Journal of Integer Sequences, Vol. 10 (2007),
Article 07.3.3

A Note on the Average Order of the gcd-sum Function

Olivier Bordellès
2, Allée de la Combe
La Boriette
43000 Aiguilhe
France
borde43@wanadoo.fr

Abstract

We prove an asymptotic formula for the average order of the gcd-sum function by using a new convolution identity.

1 Introduction and main result

In 2001, Broughan [1] studied the gcd-sum function g defined for any positive integer n by

$$g(n) = \sum_{k=1}^{n} (k, n),$$

where (a, b) denotes the greatest common divisor of a and b. The author showed that g is multiplicative, and satisfies the convolution identity

$$g = \varphi * \text{ Id}, \tag{1}$$

where φ is the Euler totient function, Id is the completely multiplicative function defined by $\text{Id}(n) = n$ and $*$ is the usual Dirichlet convolution product.

The function g appears in a specific lattice point problem [1, 6], where it can be used to estimate the number of integer coordinate points under the square-root curve. As a multiplicative function, the question of its average order naturally arises. By using the

Dirichlet hyperbola principle, Broughan [1, Theorem 4.7] proved the following result: for any real number $x \geqslant 1$, the following estimate

$$\sum_{n \leqslant x} g(n) = \frac{x^2 \log x}{2\zeta(2)} + \frac{\zeta(2)^2}{2\zeta(3)} x^2 + O\left(x^{3/2} \log x\right) \tag{2}$$

holds.

The aim of this paper is to prove another convolution identity for g, and then get a fairly more precise estimate than (2).

In what follows, τ is the well-known divisor function, μ is the Möbius function, $\mathbf{1}$ is the completely multiplicative function defined by $\mathbf{1}(n) = 1$, $F * G$ is the Dirichlet convolution product of the arithmetical functions F and G, and we denote by θ the smallest positive real number such that

$$\sum_{n \leqslant x} \tau(n) = x \log x + x(2\gamma - 1) + O_{\varepsilon}\left(x^{\theta+\varepsilon}\right) \tag{3}$$

holds for any real numbers $x \geqslant 1$ and $\varepsilon > 0$. The following inequality

$$\theta \geqslant \frac{1}{4}$$

is well-known [3]. On the other hand, Huxley [4] showed that

$$\theta \leqslant \frac{131}{416} \approx 0.3149\ldots$$

holds. Now we are able to prove the following result

Theorem 1.1. *For any real numbers $x \geqslant 1$ and $\varepsilon > 0$, we have*

$$\sum_{n \leqslant x} g(n) = \frac{x^2 \log x}{2\zeta(2)} + \frac{x^2}{2\zeta(2)} \left(\gamma - \frac{1}{2} + \log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)\right) + O_{\varepsilon}\left(x^{1+\theta+\varepsilon}\right)$$

where $\mathcal{A} \approx 1.282\,427\,129\ldots$ is the Glaisher-Kinkelin constant.

For further details about the Glaisher-Kinkelin constant, see [2, 5]. The reader interested in gcd-sum integer sequences should refer to Sloane's sequence A018804.

2 A convolution identity

The proof uses the following lemmas.

Lemma 2.1. *For any real number $z \geqslant 1$ and any $\varepsilon > 0$, we have*

$$\sum_{n \leqslant z} n\tau(n) = \frac{z^2}{2} \log z + z^2 \left(\gamma - \frac{1}{4}\right) + O_{\varepsilon}\left(z^{1+\theta+\varepsilon}\right).$$

Proof. The result follows easily from (3) and Abel's summation. □

Lemma 2.2. *We have*

$$g = \mu * (Id \cdot \tau) .$$

Proof. Since $\varphi = \mu *$ Id, we have, using (1) ,

$$g = \varphi * \mathrm{Id} = \mu * \ (\mathrm{Id} * \mathrm{Id}) = \mu * (\mathrm{Id} \cdot \tau)$$

which is the desired result. □

3 Proof of Theorem 1.1

By using Lemma 2.2, we get

$$\sum_{n \leqslant x} g(n) = \sum_{d \leqslant x} \mu(d) \sum_{k \leqslant x/d} k\tau(k)$$

and Lemma 2.1 applied to the inner sum gives

$$\begin{aligned} \sum_{n \leqslant x} g(n) &= \sum_{d \leqslant x} \mu(d) \left\{ \frac{x^2}{d^2} \left(\frac{1}{2} \log\left(\frac{x}{d}\right) + \gamma - \frac{1}{4} \right) + O_\varepsilon \left(\left(\frac{x}{d}\right)^{1+\theta+\varepsilon} \right) \right\} \\ &= x^2 \left\{ \left(\frac{1}{2} \log x + \gamma - \frac{1}{4} \right) \sum_{d \leqslant x} \frac{\mu(d)}{d^2} - \sum_{d \leqslant x} \frac{\mu(d) \log d}{2d^2} \right\} + O_\varepsilon \left(x^{1+\theta+\varepsilon} \sum_{d \leqslant x} \frac{1}{d^{1+\theta+\varepsilon}} \right) \\ &= x^2 \left\{ \left(\frac{1}{2} \log x + \gamma - \frac{1}{4} \right) \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d=1}^{\infty} \frac{\mu(d) \log d}{2d^2} + O\left(\frac{\log x}{x}\right) \right\} + O_\varepsilon \left(x^{1+\theta+\varepsilon} \right) . \end{aligned}$$

Now it is well-known that, for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$, we have

$$\frac{1}{\zeta(s)} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^s}$$

which gives by differentiation

$$\frac{\zeta'(s)}{(\zeta(s))^2} = \sum_{d=1}^{\infty} \frac{\mu(d) \log d}{d^s}$$

for $\operatorname{Re} s > 1$, and hence

$$\sum_{n \leqslant x} g(n) = \frac{x^2}{2\zeta(2)} \left(\log x - \frac{\zeta'(2)}{\zeta(2)} + 2\gamma - \frac{1}{2} \right) + O_\varepsilon \left(x^{1+\theta+\varepsilon} \right) ,$$

and we use

$$\frac{\zeta'(2)}{\zeta(2)} = \gamma - \log\left(\frac{\mathcal{A}^{12}}{2\pi} \right) .$$

The proof of the theorem is now complete.

Acknowledgment. I am grateful to the referee for his valuable suggestions on this paper.

References

[1] K. A. Broughan, The gcd-sum function, *J. Integer Sequences* **4** (2001), Art. 01.2.2.

[2] S. R. Finch, *Mathematical Constants*, Cambridge University Press, 2003, pp. 135–145.

[3] G. H. Hardy, The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, *Proc. London Math. Soc.* **15** (2) (1916), 192–213.

[4] M. N. Huxley, Exponential sums and lattice points III, *Proc. London Math. Soc.* **87** (2003), 591–609.

[5] H. Kinkelin, Über eine mit der Gammafunktion verwandte Transcendente und deren Anwendung auf die Integralrechnung, *J. Reine Angew. Math.* **57** (1860), 122–158.

[6] A. D. Loveless, The general GCD-product function, *Integers* **6** (2006), article A19, available at `http://www.integers-ejcnt.org/vol6.html`. Corrigendum: **6** (2006), article A39.

2000 *Mathematics Subject Classification*: Primary 11A25; Secondary 11N37.
Keywords: gcd-sum function, Dirichlet convolution, average order of multiplicative functions .

(Concerned with sequence A018804.)

Received July 10 2006; revised version received March 28 2007. Published in *Journal of Integer Sequences*, March 28 2007.

Return to Journal of Integer Sequences home page.