)}{n^2} = \frac{\log x}{\zeta(2)} + \frac{C_\varphi}{\zeta(2)} + O\left(\frac{\log ex}{x}\right).$$

$$(ii): \sum_{n\leqslant x}\frac{\varphi(n)}{n^2}\log\left(\frac{x}{n}\right) = \frac{(\log x)^2}{2\zeta(2)} + \frac{C_\varphi \log x}{\zeta(2)} + O(1).$$

$$(iii): \frac{1}{2}\sum_{n\leqslant x}\frac{\varphi(n)}{n^2}\left(\log\left(\frac{x}{n}\right)\right)^2 = \frac{(\log x)^3}{6\zeta(2)} + \frac{C_\varphi(\log x)^2}{2\zeta(2)} + O(\log x).$$

Proof. (i). Using $\varphi = \mu *\, \mathrm{Id}$, we get

$$\begin{aligned}
\sum_{n\leqslant x}\frac{\varphi(n)}{n^2} &= \sum_{d\leqslant x}\frac{\mu(d)}{d^2}\sum_{m\leqslant x/d}\frac{1}{m}\\
&= \sum_{d\leqslant x}\frac{\mu(d)}{d^2}\left\{\log\left(\frac{x}{d}\right)+\gamma+O\left(\frac{d}{x}\right)\right\}\\
&= (\log x+\gamma)\sum_{d\leqslant x}\frac{\mu(d)}{d^2}-\sum_{d\leqslant x}\frac{\mu(d)\log d}{d^2}+O\left(\frac{1}{x}\sum_{d\leqslant x}\frac{1}{d}\right)\\
&= \frac{\log x}{\zeta(2)}+\frac{\gamma}{\zeta(2)}-\frac{\zeta'(2)}{(\zeta(2))^2}+O\left(\frac{\log ex}{x}\right).
\end{aligned}$$

Recall that $\dfrac{\zeta'(2)}{\zeta(2)} = \gamma - C_\varphi$.

(ii) and (iii). Abel's summation and estimate (i). We leave the details to the reader. □

Now we are able to show Theorem 7.1. For any integer $n \geqslant 1$, we set

$$\mathcal{L}(n) = \sum_{j=1}^{n}\frac{1}{[n,j]}.$$

Since

$$\begin{aligned}
\mathcal{L}(n) &= \frac{1}{n}\sum_{j=1}^{n}\frac{(n,j)}{j} = \frac{1}{n}\sum_{d\mid n} d\sum_{\substack{j=1\\(j,n)=d}}^{n}\frac{1}{j}\\
&= \frac{1}{n}\sum_{d\mid n} d\sum_{\substack{k\leqslant n/d\\(k,n/d)=1}}\frac{1}{kd} = \frac{1}{n}\sum_{d\mid n}\sum_{\substack{k\leqslant n/d\\(k,n/d)=1}}\frac{1}{k},
\end{aligned}$$

we get

$$
\begin{aligned}
\sum_{n\leqslant x}\mathcal{L}(n) &= \sum_{n\leqslant x}\frac{1}{n}\sum_{d\mid n}\sum_{\substack{k\leqslant n/d\\(k,n/d)=1}}\frac{1}{k}\\
&= \sum_{d\leqslant x}\frac{1}{d}\sum_{h\leqslant x/d}\frac{1}{h}\sum_{\substack{k\leqslant h\\(k,h)=1}}\frac{1}{k}\\
&= \sum_{d\leqslant x}\frac{1}{d}\sum_{h\leqslant x/d}\frac{1}{h}\sum_{\delta\mid h}\frac{\mu(\delta)}{\delta}\sum_{m\leqslant h/\delta}\frac{1}{m}\\
&= \sum_{d\leqslant x}\frac{1}{d}\sum_{\delta\leqslant x/d}\frac{\mu(\delta)}{\delta^2}\sum_{a\leqslant x/(d\delta)}\frac{1}{a}\sum_{m\leqslant a}\frac{1}{m}\\
&= \sum_{d\leqslant x}\sum_{\delta d\leqslant x}\frac{1}{d}\frac{\mu(\delta)}{\delta^2}\sum_{a\leqslant x/(d\delta)}\frac{1}{a}\sum_{m\leqslant a}\frac{1}{m}\\
&= \sum_{n\leqslant x}\frac{1}{n^2}\sum_{d\mid n}d\mu\left(\frac{n}{d}\right)\sum_{a\leqslant x/n}\frac{1}{a}\sum_{m\leqslant a}\frac{1}{m},
\end{aligned}
$$

and the convolution identity $\varphi = \mu * \mathrm{Id}$ implies that

$$
\sum_{n\leqslant x}\mathcal{L}(n) = \sum_{n\leqslant x}\frac{\varphi(n)}{n^2}\sum_{a\leqslant x/n}\frac{1}{a}\sum_{m\leqslant a}\frac{1}{m}.
$$

Thus

$$
\begin{aligned}
\sum_{n\leqslant x}\mathcal{L}(n) &= \sum_{n\leqslant x}\frac{\varphi(n)}{n^2}\sum_{a\leqslant x/n}\frac{1}{a}\left\{\log a + \gamma + O\left(\frac{1}{a}\right)\right\}\\
&= \sum_{n\leqslant x}\frac{\varphi(n)}{n^2}\left\{\frac{1}{2}\left(\log\frac{x}{n}\right)^2 + \gamma\left(\log\frac{x}{n}\right) + O(1)\right\}\\
&= \frac{(\log x)^3}{6\zeta(2)} + \frac{C_\varphi(\log x)^2}{2\zeta(2)} + \frac{\gamma(\log x)^2}{2\zeta(2)} + O(\log x)
\end{aligned}
$$

where $C_\varphi = \log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)$, which concludes the proof.

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